Complexity of unique list colorability

Dániel Marx [1]

Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Budapest H-1521, Hungary.

Abstract

Given a list $L(v)$ for each vertex v, we say that the graph G is *L-colorable* if there is a proper vertex coloring of G where each vertex v takes its color from $L(v)$. The graph is *uniquely k-list colorable* if there is a list assignment L such that $|L(v)| = k$ for every vertex v and the graph has exactly one L-coloring with these lists. Mahdian and Mahmoodian [MM99] gave a polynomial-time characterization of uniquely 2-list colorable graphs. Answering an open question from [GM01,MM99], we show that uniquely 3-list colorable graphs are unlikely to have such a nice characterization, since recognizing these graphs is Σ_2^p-complete.

1 Introduction

List colorings were introduced in [ERT80,Viz76] as a generalization of ordinary vertex coloring. Given a *list assignment* L that assigns to each vertex v a set of colors, we say that G is *L-colorable* (or *list colorable* with lists L) if there is a proper vertex coloring of G where each vertex receives a color from its set $L(v)$ of available colors. Obviously, if every set $L(v)$ is $\{1, 2, \ldots, k\}$, then L-colorability is the same as k-colorability.

A *k-list assignment* is a list assignment in which the list of each vertex has size k. A graph G is *k-list colorable* (or *k-choosable*) if it is L-colorable for every k-list assignment L. Rubin [ERT80] gave a polynomial-time characterization of 2-list colorable graphs, while in [Gut96] it is shown that recognizing 3-list colorable graphs is Π_2^p-complete. For more information on list coloring and related problems, the reader is referred to the thorough survey of Tuza [Tuz97].

Email address: dmarx@cs.bme.hu (Dániel Marx).

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The concept of uniquely list colorable graphs was introduced independently in [DM95] and [MM99]. A graph is *uniquely k-list colorable* if there is a k-list assignment L such that G has *exactly one* L-coloring. Figure 1 shows a uniquely 3-list colorable graph (taken from [EGH02]) with a uniquely colorable 3-list assignment.

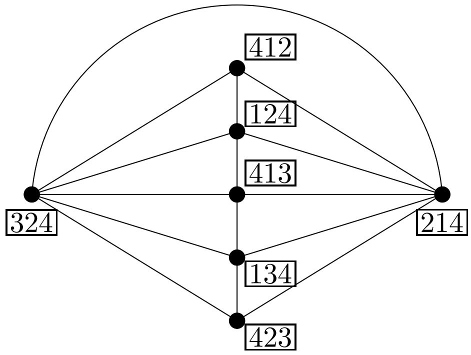

Figure 1. A uniquely 3-list colorable graph. The framed numbers at the vertices show the lists of the vertices. Taking the first color from each list is the unique list coloring.

Trivially, every graph is uniquely 1-list colorable. Mahdian and Mahmoodian [MM99] characterized uniquely 2-list colorable graphs:

Theorem 1 (Mahdian and Mahmoodian [MM99]) *A graph is uniquely 2-list colorable if and only if it contains a biconnected component which is neither a cycle, a complete graph, nor a complete bipartite graph.*

Theorem 1 implies that uniquely 2-list colorable graphs can be recognized in polynomial time. In [MM99] it is asked as an open question to characterize uniquely k-list colorable graphs for $k \geq 3$. More specifically, Ghebleh and Mahmoodian [GM01] ask what is the complexity of deciding whether a graph is uniquely 3-list colorable. The main contribution of this paper is showing that recognizing uniquely k-list colorable graphs is Σ_2^p-complete for every $k \geq 3$. Essentially, this means that the problem is as hard as possible, and we cannot hope for an NP- or coNP-characterization of these graphs.

The paper is organized as follows. Section 2 introduces notation and some preliminary results. In Section 3, we introduce a special Σ_2^p-complete satisfiability problem that will be used to obtain our hardness result. The reduction uses a number of somewhat complicated gadgets, these gadgets are described in Sections 4–6. The reduction itself is presented in Section 7.

2 Preliminaries

The formal definition of the unique k-list coloring problem is as follows:

Unique k-list colorability (UkLC)	
Input:	A graph $G(V, E)$.
Question:	Is there a k-list assignment L on the vertices of G such that G is uniquely L-colorable?

The problem does not seem be in NP: if G has a uniquely colorable k-list assignment L, then L can serve as a certificate, but it is not clear how we could certify that L has exactly one coloring. To be in coNP is even less likely: we should certify that *every* list assignment has either zero or more than one coloring. It seems that the problem lies higher in the polynomial hierarchy.

The complexity class $\Sigma_2^p = \mathrm{NP}^{\mathrm{NP}}$ contains those problems that can be solved by a polynomial-time nondeterministic Turing machine equipped with an NP-oracle. An NP-oracle can be thought of as a subroutine that is capable of solving one NP-complete problem (say, the 3-SAT problem) in constant time. Like NP, the class Σ_2^p has an equivalent characterization using certificates. A problem is in NP if there is a polynomial-size certificate for each yes-instance, and verifying this certificate is a problem in P. The definition of the class Σ_2^p is similar, but here we require only that verifying the certificate is in coNP (cf. [Pap94] for more details).

It is easy to see that unique k-list colorability is in Σ_2^p: a uniquely colorable k-list assignment L and the corresponding coloring ψ can serve as a certificate. To verify the certificate, one has to check that ψ is a proper coloring and it is the unique coloring of L. Checking whether ψ is proper can be done in polynomial time and finding an L-coloring different from ψ is a problem in NP; therefore, verifying the certificate is in coNP. This establishes the upper bound Σ_2^p on the complexity of the problem, which will turn out to be tight for $k \geq 3$.

Proposition 2 *Unique k-list colorability is in Σ_2^p.* $\square$

The following straightforward generalization of the UkLC problem was introduced in [EGH02]. Instead of requiring $|L(v)| = k$ for every vertex v, the input contains a function $f\colon V \to \mathbb{N}$, and the question is to find a uniquely colorable list assignment L such that $|L(v)| = f(v)$ for each $v \in V$. For a given function f, we say that G is uniquely f-list colorable if it has such a list assignment. Clearly, unique k-list colorability is the special case $f(v) \equiv k$.

We denote by U$(2,3)$LC the special case of unique f-list colorability where $f(v) \in \{2, 3\}$ for every $v \in V$. In Section 7 we show that U$(2,3)$LC is Σ_2^p-hard. As the following lemma shows, this implies that UkLC is also Σ_2^p-hard for $k \geq 3$:

Lemma 3 *For every $k \geq 3$, $U(2,3)LC$ can be reduced to UkLC in polynomial time.*

PROOF. Given a graph $G(V, E)$ and a function $f(v) \in \{2, 3\}$, we construct a graph G' that is uniquely k-list colorable if and only if G is uniquely f-list colorable. Graph G' is constructed as follows. We use the fact that for every $k \geq 1$ there is a uniquely k-list colorable graph G_k (see Figure 1 and [EGH02,GM01,MM99] for examples). Let $V = \{v_1, \ldots, v_n\}$. For each $1 \leq i \leq n$, we add $k - f(v_i)$ copies of G_k to the graph; denote these copies by $G_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq k - f(v_i)$. Let us fix an arbitrary vertex $v_{i,j}$ of each graph $G_{i,j}$. For each $1 \leq j \leq k - f(v_i)$, vertex $v_{i,j}$ is connected to vertex v_i.

Assume that L is a uniquely f-list colorable list assignment of G. Let α_1, $\ldots$, α_{k-2} be colors not appearing in L. For each vertex $v_i \in V$, let $L'(v_i) = L(v_i) \cup \{\alpha_1, \ldots, \alpha_{k-f(v_i)}\}$. Furthermore, define L' on the graph $G_{i,j}$ such that this copy has a unique coloring, and in this unique coloring vertex $v_{i,j}$ receives color α_j. It is easy to see that L' on G' has a unique coloring: the coloring of each copy of each $G_{i,j}$ is uniquely determined, the color of $v_{i,j}$ is α_j, hence vertex v_i cannot receive any of the colors α_1, $\ldots$, $\alpha_{k-f(v_i)}$. Therefore, a coloring of G' with L' induces a coloring of G with L, which is unique by assumption.

Now assume that L' is a uniquely k-list colorable list assignment of G'. Let ψ' be the unique coloring of G' with L'. For each $v_i \in V$, let $L(v_i) = L'(v_i) \setminus (\psi'(v_{i,1}) \cup \cdots \cup \psi'(v_{i,k-f(v_i)}))$. Clearly, $|L(v_i)| \geq f(v_i)$. Moreover, the coloring ψ induced by ψ' on G is the unique coloring of G with L: if there were a coloring different from ψ, then ψ' could be modified accordingly to obtain a different coloring of G' with L', a contradiction.

