The Complexity of Tree Multicolorings

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Minimum sum multicoloring

- **Given:** a graph $G(V, E)$, and demand function $x : V \to \mathbb{N}$
- **Find:** an assignment Ψ of $x(v)$ colors (integers) to every vertex v, such that neighbors receive disjoint sets
- **Goal:** The *finish time* $f(v)$ of vertex v is the largest color (integer) assigned to it in the coloring. Minimize $\sum_{v \in V} f(v)$, the *sum of the coloring*.

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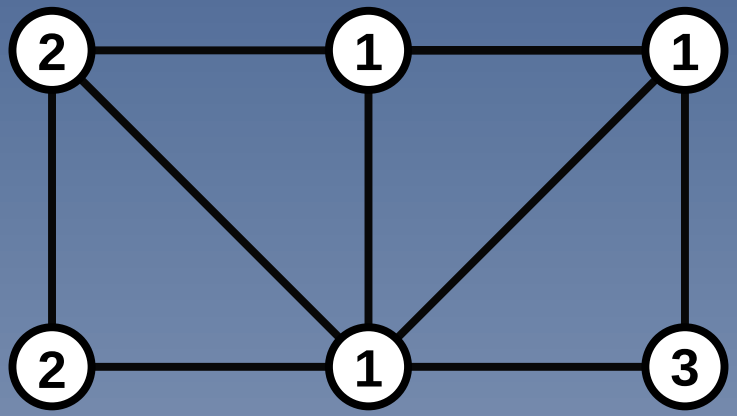

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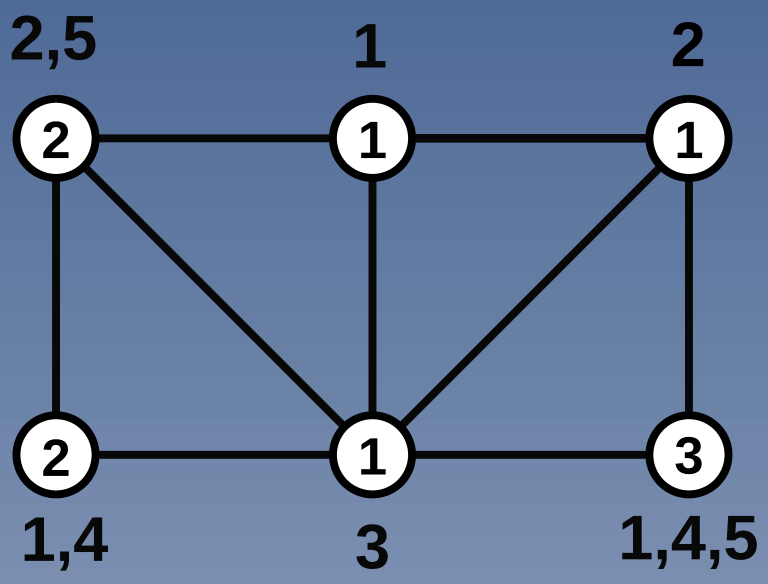

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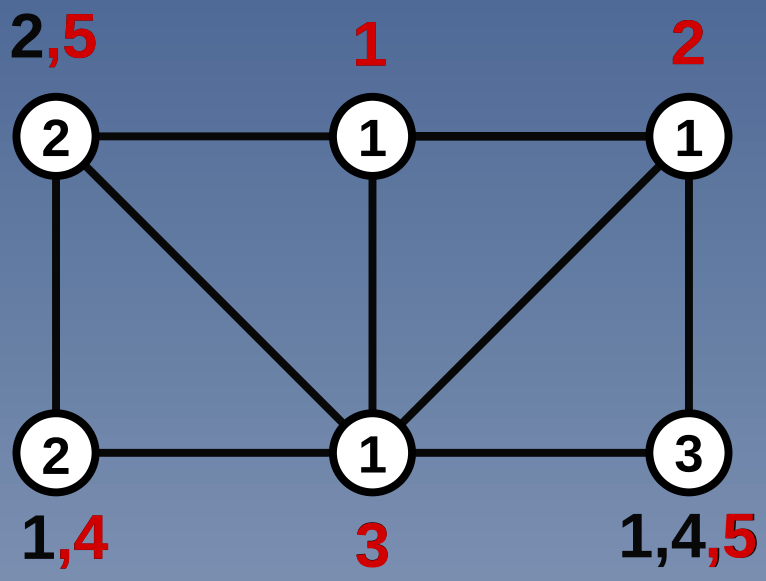