In the rest of the paper, we consider only the U(2,3)LC problem. Therefore, all the graphs appearing in the following are equipped with a list size function $f(v) \in \{2, 3\}$. In the figures to follow, the list sizes are shown by small numbers inside the vertices.

In Sections 4–6, we construct gadgets (building blocks) to be used in the reduction of Section 7. Each gadget is a graph with some distinguished *special vertices.* In the reduction a larger graph is built from these gadgets. The large graph is constructed in such a way that a gadget is connected to the rest of the graph only through its special vertices.

One direction in the proof of the reduction starts with the assumption that the constructed graph has a uniquely colorable k-list assignment L having coloring ψ as its unique coloring. For each gadget embedded in the graph, coloring ψ assigns some colors to its special vertices. What is important to notice is that the gadget has exactly one coloring in L with these combination of colors appearing on the special vertices. If there were multiple such colorings, then the gadget could be recolored, and since it is connected to the rest of the graph through its special vertices (whose colors are not changed), the resulting coloring would also be a proper coloring of the graph. However, this would contradict the assumption that ψ is the unique coloring of L. Thus the gadget has exactly one coloring with the given combination of colors on the special vertices; we will say that coloring ψ on the special vertices has a *unique extension* to the gadget. In the following, we will use this observation repeatedly. The gadgets are constructed in such a way that if some coloring of the special vertices has a unique extension in L, then L must satisfy certain properties.

3 Unique satisfiability

In the satisfiability (SAT) problem, we are given a boolean formula $\phi(\mathbf{x})$ where $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector of variables, and it has to be decided whether there is a variable assignment $\mathbf{x}$ satisfying ϕ. In the *unique satisfiability problem (USAT)*, we have to decide whether there is *exactly* one variable assignment $\mathbf{x}$ that satisfies ϕ. The USAT problem does not seem to be in either NP or coNP. On the other hand, USAT is in DP $= \{L_1 \cap L_2 : L_1 \in \mathrm{NP},\ L_2 \in \mathrm{coNP}\}$: let $L_1 \in$ NP be the set of satisfiable formulas, and let $L_2 \in$ coNP be the set of formulas with at most one satisfying assignment. However, USAT is not believed to be complete for DP: Blass and Gurevich [BG82] have given an oracle relative to which USAT is not DP-complete.

It is easy to show that USAT is coNP-hard: a formula $\phi(x_1, \ldots, x_n)$ is unsatisfiable if and only if

$$(x \vee \phi(x_1, \ldots, x_n)) \wedge (\bar{x} \vee x_1) \wedge \cdots \wedge (\bar{x} \vee x_n) \tag{1}$$

has exactly one satisfying assignment (namely, the assignment where every variable is true). USAT is not known to be NP-hard, but Valiant and Vazirani [VV86] have shown that USAT is NP-hard for randomized reductions (see [CR90] for a discussion on the precise meaning of randomized reductions in this context).

The QSAT_2 problem is the counterpart of SAT on the second level of the polynomial hierarchy:

> 2-QUANTIFIED SAT (QSAT_2)
>
> *Input:* A boolean formula $\phi(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ and $\mathbf{y}$ are vectors of variables.
>
> *Question:* Is it true that "$\exists \mathbf{x} \forall \mathbf{y} \phi(\mathbf{x}, \mathbf{y})$"? That is, is there an assignment $\mathbf{x}$ such that $\phi(\mathbf{x}, \mathbf{y})$ is true for every assignment $\mathbf{y}$?

QSAT_2 is the canonical complete problem for the complexity class Σ_2^p (see e.g. [Pap94]):

Theorem 4 QSAT_2 *is* Σ_2^p*-complete even if* ϕ *is required to be in* 3*-DNF form.*

Recall that a formula is in DNF (disjunctive normal form) if it is the disjunction of terms. 3-DNF means that each term is the conjunction of exactly 3 literals. Besides QSAT_2, the class Σ_2^p has many natural complete problems, see [SU02] for a compendium of complete problems.

We introduce a new variant of QSAT_2: the quantifier "for all $\mathbf{y}$" is replaced by "for exactly one $\mathbf{y}$."

> $\exists\exists!$-SAT
>
> *Input:* A boolean formula $\phi(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ and $\mathbf{y}$ are vectors of variables.
>
> *Question:* Is it true that "$\exists \mathbf{x} \exists! \mathbf{y} \phi(\mathbf{x}, \mathbf{y})$"? That is, is there an assignment $\mathbf{x}$ such that $\phi(\mathbf{x}, \mathbf{y})$ is true for exactly one assignment $\mathbf{y}$?

It will be convenient to use this problem for determining the complexity of unique list coloring, since $\exists\exists!$-SAT and UkLC have a similar quantifier structure. In the unique list coloring problem we have to decide whether there *exists* a list assignment L such that there is *exactly one* coloring of L; that is, an "exists" quantifier is followed by a "uniquely exists" quantifier, as in $\exists\exists!$-SAT.

Unlike USAT, which does not seem to fit into the complexity classes of the polynomial hierarchy, $\exists\exists!$-SAT has the same complexity as QSAT_2 (recall that a formula is in 3-CNF form if it is the conjunction of clauses and each clause is the disjunction of 3 literals):

Theorem 5 $\exists\exists!$*-SAT is* Σ_2^p*-complete even if* ϕ *is required to be in* 3*-CNF form.*

PROOF. The formula "$\exists\mathbf{x}\exists!\mathbf{y}\phi(\mathbf{x},\mathbf{y})$" can be written as "$\exists\mathbf{x},\mathbf{y}_0\forall\mathbf{y}\phi(\mathbf{x},\mathbf{y}_0)\wedge(\mathbf{y}\neq\mathbf{y}_0\Rightarrow\neg\phi(\mathbf{x},\mathbf{y}))$": there is an assignment $\mathbf{x}$ such that $\psi(\mathbf{x},\mathbf{y}_0)$ is true for some $\mathbf{y}_0$, but $\psi(\mathbf{x},\mathbf{y})$ is false for every $\mathbf{y}\neq\mathbf{y}_0$. Therefore, $\exists\exists!$-SAT can be reduced to QSAT_2, which shows that $\exists\exists!$-SAT is in Σ_2^p.

By a reduction from QSAT_2, we show that $\exists\exists!$-SAT is hard for Σ_2^p. One can write "$\exists\mathbf{x}\forall\mathbf{y}\phi(\mathbf{x},\mathbf{y})$" as "$\exists\mathbf{x}\neg\exists\mathbf{y}\neg\phi(\mathbf{x},\mathbf{y})$." Furthermore, introducing a new variable y, we can rewrite $\neg\exists\mathbf{y}\neg\phi(\mathbf{x},\mathbf{y})$ (using the same trick as in (1) above) to obtain

$$\exists\mathbf{x}\exists!y,\mathbf{y}:(y\vee\neg\phi(\mathbf{x},\mathbf{y}))\wedge(\bar{y}\vee y_1)\wedge\cdots\wedge(\bar{y}\vee y_m), \tag{2}$$

where y_1, ..., y_m are the variables in $\mathbf{y}$.

By Theorem 4, it can be assumed that ϕ is in 3-DNF form. Therefore, by applying De Morgan's law, $\neg\phi(\mathbf{x},\mathbf{y})$ can be written in 3-CNF form, let C_1, ..., C_r be its clauses. Now $y\vee\neg\phi(\mathbf{x},\mathbf{y})=(y\vee C_1)\wedge\cdots\wedge(y\vee C_r)$, hence (2) can be written as a formula in conjunctive normal form, where there are clauses of size two (namely, $\bar{y}\vee y_i$) and clauses of size four (namely, $y\vee C_j$). However, we want to prove that $\exists\exists!$-SAT is Σ_2^p-complete even if the formula is 3-CNF. For each clause $(y\vee C_j)$ of size 4, we proceed as follows. Let $C_j=(\ell_{j,1}\vee\ell_{j,2}\vee\ell_{j,3})$ for some literals $\ell_{j,1}$, $\ell_{j,2}$, $\ell_{j,3}$. For every $1\le j\le r$, we introduce a new variable y'_j, which is bounded by the $\exists!$ quantifier. The clause $(y\vee C_j)$ can be replaced by the following 4 clauses:

$$(y\vee\ell_{j,1}\vee y'_j)\wedge(\ell_{j,2}\vee\ell_{j,3}\vee\bar{y}'_j)\wedge(\bar{\ell}_{j,2}\vee y'_j)\wedge(\bar{\ell}_{j,3}\vee y'_j) \tag{3}$$

It is clear that if (3) holds, then clause $(y\vee C_j)$ is satisfied: if y, $\ell_{j,1}$, $\ell_{j,2}$, $\ell_{j,3}$ are all false, then the first two clauses could not be satisfied simultaneously. Moreover, if a variable assignment satisfies $(y\vee C_j)$, then this uniquely determines the value of y'_j: if both $\ell_{j,2}$ and $\ell_{j,3}$ are false, then y'_j is false; if at least one of them is true, then y'_j has to be true. Therefore, this replacement does not change the solution to the $\exists\exists!$-SAT problem. The clauses of size 2 can be easily taken care of: we can simply duplicate one of the literals. Hence (2) can be transformed to a 3-CNF formula, completing the proof.

4 Implication gadgets

The implication gadgets will be useful building blocks in our reduction. The following lemma summarizes the properties required from such a graph. We prove that the graph shown in Figure 2a satisfies these requirements. When building larger graphs that contain implication gadgets, then we use the shorthand notation shown in Figure 2b for each copy of the implication gadget.

Lemma 6 (Implication gadget) *Let x and y be two vertices, and let L be a list assignment on these vertices. Then x and y can be connected by a graph F (called the* implication gadget with input x and output y*) such that the following statements hold:*

(1) For arbitrary colors $c \in L(x)$ and $d \in L(y)$, the list assignment L can be extended to F such that
  (a) If $\psi(x) = c$ and $\psi(y) = d$, then ψ has a unique extension to F.
  (b) If $\psi(x) = c$ and $\psi(y) \neq d$, then ψ cannot be extended to F.
  (c) If $\psi(x) \neq c$ and $\psi(y)$ is arbitrary, then ψ can be extended to F.

(2) Let $c \in L(x)$ and $d \in L(y)$ be arbitrary colors. Assume that the list assignment L is extended to F in such a way that color c on x, and color d on y uniquely determines the coloring of F. If $c' \in L(x)$ and $d' \in L(y)$ are arbitrary colors with $c \neq c'$, then there is a coloring ϕ with $\phi(x) = c'$ and $\phi(y) = d'$.

Intuitively, the first statement says that the lists of F can be set up in such a way that using color c on the input vertex x forces the use of colors d on the output y (properties 1a and 1b). On the other hand, if the color of x is different from c, then the gadget is "turned off": there is no restriction on the color of y (property 1c). Therefore, there is only one color at x that has any effect on the colors assignable to y (we say that only color c activates the gadget).

It is possible that in a given list assignment more than one color at x can activate the gadget. However, the second statement says that if the gadget is part of a larger graph that has a uniquely colorable list assignment L, and vertex x receives color c in the unique coloring, then every color c' different from c turns off the gadget.

PROOF. We show that the graph F shown in Figure 2 satisfies the requirements. To prove the first statement of the lemma, assume that $L(y) = \{d, \lambda_1, \lambda_2\}$ and consider the following list assignment (see Figure 2):

- $L(r'_j) = L(s'_j) = \{c, \gamma_1\}$ and $L(r''_j) = L(s''_j) = \{c, \gamma_2\}$ for $j = 1, 2$,
- $L(r_1) = L(s_1) = \{\gamma_1, \gamma_2, \delta_1\}$ and $L(r_2) = L(s_2) = \{\gamma_1, \gamma_2, \delta_2\}$,
- $L(r) = \{\delta_1, \delta_2, \lambda_1\}$ and $L(s) = \{\delta_1, \delta_2, \lambda_2\}$.

If $|L(y)| = 2$, then we can take $\lambda_1 = \lambda_2$. This will not cause any difficulties in the proof.

In list assignment L, if x is colored with color c, then this has a unique extension to the gadget, and it forces vertex y to have color d. Color c at x forces vertices r'_j, s'_j to color γ_1, and vertices r''_j, s''_j to color γ_2, which, in turn, implies that vertices r_1 and s_1 have color δ_1, and vertices r_2 and s_2 have color

δ_2. Therefore, vertex r has color λ_1, vertex s has color λ_2, hence the only color remaining for y is d, as required. To show part (c) of the statement, notice that if the color of x is different from c, then vertices r'_j, r''_j, s'_j, and s''_j can receive color c. Assign to vertex r and s colors different from the color of y. Now vertex r_1 can receive a color different from c and the color assigned to r. Vertices r_2, s_1, s_2 can be assigned a color similarly, proving part (c).

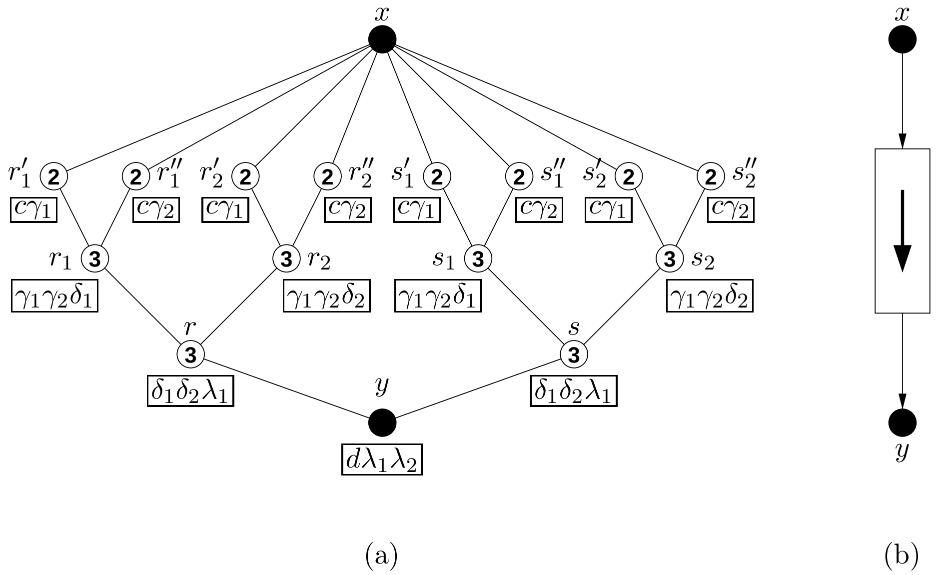

Figure 2. The implication gadget (a), and its simplified notation (b).

To prove the second statement, assume that we are given a list assignment L, and ψ is the unique coloring with $\psi(x) = c \in L(x)$ and $\psi(y) = d \in L(y)$. For every $c' \in L(x) \setminus \{c\}$ and $d' \in L(y)$, it has to be shown that there is a coloring ϕ with $\phi(x) = c'$ and $\phi(y) = d'$.

We claim that either color c is present in both of $L(r'_1)$ and $L(r''_1)$, or c is present in both of $L(r'_2)$ and $L(r''_2)$. If not, then without loss of generality it can be assumed that $c \notin L(r'_1)$ and $c \notin L(r'_2)$. We modify coloring ψ in such a way that it remains the same on x and y, contradicting the assumption that there is exactly one coloring with color c at x, and color d at y. The list $L(r)$ contains 3 colors; therefore, it contains a color α different from $\psi(r)$ and from $\psi(y) = d$. Assign this color α to r. Furthermore, assign to r_1 a color different from α and $\psi(r''_1)$, denote this color by κ_1. The list $L(r'_1)$ contains a color ω_1 different from κ_1, assign this color to r'_1. Since $\omega_1 \neq c$, there is no conflict between r'_1 and x. Similarly, we can assign a color κ_2 different from α and $\psi(r''_2)$ to vertex r_2. Vertex r'_2 can receive a color ω_2 different from κ_2, which cannot be c. Therefore, the resulting coloring is a proper list coloring of the gadget. This coloring is different from ψ (since the color of r was changed), but it assigns the same colors to the input and output, a contradiction.

Assume therefore that, without loss of generality, $c \in L(r'_1)$ and $c \in L(r''_1)$. We show that color $c' \neq c$ at x, and color d' at y can be extended to the gadget. Let $\phi(r'_2) \in L(r'_2)$ and $\phi(r''_2) \in L(r''_2)$ be colors different from c', and let $\phi(r_2) \in L(r_2)$ be a color different from $\phi(r'_2)$ and $\phi(r''_2)$. Let $\phi(r) \in L(r)$ be a color different from $\phi(r_2)$ and d'. Set $\phi(r'_1) = \phi(r''_1) = c$, and let $\phi(r_1) \in L(r_1)$ be a color different from c and from $\phi(r)$. We have shown how to determine the colors assigned to the vertices r, r_j, r'_j, r''_j; the vertices s, s_j, etc. can be handled analogously. The way ϕ was constructed ensures that it is a proper list coloring.

The multi-implication gadget is the more advanced version of the implication gadget, having several input and output vertices. The multi-implication gadget is not a single gadget, but a family of gadgets: the number of input and output vertices can be arbitrary.