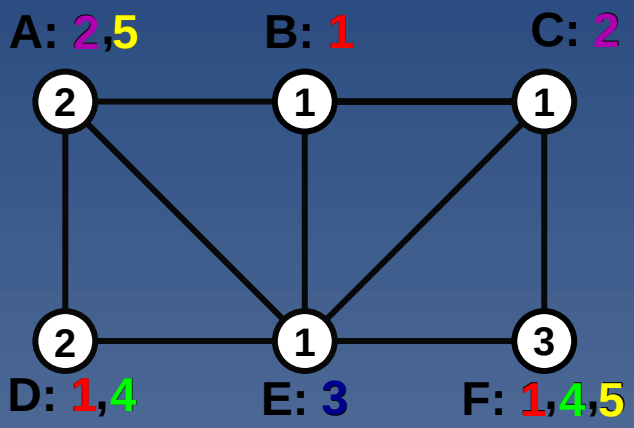
Sum of the coloring:
5 + 1 + 2 + 4 + 3 + 5 = 20

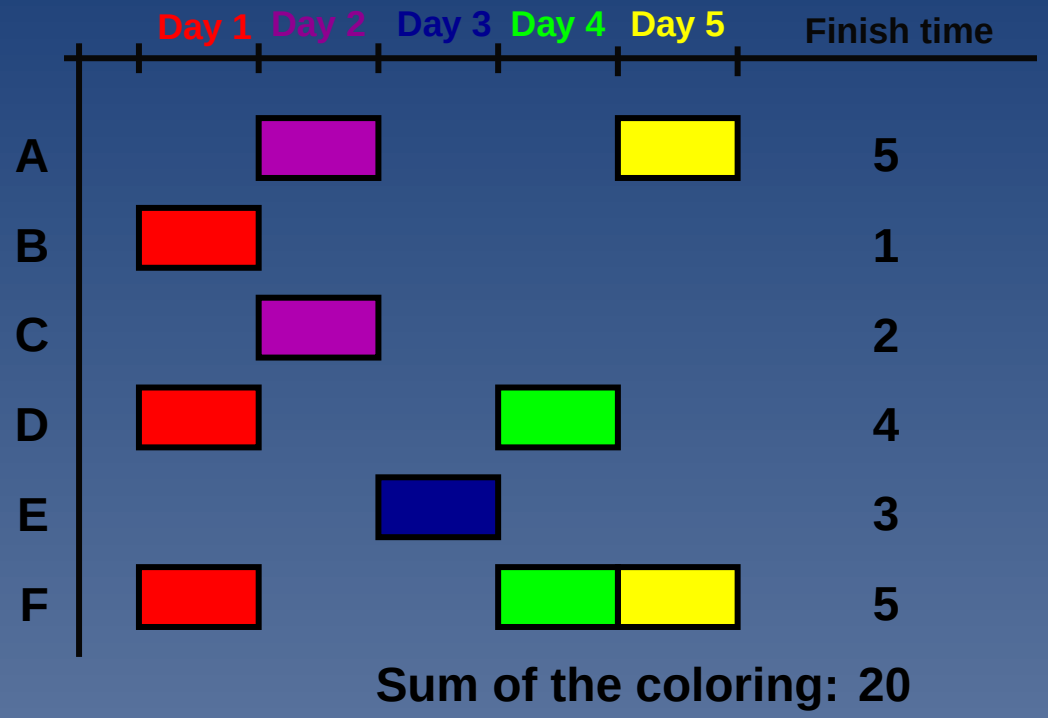
Application in scheduling

Scheduling of interfering jobs, minimizing the sum of completion times (same as minimizing the average completion times)

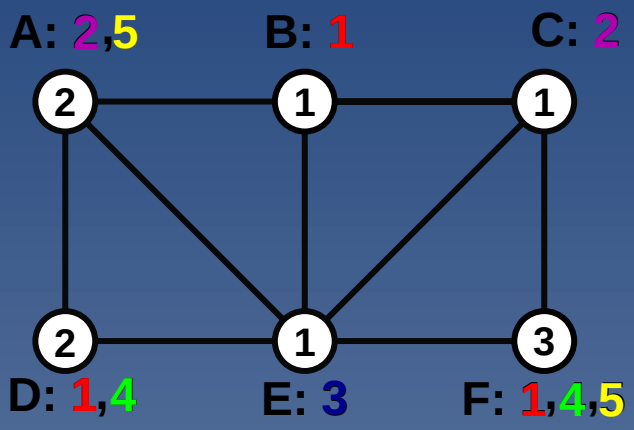
vertices	$\Longleftrightarrow$	jobs
demands	$\Longleftrightarrow$	days required
edges	$\Longleftrightarrow$	interfering pairs of jobs
colors	$\Longleftrightarrow$	days
assignment of colors	$\Longleftrightarrow$	assignment of days
finish time of a vertex	$\Longleftrightarrow$	day when the job is finished
sum of the coloring	$\Longleftrightarrow$	sum of the completion times