Lemma 7 (Multi-implication gadget) *Let $I = \{x_1, x_2, \dots, x_n\}$ and $O = \{y_1, y_2, \dots, y_m\}$ be two sets of vertices, and let L be a list assignment on these sets. Then I and O can be connected by a graph $F_{n,m}$ (called the* multi-implication gadget with n inputs and m outputs*) such that the following statements hold:*

(1) For arbitrary colors $c_i \in L(x_i)$ and $d_j \in L(y_j)$ $(1 \le i \le n,\ 1 \le j \le m)$, the list assignment L can be extended to $F_{n,m}$ such that
  - *(a) If $\psi(x_i) = c_i$ and $\psi(y_i) = d_i$ for $1 \le i \le n$, $1 \le j \le m$, then ψ has a unique extension to F.*
  - *(b) If $\psi(x_i) = c_i$ and $\psi(y_j) = d'_j$ for $1 \le i \le n$, $1 \le j \le m$, and $d'_{j'} \ne d_{j'}$ for at least one j' $(1 \le j' \le m)$, then ψ cannot be extended to F.*
  - *(c) If $\psi(x_i) = c'_i$ and $\psi(y_j) = d'_j$ for $1 \le i \le n$, $1 \le j \le m$, and $c'_{i'} \ne c_{i'}$ for at least one i' $(1 \le i' \le n)$, then ψ can be extended to F.*

(2) Let $c_i \in L(x_i)$ and $d_j \in L(y_j)$ $(1 \le i \le n,\ 1 \le j \le m)$ be arbitrary colors. Assume that the list assignment L is extended to $F_{n,m}$ in such a way that there is a unique coloring ψ with $\psi(x_i) = c_i$ and $\psi(y_j) = d_j$ for every $1 \le i \le n$, $1 \le j \le m$. Let $c'_i \in L(x_i)$ and $d'_j \in L(y_j)$ $(1 \le i \le n,$ $1 \le j \le m)$ be arbitrary colors with $c_{i'} \ne c'_{i'}$ for at least one $1 \le i' \le n$. Then there is a coloring ϕ with $\phi(x_i) = c'_i$ and $\phi(y_j) = d'_j$ for every $1 \le i \le n$, $1 \le j \le m$.

The idea is the same as in the implication gadget, but here a particular combination of colors on the input vertices forces a particular combination of colors on the output vertices, and every other combination of colors has no effect on the output vertices.

PROOF. The construction of $F_{n,m}$ starts with a path $b_1, b_2, \dots, b_n$, vertex b_1 has list size 2, while the other vertices have size 3 (see Figure 3). A vertex a_i with list size 2 is attached to each vertex b_i. Input vertex x_i is connected to a_i via an implication gadget F_i^{in} (x_i is the input, a_i is the output of the gadget). Finally, vertex b_n is connected to each output vertex y_j via a copy of the implication gadget; denote by F_j^{out} the gadget with input b_n and output y_j. This completes the construction of the graph $F_{n,m}$.

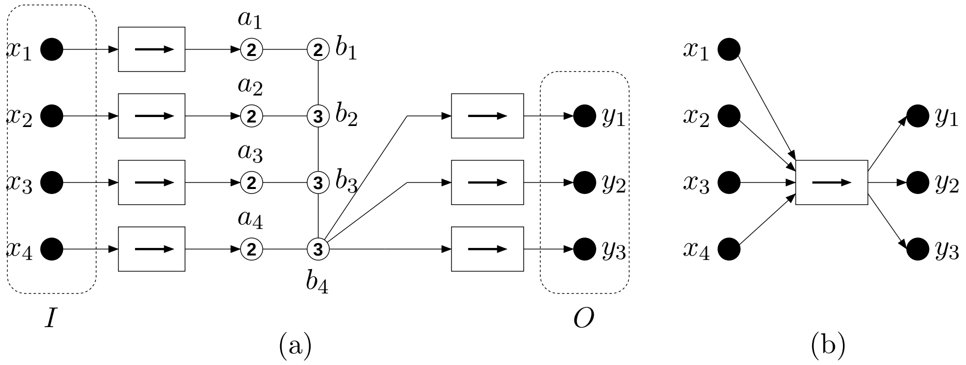

Figure 3. The multi-implication gadget with 4 inputs and 3 outputs (a), and its simplified notation.

To prove the first statement of the lemma, consider the following list assignment:

- $L(a_i) = \{\alpha_i, \gamma\}$ for $1 \leq i \leq n$,
- $L(b_1) = \{\alpha_1, \beta_1\}$,
- $L(b_i) = \{\alpha_i, \beta_{i-1}, \beta_i\}$ for $1 < i \leq n$,

Moreover, the lists of the vertices in F_i^{in} are set up in such a way that color c_i at x_i forces color α_i on vertex a_i (by Statement 1 of Lemma 6, such a list assignment exits). Similarly, the list assignment of the implication gadget F_j^{out} ensures that color β_n at b_n forces color d_j on output vertex y_j. Therefore, if $\psi(x_i) = c_i$ for every $1 \leq i \leq n$, then this implies $\psi(a_i) = \alpha_i$, and consequently, $\psi(b_i) = \beta_i$ for every $1 \leq i \leq n$. Moreover, $\psi(b_n) = \beta_n$ implies $\psi(y_j) = d_j$ for every $1 \leq j \leq m$, proving Statements 1a and 1b.

To prove Statement 1c, assume that $c'_{i'} \neq c_{i'}$ for some $1 \leq i' \leq n$, and consider the following coloring:

- $\psi(x_i) = c'_i$ for $1 \leq i \leq n$,
- $\psi(y_i) = d'_i$ for $1 \leq j \leq m$,
- $\psi(a_i) = \alpha_i$ for $i \neq i'$,
- $\psi(a_{i'}) = \gamma$ and $\psi(b_{i'}) = \alpha_i$,
- $\psi(b_i) = \beta_i$ for $1 \leq i < i'$,
- $\psi(b_i) = \beta_{i-1}$ for $i' < i \leq n$.

This coloring can be extended to $F_{i'}^{\text{in}}$: since $\psi(x_{i'}) \neq c_i$, the gadget is turned off (Statement 1c of Lemma 6). For every $i \neq i'$, the color of a_i is α_i, hence the coloring can be extended to F_i^{in} as well, regardless of the color of x_i. Finally, the color of b_n is different from β_n, thus the gadgets F_j^{out} are also turned off, and the coloring can be extended to the whole gadget.

Now assume that the conditions of Statement 2 hold for list assignment L. A coloring ϕ with $\phi(x_i) = c'_i$ and $\phi(y_j) = d_j$ is constructed as follows. In the following we assume for convenience that $1 < i' < n$, it is straightforward to adapt the proof for the cases $i' = 1$ and $i' = n$. Let $\phi(a_i) = \psi(a_i)$ for $i \neq i'$. Let $\phi(b_n)$ be a color different from $\psi(b_n)$ and $\phi(a_n)$. For $i = n-1, n-2, \ldots, i'+1$, let $\psi(b_i)$ be a color different from $\phi(a_i)$ and $\phi(b_{i+1})$. Let $\phi(b_1)$ be a color different from $\phi(a_1)$, and for $i = 2, 3, \ldots, i'-1$, let $\phi(b_i) \in L(b_i)$ be a color different from $\phi(b_{i-1})$ and $\phi(a_i)$. Let $\phi(b_{i'}) \in L(b_{i'})$ be a color different from $\phi(b_{i'-1})$ and $\phi(b_{i'+1})$. Finally, let $\phi(a_{i'})$ be a color different from $\phi(b_{i'})$.

We show that coloring ϕ can be extended to the implication gadgets, which proves Statement 2. Notice that with the assumed list assignment L, the conditions in Statement 2 of Lemma 6 hold for each implication gadget. For example, if there were two different colorings of F_i^{in} with color $\psi(x_i)$ on the input and color $\psi(a_i)$ on the output, then there would be another coloring of $F_{n,m}$ with colors c_i on the inputs and colors d_j on the outputs, which would contradict the assumption of Statement 2 of the lemma being proved. This means that coloring ϕ can be extended to $F_{i'}^{\text{in}}$, since $\phi(x_{i'}) = c'_i \neq c_i$ implies that $F_{i'}^{\text{in}}$ is turned off. The coloring can be extended also to each F_i^{in} for $i \neq i'$, since $\phi(a_i) = \psi(a_i)$. Finally, $\phi(b_n) \neq \psi(b_n)$ implies that the gadgets F_j^{out} are turned off, the coloring can be extended regardless of the colors assigned to the vertices y_j.

5 L-variable gadgets

Two different types of gadgets represent the variables in the reduction: the L-variable gadgets defined in this section, and the C-variable gadgets to be introduced in Section 6. "L" stands for "list": intuitively, a list assignment on the L-variable gadget can be used to determine a truth value. The "C" stands for "coloring" in the C-variable gadget: in every list assignment of the gadget, its colorings can be divided into colorings representing *true*, and colorings representing *false*.

The following lemma describes what properties are required from an L-variable gadget; the proof shows how to construct such a graph.

Lemma 8 *Given $n+m$ vertices $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_m$ with list size 2, they can be connected by a graph $H_{n,m}$ (called the L-variable gadget with $n+m$ outputs) such that*

(1) There are two list assignments L_1 and L_2, such that $L_1(x_i) = L_1(\bar{x}_j) = L_2(x_i) = L_2(\bar{x}_j) = \{1,2\}$ for $1 \le i \le n$ and $1 \le j \le m$, and

(a) $\psi(x_i) = 1$ ($1 \le i \le n$) for every coloring ψ of list assignment L_1, and every combination of colors 1 and 2 on the vertices $\bar{x}_1, \ldots, \bar{x}_m$ can be extended to the gadget in a unique way.

(b) $\psi(\bar{x}_j) = 1$ ($1 \le j \le m$) for every coloring ψ of list assignment L_2, and every combination of colors 1 and 2 on the vertices $x_1, \ldots, x_n$ can be extended to the gadget in a unique way.

(2) Let L be a list assignment of the gadget, and let $\psi(x_1), \ldots, \psi(x_n), \psi(\bar{x}_1), \ldots, \psi(\bar{x}_m)$ be colors such that the gadget has a unique extension ψ if these colors appear on the outputs. Then one of the following holds:

(a) For arbitrary colors $\bar{c}_j \in L(\bar{x}_j)$ ($1 \le j \le m$), there is a coloring of the gadget such that color $\psi(x_i)$ appears on x_i, and color $\bar{c}_j$ appears on $\bar{x}_j$ ($1 \le i \le n$, $1 \le j \le m$).

(b) For arbitrary colors $c_i \in L(x_i)$ ($1 \le i \le n$), there is a coloring of the gadget such that color c_i appears on x_i, and color $\psi(\bar{x}_j)$ appears on $\bar{x}_j$ ($1 \le i \le n$, $1 \le j \le m$).

The vertices $x_1, \ldots, x_n$ are called the *left side* of the gadget, while vertices $\bar{x}_1, \ldots, \bar{x}_m$ form the *right side*. Statement 1a says that there is a list assignment that forces every vertex of the left side to color 1, but has no effect on the right side, those vertices can be colored arbitrarily. Conversely, there is a list assignment that forces the right side to color 1, but has no effect on the left side. Statement 2 considers list assignments where the outputs can force a unique coloring on the gadget. Statement 2 requires that there is no such list assignment that forces vertices on both sides: in every list assignment, either the first or the right side can be recolored arbitrarily. In our reduction, we use Statement 1 to chose a list assignment for the gadget based on the truth value of the variable. In the other direction of the reduction, Statement 2 is used to deduce a value for the variable, based on whether 2(a) or 2(b) is satisfied by the list assignment.

PROOF. The gadget is constructed as follows (see Figure 4). The seven vertices v, v_1, v_2, v_3, $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$ form the *core* of the gadget, denote the set of these vertices by K. The edges induced by the core are shown in bold in Figure 4. We add $3n + 3m$ new vertices s_i, t_i, u_i, $\bar{s}_j$, $\bar{t}_j$, $\bar{u}_j$ ($1 \le i \le n$, $1 \le j \le m$). Connect vertex t_i to v_1, v_3, and s_i; connect s_i to x_i and u_i. Vertices $\bar{t}_j$, $\bar{v}_1$, $\bar{v}_3$, $\bar{s}_j$, $\bar{x}_j$, and $\bar{u}_j$ are connected in a similar way. Finally, add a multi-implication gadget with seven inputs and $n + m$ outputs: the inputs are the vertices of the core, the outputs are the vertices u_i, $\bar{u}_j$ ($1 \le i \le n$, $1 \le j \le m$). The list size of the vertices are as shown in Figure 4.

We show how to construct the list assignment L_1 required by the first statement, the existence of L_2 follows by symmetry. Set $L(u_i) = L(\bar{u}_j) = \{3, 4\}$,

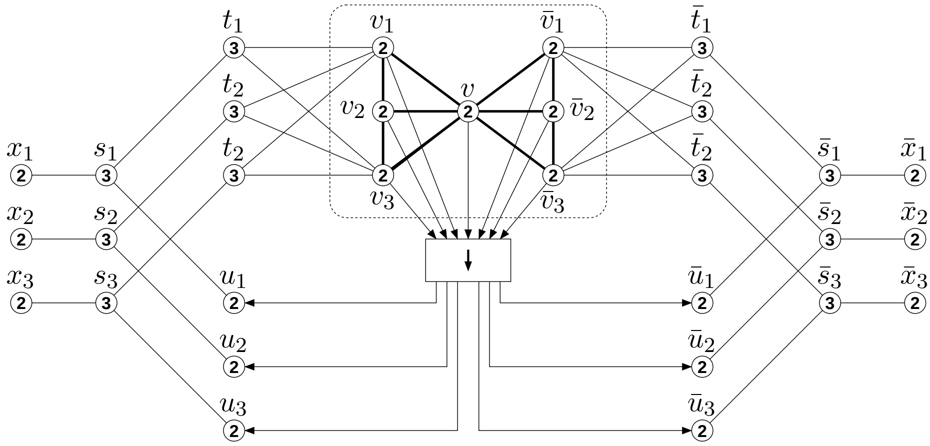

Figure 4. The L-variable gadget with $3 + 3$ outputs.

and consider the list assignment of the core K shown in Figure 5. It is easy to verify that this list assignment admits exactly one coloring of the core, namely the coloring where every vertex receives the first color from its list. The lists of the vertices in the multi-implication gadget can be set in such a way that this particular combination of colors on the core forces color 3 on each of the vertices u_i, $\bar{u}_j$. Since there is only one coloring of the core, the vertices u_i, $\bar{u}_j$ have color 3 in every coloring. The lists of the remaining vertices are set as follows:

- $L(t_i) = \{1, 3, 4\}$, $L(s_i) = \{2, 3, 4\}$ for $1 \leq i \leq n$, and
- $L(\bar{t}_j) = L(\bar{s}_j) = \{1, 2, 3\}$ for $1 \leq j \leq m$.

Vertices v_1 and v_3 force vertex t_i to have color 4, thus vertices t_i and u_i force vertex s_i to receive color 2, and vertex x_i receives color 1.

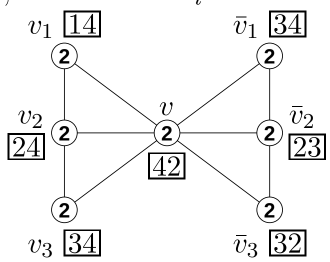

Figure 5. The core of the L-variable gadget with a uniquely colorable list assignment.

Since $\bar{v}_1$ and $\bar{v}_3$ have color 3, both color 1 and 2 are still available at $\bar{t}_j$. The color of $\bar{u}_j$ is 3, hence only colors 1 and 2 are available at $\bar{s}_j$. Therefore, the color of $\bar{x}_j$ can be either 1 or 2, and setting the color of $\bar{x}_j$ uniquely determines the colors of $\bar{s}_j$ and $\bar{t}_j$. This proves Statement 1 of the lemma.

To prove Statement 2, assume that list assignment L and coloring ψ satisfy the conditions. We consider two cases depending on whether L restricted to the core K is uniquely colorable or not. Assume first that the core has a coloring ψ' that is different from the coloring induced by ψ. Notice that the multi-implication gadget together with the list assignment L satisfy the conditions in Statement 2 of Lemma 7 (it is "turned on"): if the multi-implication gadget had another coloring with the same colors on the inputs and outputs, then the L-variable gadget would have another coloring with the same colors on its outputs. Therefore, if we recolor the core using coloring ψ', then this turns off the multi-implication gadget, which means that the coloring can be extended to the gadget, regardless of the colors at the vertices u_i, $\bar{u}_j$. We show that now both (a) and (b) of Statement 2 hold, in fact, any coloring of the outputs can be extended to the L-variable gadget. First recolor the core, as described above. Assign to vertex t_i a color different from the colors of v_1 and v_3; assign to s_i a color different from the colors of t_i and x_i; and assign to u_i a color different from the color of s_i. Similar assignments can be done on the other side of the gadget. Since recoloring the core turns of the multi-implication gadget, the coloring described so far can be extended to the whole gadget, as required.

Therefore, we can assume that the core is uniquely colorable in L. We claim that either $\psi(v_1) = \psi(v_3)$ or $\psi(\bar{v}_1) = \psi(\bar{v}_3)$ holds. If not, then consider the graph that is the same as the core, but it has the edges v_1v_3 and $\bar{v}_1\bar{v}_3$ in addition. This new graph is also uniquely colorable, since adding edges cannot increase the number of colorings, and ψ remains a proper coloring. However, after the addition of these two new edges, both biconnected components of the core become complete graphs, thus by Theorem 1, the graph cannot be uniquely 2-list colorable, a contradiction.

We show that if $\psi(\bar{v}_1) = \psi(\bar{v}_3)$, then (a) of Statement 2 holds; by a similar argument one can show that $\psi(v_1) = \psi(v_3)$ implies (b). We modify ψ such that color c_j appears on vertex $\bar{x}_j$. Assign to vertex $\bar{s}_j$ a color different from the colors appearing on $\bar{x}_j$ and $\bar{u}_j$; assign to $\bar{t}_j$ a color different from the color assigned to $\bar{s}_j$ and from $\psi(v_1) = \psi(v_3)$. This yields a coloring satisfying the requirements of (a), proving the lemma.