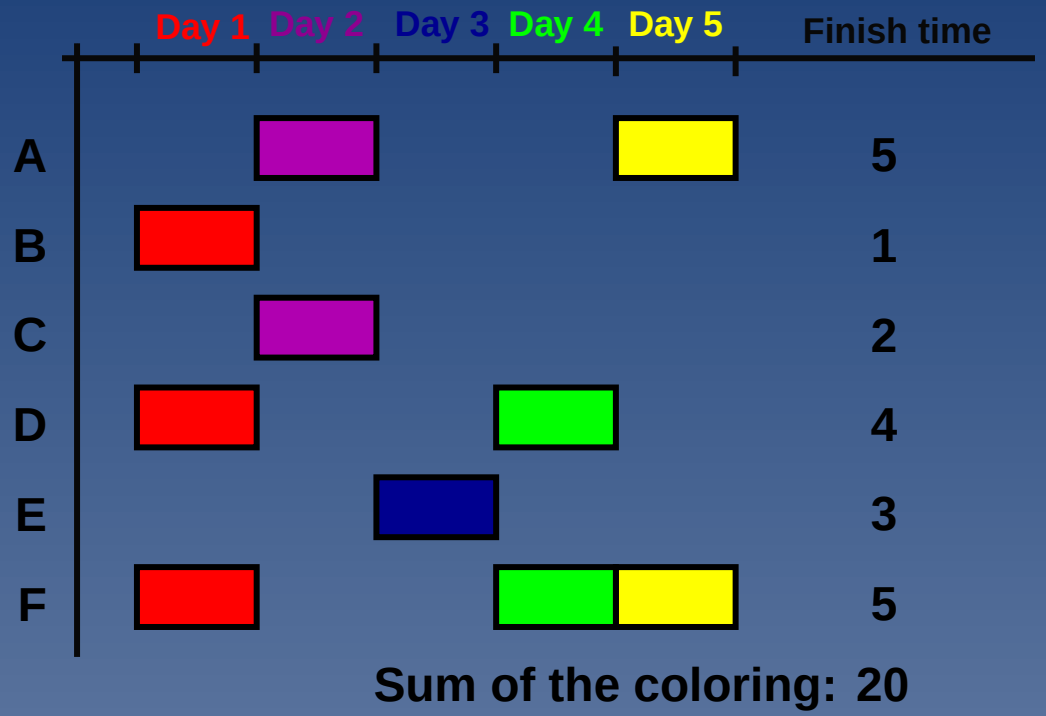
Example


A: 2,5
B: 1
C: 2
2
1
1
2
1
3
D: 1,4
E: 3
F: 1,4,5


Day 1
Day 2
Day 3
Day 4
Day 5
Finish time
A
B
C
D
E
F
5
1
2
4
3
5
Sum of the coloring: 20

Example





Preemptive scheduling: the jobs can be interrupted

Known results

Special case, the chromatic sum problem: $x(v) = 1,\ \forall v \in V$

- **General graphs:**
  - ⋆ cannot be approximated within $n^{1-\epsilon}$ even if every demand is 1 (unless **NP** $=$ **ZPP**) [Bar-Noy et al., 1998],
  - ⋆ $O(n/log^2 n)$ approximation for sum multicoloring [Bar-Noy et al., 2000]
- **Bipartite graphs:**
  - ⋆ 1.5-approximation for sum multicoloring [Bar-Noy and Kortsarz, 1998]
  - ⋆ **APX**-hard, even if every demand is 1

Known results

- **Planar graphs:**
  - $(1+\epsilon)$-approximation for sum multicoloring [Halldórsson and Kortsarz, 1999]
  - **NP**-complete even if every demand is 1
- **Trees:**
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New result: Minimum sum multicoloring is **NP**-hard on binary trees, even if every demand is polynomially bounded (in the size of the tree)

List multicoloring

As a first step of the proof, we introduce another problem where trees are difficult to color:

List multicoloring

- **Given:** a graph $G(V, E)$, a demand function $x : V \to \mathbb{N}$, and a set of available colors $L(v)$ for every vertex
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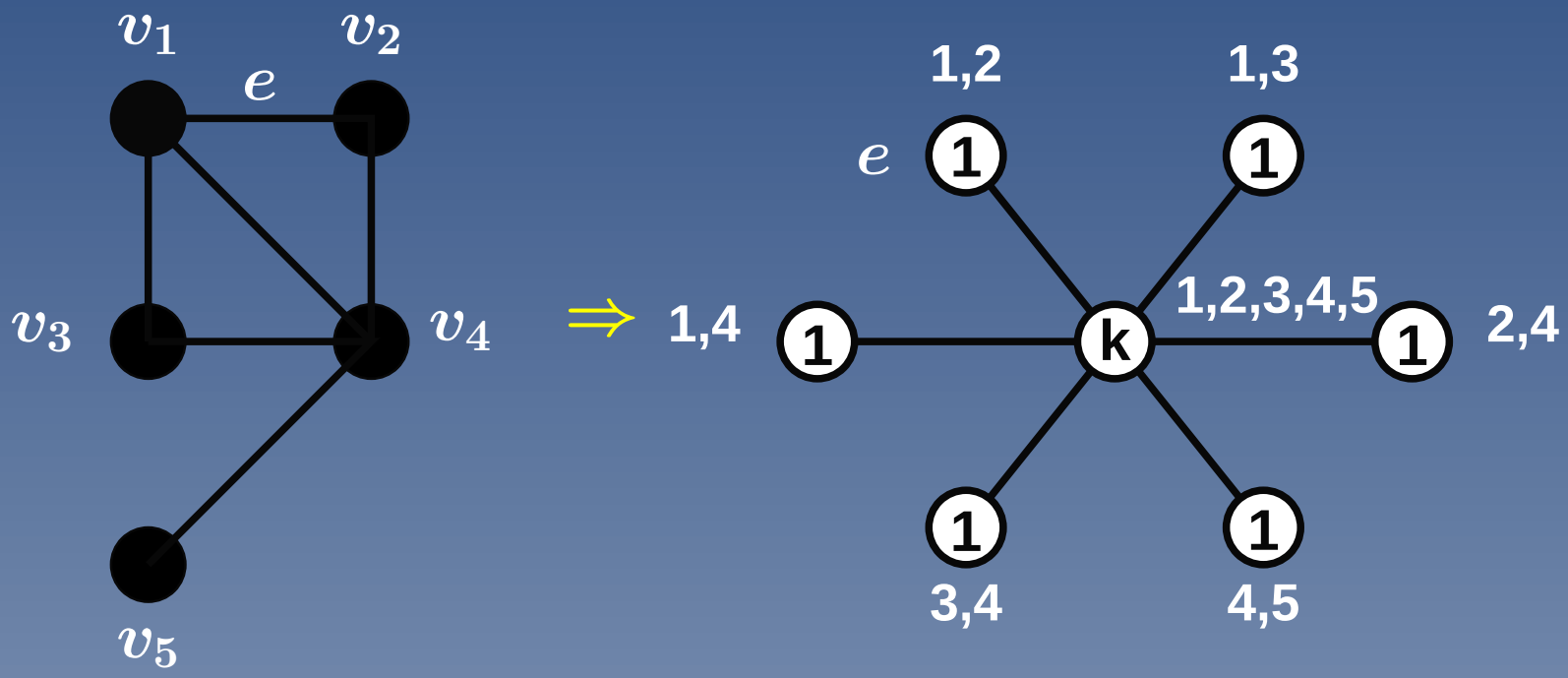
New result: List multicoloring is **NP**-complete in binary trees.

Theorem: List multicoloring is **NP**-complete in trees.

(Sketch of proof) Reduction from the maximum independent set problem ("Is there an independent set of size k?")

The tree is a star with one leaf for each edge.
For every edge $v_x v_y$, let $\{x, y\}$ be the list of the corresponding leaf.
The list of the central node v contains every color.



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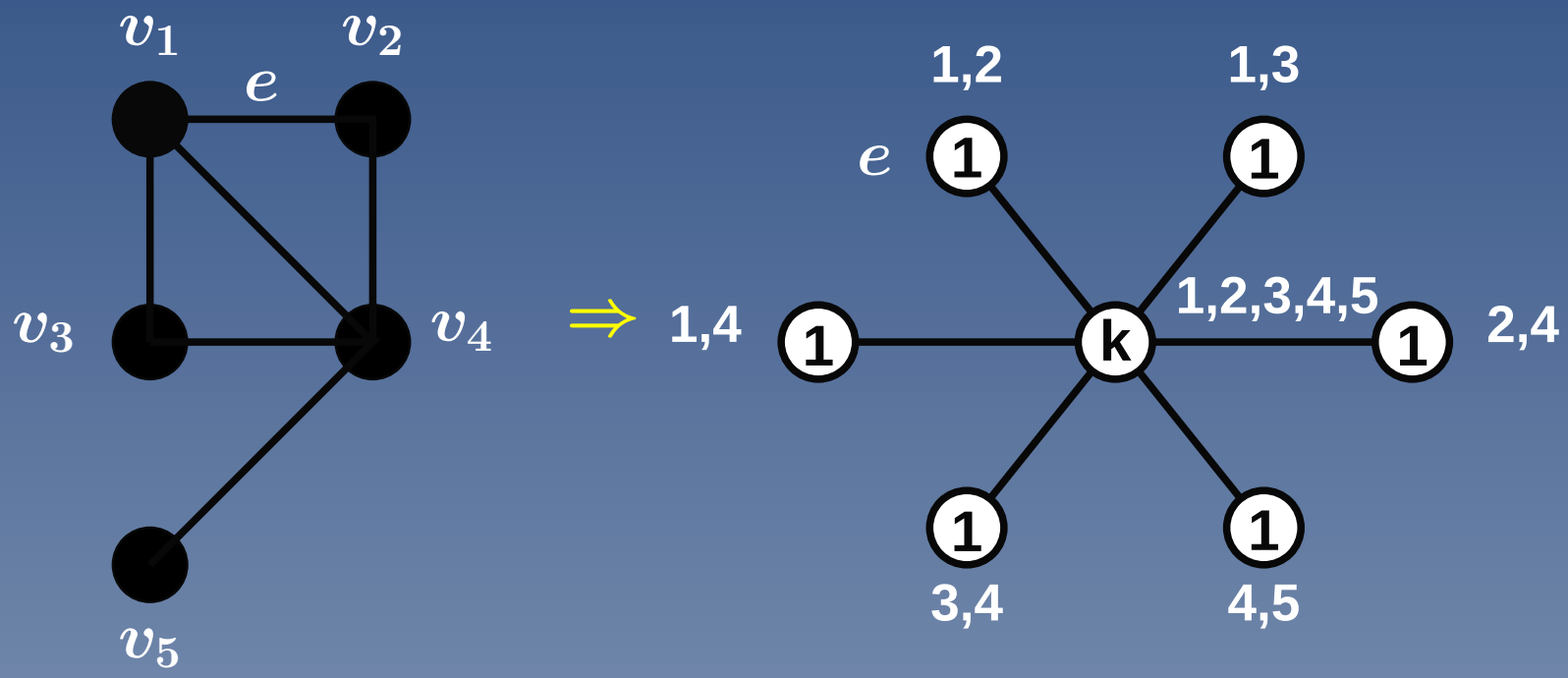