6 C-variable gadgets

The second type of variable gadget used in the reduction is the C-variable gadget, which is somewhat more complex than the L-variable gadget.

Lemma 9 *Given vertices $x_1, \ldots, x_n, \bar{x}, \ldots, \bar{x}_m, v_1, \ldots, v_7$, and u with list size* 2*, one can connect these $n+m+2$ vertices with a graph $C_{n,m}$ (called the* C-variable gadget with $n+m$ outputs, core vertices $v_1, \ldots, v_7$, and control vertex *u) that satisfies the following requirements:*

(1) There are list assignments L and $\bar{L}$ such that the lists of the vertices x_i,

$\bar{x}_j$, v_1, u are $\{1,2\}$ in both list assignments, and the following properties hold:

(a) In list assignment L, if vertex v_1 receives color 1, then every x_i receives color 1; if vertex v_1 receives color 2, then every $\bar{x}_j$ receives color 1.

(b) In L there is exactly one coloring ψ with $\psi(x_i) = 1$, $\psi(\bar{x}_j) = 2$, and $\psi(u) = 1$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$.

(c) In L there is a coloring $\bar{\psi}$ with $\bar{\psi}(v_1) = 2$, $\bar{\psi}(x_i) = 2$, and $\bar{\psi}(\bar{x}_j) = 1$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$.

(d) In list assignment $\bar{L}$, if vertex v_1 receives color 1, then every x_i receives color 1; if vertex v_1 receives color 2, then every $\bar{x}_j$ receives color 1.

(e) In $\bar{L}$ there is exactly one coloring $\bar{\psi}$ with $\bar{\psi}(x_i) = 2$, $\bar{\psi}(\bar{x}_j) = 1$, and $\bar{\psi}(u) = 1$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$.

(f) In $\bar{L}$ there is a coloring ψ with $\psi(v_1) = 1$, $\psi(x_i) = 1$, and $\psi(\bar{x}_j) = 2$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$.

(2) Let L be a list assignment and ψ a coloring of L such that in L the colors assigned by ψ to the vertices x_i, $\bar{x}_j$, $v_1, \ldots, v_7$, u uniquely determine the color of every other vertex in the gadget. Then

(a) There is a coloring ϕ of the vertices $v_1, \ldots, v_7, \bar{x}_1, \ldots, \bar{x}_m$ such that for every possible combination of colors on $x_1, \ldots, x_n$, coloring ϕ can be extended to the whole gadget with this combination on those vertices.

(b) There is a coloring $\bar{\phi}$ of the vertices $v_1, \ldots, v_7, x_1, \ldots, x_n$ such that for every possible combination of colors on $\bar{x}_1, \ldots, \bar{x}_m$, coloring $\bar{\phi}$ can be extended to the whole gadget with this combination on those vertices.

(c) Colorings ϕ and $\bar{\phi}$ differ on at least one of the vertices $v_1, \ldots, v_7$.

Let us try to make sense of these technical requirements. We call the vertices $x_1, \ldots, x_n$ the *lower side* of the gadget, while vertices $\bar{x}_1, \ldots, \bar{x}_m$ form the *upper side*. The two list assignments defined in Statement 1 are almost the same. In both of L and $\bar{L}$, if vertex v_1 has color 1, then this forces the lower side ($x_1, \ldots, x_n$) to have color 1; while if there is color 2 on v_1, then the upper side ($\bar{x}_1, \ldots, \bar{x}_m$) is forced to color 1. The output vertices not forced by the color of v_1 can be colored with 2 (possibly other combination of colors can also appear on these vertices, but it will not be relevant). The color of vertex v_1 will correspond to the two possible truth values of a given variable. The difference between L and $\bar{L}$ appears only if we consider the uniqueness of the colorings. In both of L and $\bar{L}$, there is a coloring that assigns color 1 to v_1, color 1 to the lower side, and color 2 to the upper side. Possibly there are several such colorings, but we know that in L there is exactly one such coloring that also assigns color 1 to control vertex u. Similarly, in $\bar{L}$ there is exactly one coloring where v_1 has color 2, the lower side has color 2, the upper side has color 1, and in addition, control vertex u has color 1.

If the C-variable gadget is part of a larger graph having a uniquely colorable list assignment L, then Statement 2 can be used. If ψ is the unique coloring of L, then clearly the colors assigned by ψ to the vertices x_i, $\bar{x}_j$, v_1, ..., v_7, and u have to force a unique coloring on the rest of the gadget, otherwise the gadget could be recolored, which would result in another coloring of L. The important thing in Statement 2 is that there is a coloring of the core vertices that does not force any restriction on the coloring of the lower side, and there is another coloring that does not force any restriction on the upper side.

PROOF. The construction of the gadget starts with the 11 vertices $v_1, \ldots, v_7, w_1, \ldots, w_4$ (see Figure 6). The subgraph induced by $v_1, \ldots, v_7$ will be called the *core* of the gadget (shown by bold edges in the figure). For $1 \leq \ell \leq 7$, the vertices u_ℓ and $\bar{u}_\ell$ are connected to vertex v_ℓ (note that these edges are not fully drawn in Figure 6).

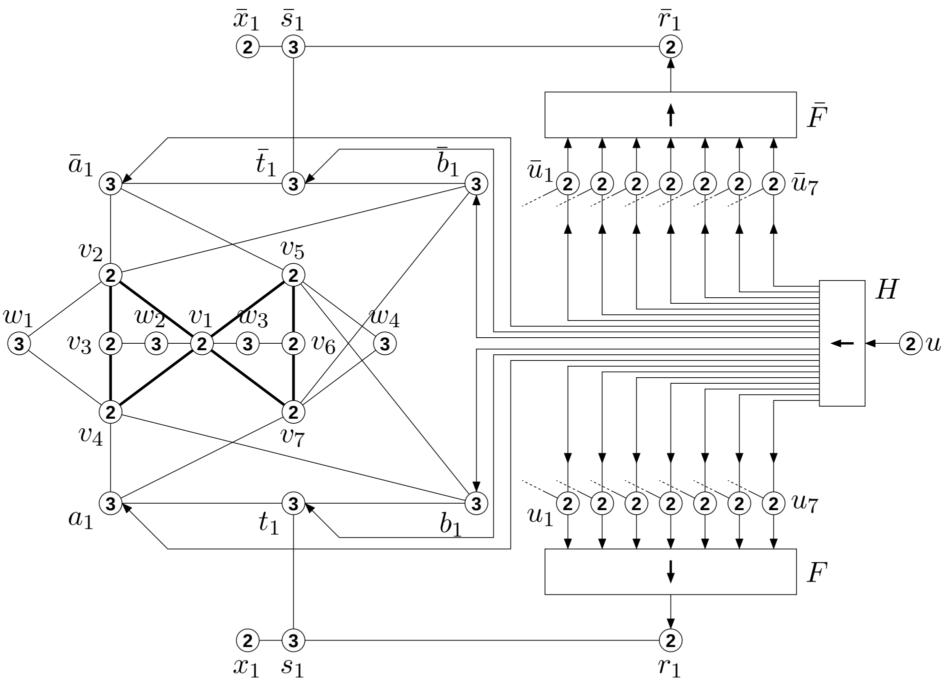

Figure 6. The C-variable gadget with $1+1$ outputs, core vertices v_1, ..., v_7, and control vertex u.

For each $1 \leq i \leq n$, we add the following vertices:

- a vertex a_i connected to v_4 and v_7,
- a vertex b_i connected to v_4 and v_5,
- a vertex t_i connected to a_i and b_i,
- a vertex s_i connected to x_i and t_i, and
- a vertex r_i connected to s_i.

For each $1 \leq j \leq m$, we add the following vertices:

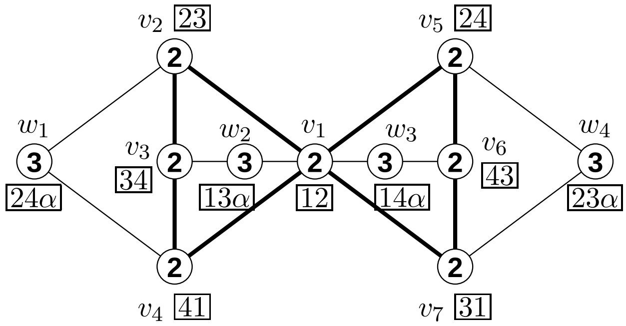

Figure 7. The core of the C-variable gadget with a list assignment that has exactly two colorings.