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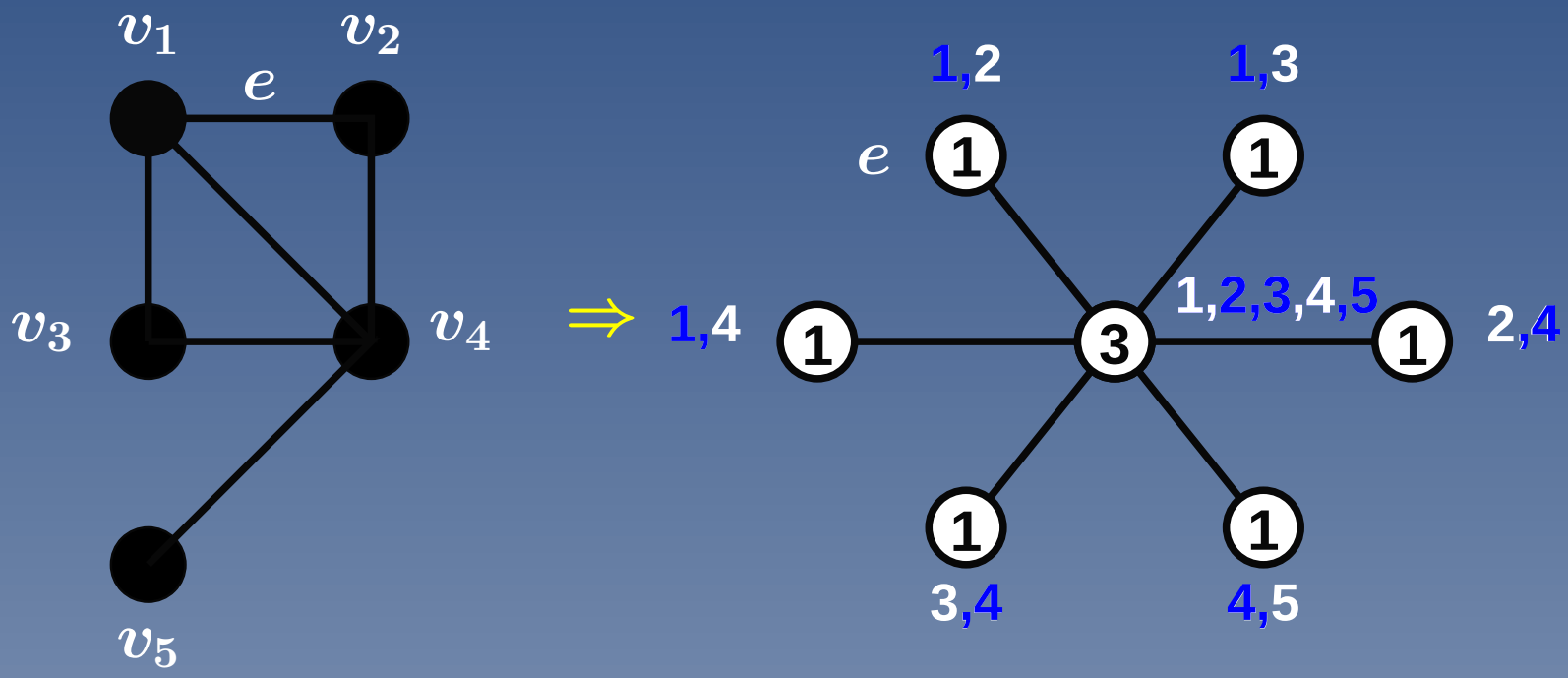

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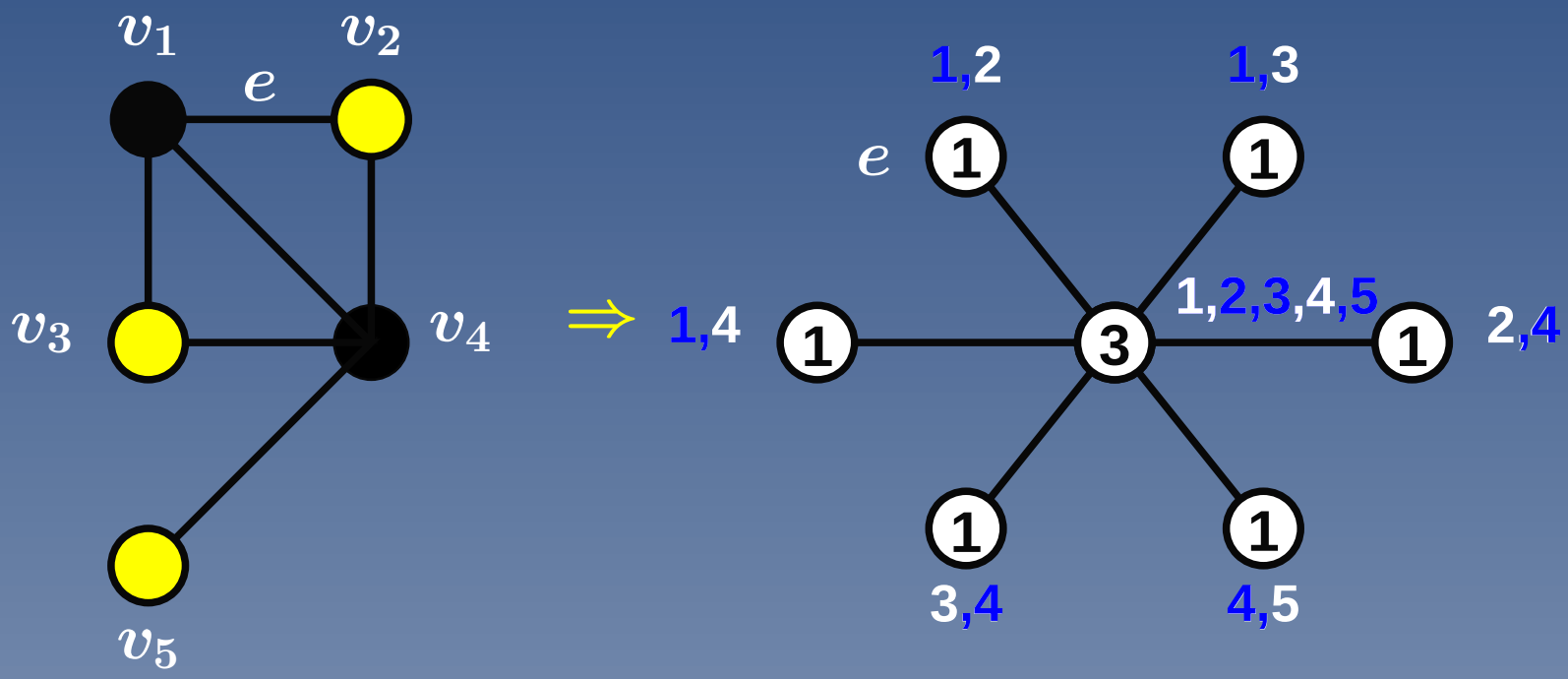

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Returning to minimum sum multicoloring. (There are no lists, the goal is to minimize $\sum_{v \in V} f(v)$, where $f(v)$ is the largest color assigned to v.)

The **NP**-hardness of minimum sum coloring in trees is proved by a similar reduction. The lists are simulated by **“penalty gadgets”**.

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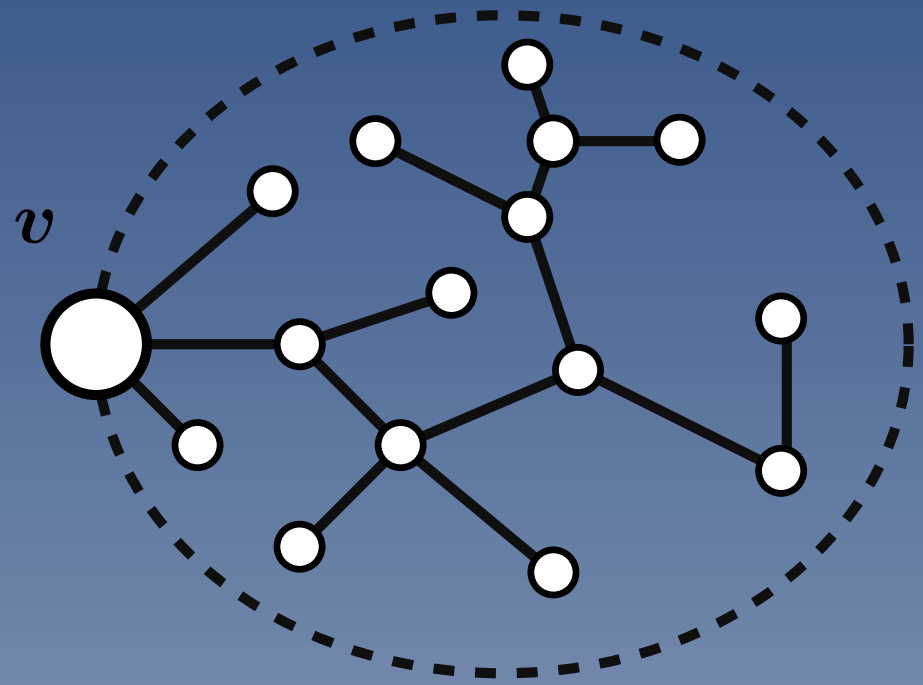
Illustration: forcing vertex v to use only colors greater than a