- a vertex $\bar{a}_j$ connected to v_2 and v_5,
- a vertex $\bar{b}_j$ connected to v_2 and v_7,
- a vertex $\bar{t}_j$ connected to $\bar{a}_j$ and $\bar{b}_j$,
- a vertex $\bar{s}_j$ connected to $\bar{x}_j$ and $\bar{t}_j$, and
- a vertex $\bar{r}_j$ connected to $\bar{s}_j$.

There is a multi-implication gadget F whose inputs are the vertices $u_1, \ldots, u_7$, and the outputs are the vertices $r_1, \ldots, r_n$. Similarly, a multi-implication gadget $\bar{F}$ connects the 7 vertices $\bar{u}_1, \ldots, \bar{u}_7$ to vertices $\bar{r}_1, \ldots, \bar{r}_m$. Finally, there is a multi-implication gadget H whose only input is vertex u, and has $14+3n+3m$ outputs: vertices u_ℓ, $\bar{u}_\ell$, a_i, b_i, t_i, $\bar{a}_j$, $\bar{b}_j$, $\bar{t}_j$ for $1 \le \ell \le 7$, $1 \le i \le n$, $1 \le j \le m$. This completes the description of the gadget.

The list assignments L and $\bar{L}$ are defined as follows. We describe only L and prove properties (a)–(c). List assignment $\bar{L}$ and properties (d)–(f) follow from symmetry. Notice the inherent symmetry in the construction: the connections are the same on both sides of the gadget (in the core, vertices v_2 and v_5 play the same role in the upper side as v_4 and v_7 play in the lower side).

The list assignment on the vertices $v_1, \ldots, v_7, w_1, \ldots, w_4$ is shown in Figure 7. The core has exactly two colorings with these lists: either every vertex v_ℓ receives the first color from its list, or every vertex receives the second color. For every $1 \le \ell \le 7$, the list of u_ℓ contains color α and the first color in $L(v_\ell)$, while the list of $\bar{u}_\ell$ contains α and the second color of $L(v_\ell)$. Furthermore, for every $1 \le i \le n$ and $1 \le j \le m$

- $L(r_i) = L(\bar{r}_j) = \{\alpha, \beta\}$,
- $L(s_i) = L(\bar{s}_j) = \{1, 2, \alpha\}$,
- $L(t_i) = L(\bar{t}_j) = \{1, \alpha, \beta\}$,
- $L(a_i) = \{3, 4, \alpha\}$, $L(b_i) = \{2, 4, \beta\}$, $L(\bar{a}_j) = \{3, 4, \alpha\}$, $L(\bar{b}_j) = \{1, 3, \beta\}$.

The lists in multi-implication gadget F are set up such that if all 7 vertices u_ℓ have color α, then this forces color α on the vertices r_i. Similarly, the gadget $\bar{F}$ is set up to ensure that color α on vertices $\bar{u}_\ell$ forces color α on the vertices $\bar{r}_j$. Finally, the list assignment of gadget H is set in such a way that if u has color 1, then this forces

- color α on vertices u_ℓ and $\bar{u}_\ell$ ($1 \leq \ell \leq 7$),
- color α on vertices a_i ($1 \leq i \leq n$),
- color β on vertices b_i ($1 \leq i \leq n$),
- color 1 on vertices t_i ($1 \leq i \leq n$),
- color 3 on vertices $\bar{a}_j$ ($1 \leq j \leq m$),
- color β on vertices $\bar{b}_j$ ($1 \leq i \leq m$),
- color α on vertices $\bar{t}_j$ ($1 \leq i \leq m$).

To verify (a) of Statement 1, assume first that vertex v_1 has color 1, this determines the coloring of the core: each vertex must receive the first color from its list. Because of the edge between v_ℓ and u_ℓ, each vertex u_ℓ has to receive color α. Thus the multi-implication gadget F is turned on, and it forces vertex r_i to color α. Vertices v_4 and v_7 force vertex a_i to color α, while vertices v_4 and v_5 force vertex b_i to color β, thus vertex t_i has to receive color 1. Therefore, vertices r_i and t_i force vertex s_i to color 2, and vertex x_i receives color 1, as required. A similar argument shows that color 2 on vertex v_1 forces color 1 on vertex $\bar{x}_j$.

Next we prove property (b). By property (a), $\psi(\bar{x}_j) = 1$ implies $\psi(v_1) = 1$. Having color 1 at vertices v_1 and u uniquely determines the color of every vertex except the vertices $\bar{s}_j$ and $\bar{x}_j$. Indeed, color 1 at vertex v_1 determines the coloring of the core, and this coloring forces color α on each of the vertices $w_1, \ldots, w_4$. Color 1 at u turns on the gadget H, setting the color on the outputs of H. In particular, every u_ℓ, $\bar{u}_\ell$ is forced to color α, hence gadgets F and $\bar{F}$ are also turned on, giving color α to vertices r_i and $\bar{r}_j$. Color α on r_i and color 1 on x_i and t_i force s_i to have color 2. Similarly, color α on $\bar{r}_j$ and color 2 on $\bar{x}_j$ force $\bar{s}_j$ to color 1. Therefore, in this case the coloring of the gadget is uniquely determined by the coloring of the vertices $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_m, u$.

To prove (c), first assign color 2 to vertex u, this turns off the multi-implication gadget H. For each vertex of the core, assign to it the second color from its list, hence vertex v_1 receives color 2. Now vertex a_i can receive color 3 (since both v_4 and v_7 have color 1), vertex b_i can receive color β, hence t_i can receive color α. This means that s_i can use color 1, thus there can be color 2 on vertex x_i. As we have seen in property (a), if vertex v_1 has color 2, then the vertices $\bar{x}_j$ receive color 1, as required. This proves property (c), since it is easy to assign colors to the vertices not considered in this paragraph.

To prove the second part of the lemma, assume that L and ψ satisfy the assumptions. In the colorings ϕ and $\bar{\phi}$ required by the lemma, we assign to vertex u a color different from $\psi(u)$, which turns off gadget H; therefore, this gadget does not play any role in the rest of the proof. A coloring of the core can force some restriction on the lower side via vertices a_i, b_i, and the gadget F. We show that there is coloring ϕ of the core that either does not force a_i, or does not force b_i, or does not turn on gadget F, hence the lower side can be colored arbitrarily. Similarly, we construct another coloring $\bar{\phi}$ of the core, which either does not force $\bar{a}_i$, or does not force $\bar{b}_i$, or does not turn on $\bar{F}$, allowing any combination of colors on the upper side.

If $\phi(v_4) = \phi(v_7)$, then coloring ϕ of the core does not force a_i, and if $\phi(v_4) = \phi(v_5)$, then coloring ϕ of the core does not force b_i. Similarly, if $\bar{\phi}(v_2) = \bar{\phi}(v_5)$, then $\bar{a}_i$ is not forced, and $\bar{\phi}(v_2) = \bar{\phi}(v_7)$ means that $\bar{b}_i$ is not forced. How can we ensure that a coloring of the core does not turn on gadget F or $\bar{F}$? Let α_ℓ be the color $L(u_\ell) \setminus \psi(u_\ell)$, and let $\bar{\alpha}_\ell$ be $L(\bar{u}_\ell) \setminus \psi(\bar{u}_\ell)$. By Lemma 7, Statement 2, the only combination of colors on the vertices $u_1, \ldots, u_7$ that turns on gadget F is the colors assigned by ψ. Therefore, if color α_ℓ appears on v_ℓ for each $1 \leq \ell \leq 7$, then this forces color $\psi(u_\ell)$ on vertex u_ℓ, which turns on the gadget F. Moreover, this coloring of the core is the only combination of colors that turns on gadget F. Similarly, the only combination of colors on the core that turns on gadget $\bar{F}$ is having color $\bar{\alpha}_\ell$ on v_ℓ for every $1 \leq \ell \leq 7$. The following lemma shows that there is a coloring ϕ of the core that does not force the lower side (since it satisfies one of the three properties discussed above) and a coloring $\bar{\phi}$ that does not force the upper side (for a similar reason).

Claim 10 *The core has two different colorings ϕ and $\bar{\phi}$ such that the following two statements hold:*

(1) Either
- *coloring ϕ is different from coloring $\alpha_1, \ldots, \alpha_\ell$,*
- *$\phi(v_4) = \phi(v_5)$ holds, or*
- *$\phi(v_4) = \phi(v_7)$ holds.*

(2) Either
- *coloring $\bar{\phi}$ is different from coloring $\bar{\alpha}_1, \ldots, \bar{\alpha}_\ell$,*
- *$\bar{\phi}(v_2) = \bar{\phi}(v_5)$ holds, or*
- *$\bar{\phi}(v_2) = \phi(v_7)$ holds.*

PROOF. By Theorem 1, the core induces a graph that is not uniquely 2-list colorable. Since the list size is 2 for every vertex of the core, this means that the core has at least two different colorings in L. Assume first that the core has at least three different colorings. In this case one of these colorings is different from coloring $\alpha_1, \ldots, \alpha_\ell$, and one of the remaining at least two colorings is different from coloring $\bar{\alpha}_1, \ldots, \bar{\alpha}_\ell$. Thus we can define ϕ and $\bar{\phi}$ as required.