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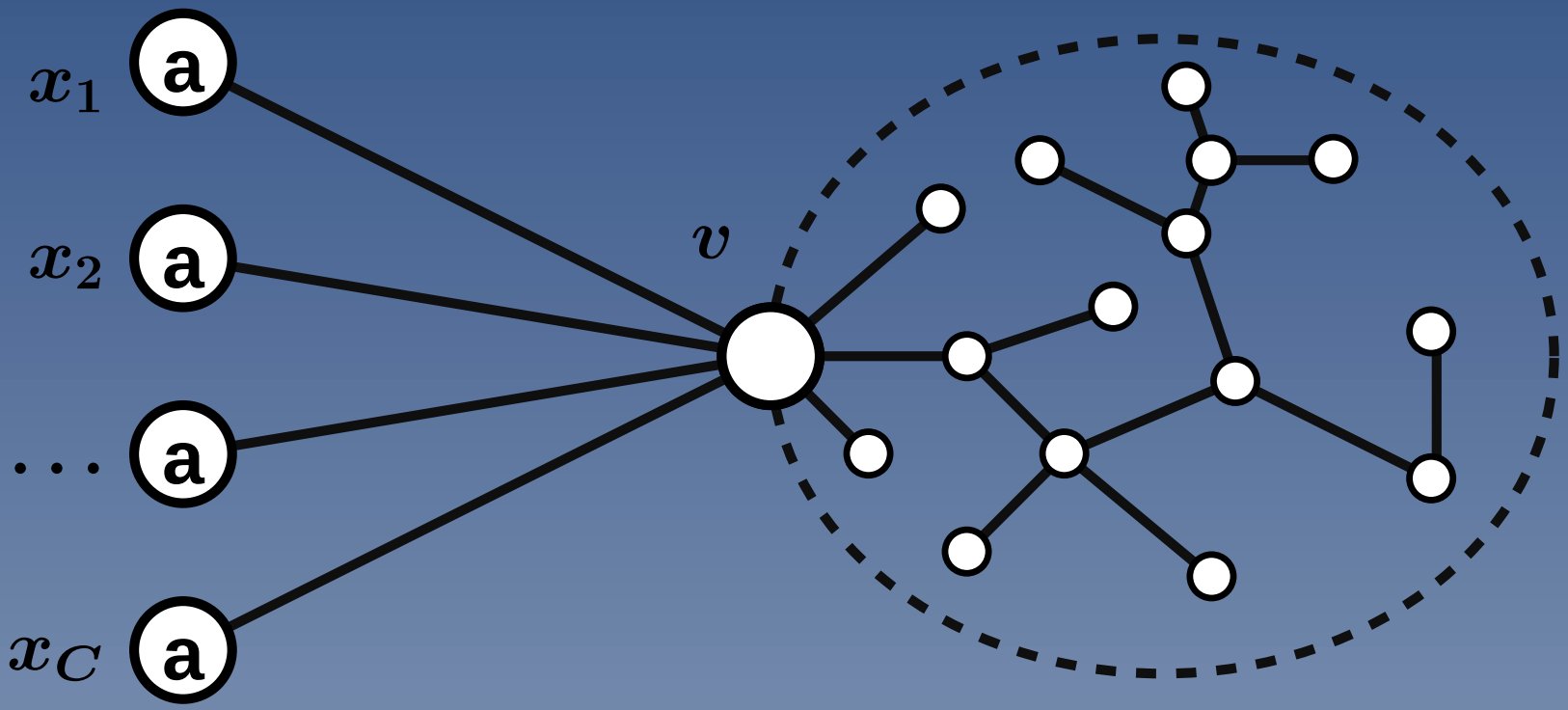

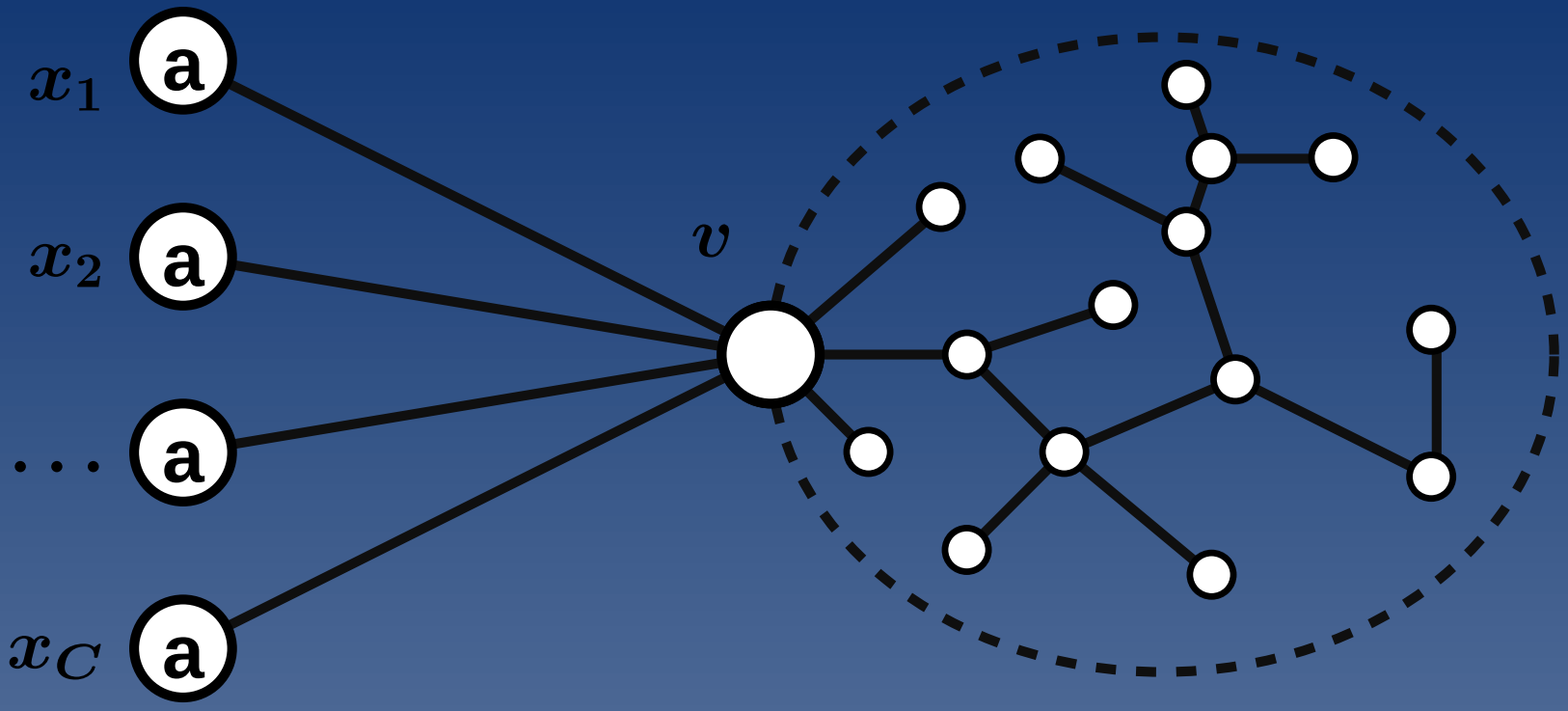

Every vertex x_i has demand $a \Rightarrow$ the sum of vertices x_i is at least aC. If C is "very large", then this forces v to have only colors greater than a:

- If v has only colors greater than $a \Rightarrow$ every vertex x_i can receive colors $\{1, \dots, a\} \Rightarrow$ their total sum is aC.
- If v has a color $\leq a \Rightarrow$ every x_i has a color greater than $a \Rightarrow$ their total sum is at least $aC + C$.

Remaining steps

- A similar gadget can force v to have only colors less than b
- Using these two gadgets, we can force v to have colors from a given set
  $\Rightarrow$ we can prove that minimum sum multicoloring is **NP**-complete in trees
- With a more complicated construction, we can make penalty gadgets with maximum degree 3
  $\Rightarrow$ we can prove that minimum sum multicoloring is **NP**-complete in **binary** trees

Summary

- **Coloring problem:** Minimum sum multicoloring (minimize the sum of the finish times)

- **Previous positive result:** Minimum sum multicoloring is polynomial in trees if every demand is 1 (or bounded by a constant)

  More general result: if every demand is at most p, then the problem can be solved in $O(n \cdot (p \log n)^p)$ time $\Rightarrow$ polynomial time, if every demand is $O(\log n / \log \log n)$

- **Previous positive result:** $(1+\epsilon)$-approximation algorithm for minimum sum multicoloring in trees.

- **New negative result:** Minimum sum multicoloring is **NP**-complete in binary trees.

- List multicoloring is **NP**-complete in binary trees.