Now assume that the core has exactly two colorings. Let ψ_1 be the coloring induced by ψ and let ψ_2 be the other coloring. It turns out that in this case the list assignment of the core has to be essentially the same as in Figure 7.

That is, in one of the colorings, v_2 has the same color as v_5 or v_7, implying that the core does not force a color on $\bar{t}_j$; in the other coloring, v_4 has the same color as v_5 or v_7, implying that the core does not force a color on t_i. In order to show this, we have to use the fact that $\psi_1(v_2) \neq \psi_1(v_4)$, otherwise there would be at least two different possible colors for w_1, contradicting the assumptions of Statement 2 of Lemma 9. The vertices w_2, w_3, w_4 can be used to obtain similar requirements on ψ_1.

First we show that $\psi_1(v_1) \neq \psi_2(v_1)$. Assume that $\psi_1(v_1) = \psi_2(v_1)$, this means that color $c = L(v_1) \setminus \psi_1(v_1)$ on vertex v_1 cannot be extended to a coloring of the core. If color c on vertex v_1 can be extended to a coloring of the vertices v_1, v_2, v_3, v_4, and it can be extended to a coloring of v_1, v_5, v_6, v_7, then there is a coloring of the core with color c on vertex v_1. We assumed that ψ_1 and ψ_2 are the only colorings of the core; therefore, it can be assumed, without loss of generality, that color c on vertex v_1 cannot be extended to vertices v_2, v_3, v_4. This is only possible if $L(v_2) = \{c, \alpha\}$, $L(v_4) = \{c, \beta\}$ and $L(v_3) = \{\alpha, \beta\}$ for some distinct colors α, β different from c. However, in this case, color $\psi_1(v_1) \neq c$ on vertex v_1 can be extended to vertices v_2, v_3, v_4 in at least three different ways, as follows. Color $\psi_1(v_1)$ is different from one of α and β, assume that $\psi_1(v_1) \neq \alpha$. The following three colorings are compatible with color $\psi_1(v_1)$ on v_1, hence there are at least three colorings of the core, a contradiction:

- $\psi(v_2) = c$, $\psi(v_3) = \alpha$, $\psi(v_4) = c$,
- $\psi(v_2) = c$, $\psi(v_3) = \beta$, $\psi(v_4) = c$,
- $\psi(v_2) = \alpha$, $\psi(v_3) = \beta$, $\psi(v_4) = c$.

Thus each of the two colors in $L(v_1)$ has a unique extension to v_1, v_2, v_3, v_4, and a unique extension to v_1, v_5, v_6, v_7. Let $L(v_1) = \{1, 2\}$. At least one of $L(v_2)$ and $L(v_4)$ has to contain color 1, otherwise color 1 on vertex v_1 can be extended to v_2, v_3, v_4 in more than one way. Similarly, one of $L(v_2)$ and $L(v_4)$ contains color 2. We show that for some $i_1, i_2 \in \{2, 4\}$, $i_1 \neq i_2$, we have that $\psi_1(v_{i_1}), \psi_2(v_{i_2}) \in \{1, 2\}$ and $\psi_1(v_{i_1}) \neq \psi_2(v_{i_2})$. That is, one of v_2 or v_4 has color 1 or 2 in ψ_1, and the other vertex has the other color in ψ_2. We consider the following cases:

Case 1: $1 \in L(v_2) \cap L(v_4)$. If $1 \notin L(v_3)$, then there is more than one way of extending color 2 on v_1 to v_2, v_3, v_4: assign color 1 to v_2 and v_4, and assign either color of $L(v_3)$ to v_3. Thus, $L(v_2) = \{1, \alpha\}$, $L(v_4) = \{1, \beta\}$, $L(v_3) = \{1, \gamma\}$, where α, β, γ need not be distinct colors (but they are all different from 1). Now color 1 on vertex v_1 can be extended by assigning color α to v_2, color 1 to v_3, and color β to v_4; while color 2 on vertex v_1 can be extended by assigning color 1 to v_2 and v_4, and color γ to v_3. Since the core has only two colorings, these are the unique extensions of color 1 and 2 on v_1 to the vertices v_2, v_3, v_4. However, neither of these colorings satisfies the requirements on ψ_1: if $\psi_1(v_1) = 1$, then $\psi_1(v_1) = \psi_1(v_3) = 1$ follows; if $\psi_1(v_1) = 2$, then $\psi_1(v_2) = \psi_1(v_4) = 1$.

Case 2: $2 \in L(v_2) \cap L(v_4)$. Similar to Case 1.

Case 3: $L(v_2) = \{1, 2\}$. It can be assumed that $1, 2 \notin L(v_4)$, otherwise we are in Case 1 or 2. Therefore, the edge between v_4 and v_1 can be disregarded, since their lists are disjoint. Assigning either color 1 or 2 to vertex v_1 has to force a unique coloring on the path v_1, v_2, v_3, v_4, this is only possible if $L(v_1) = L(v_2) = L(v_3) = L(v_4) = \{1, 2\}$. However, in this case the color of v_1 and v_3 are the same in every coloring, contradicting the assumption $\psi_1(v_1) \neq \psi_1(v_3)$.

Case 4: $L(v_4) = \{1, 2\}$. Similar to Case 3.

Case 5: $L(v_2) \cap L(v_4) \neq \emptyset$. If we are not in Case 1 or 2, then it can be assumed without loss of generality that $L(v_2) = \{1, \alpha\}$ and $L(v_4) = \{2, \alpha\}$. The list $L(v_3)$ has to contain color α, otherwise any color on v_1 could be extended to v_2, v_3, v_4 in more than one way: assign color α to v_2, v_4, and assign any color of $L(v_3)$ to v_3. Let $L(v_3) = \{\alpha, \beta\}$. Color 1 on v_1 forces vertex v_2 to color α, which forces color β on vertex v_3. Color β has to force some color on vertex v_4. Since β is different from α, this is only possible if $\beta = 2$. However, by a symmetrical argument (considering the extension of color 2 on v_1), one can show that $\beta = 1$, a contradiction.

Case 6: $L(v_2) \cap L(v_4) = \emptyset$. If we are not in Case 3 or 4, then it can be assumed without loss of generality that $L(v_2) = \{1, \alpha\}$ and $L(v_4) = \{2, \beta\}$ for some distinct colors α, β different from 1 and 2. We know that color 1 on v_1 has a unique extension to v_2, v_3, v_4. Since color 1 on v_1 forces color α on v_2, this is only possible if color α on vertex v_2 forces some color on vertex v_3, which forces some color on v_4. Therefore, $L(v_3)$ has to contain α. By a similar argument, $L(v_3)$ has to contain also β, thus $L(v_3) = \{\alpha, \beta\}$. Therefore, in the coloring that assigns color 1 to vertex v_1, vertex v_4 has to receive color 2, while in the other coloring, which assigns color 2 to v_1, vertex v_2 has to receive color 1. Thus one can set i_1 and i_2, as required.

A symmetrical argument shows that for some $j_1, j_2 \in \{5, 7\}$, $j_1 \neq j_2$, we have that $\psi_1(v_{j_1}), \psi_2(v_{j_2}) \in \{1, 2\}$ and $\psi_1(v_{j_1}) \neq \psi_2(v_{j_2})$. Notice that $\psi_1(v_{i_1}) = \psi_1(v_{j_1})$: they are both 1 or 2, but different from $\psi_1(v_1) \in \{1, 2\}$. Similarly, $\psi_2(v_{i_2}) = \psi_2(v_{j_2})$. If $i_1 = 2$, then let $\phi := \psi_1$, and $\bar{\phi} := \psi_2$, In this case $\phi(v_2) = \phi(v_5)$ (if $j_1 = 5$) or $\phi(v_2) = \phi(v_7)$ (if $j_1 = 7$). Similarly, $\bar{\phi}(v_4) = \bar{\phi}(v_5)$ (if $j_2 = 5$) or $\bar{\phi}(v_4) = \bar{\phi}(v_7)$ (if $j_2 = 7$), what we had to prove. The case $i_1 = 4$ is similar. This completes the proof of Claim 10.

To proceed with the proof of Lemma 9 and prove Statement 2(a), we show that for arbitrary colors $d_i \in L(x_i)$, one can extended coloring ϕ of the core defined by Claim 10 to the whole gadget such that $\phi(x_i) = d_i$ holds for every $1 \leq i \leq n$. Moreover, the color assigned to vertex $\bar{x}_j$ will not depend on the colors d_i, hence color $\phi(\bar{x}_j)$ of Statement 2 can be defined as this color.

Coloring ϕ of the core can be extended to vertices w_1, w_2, w_3, w_4: these vertices have only two neighbors, but their lists contain three colors. Similarly, the vertices $u_1, \ldots, u_7, \bar{u}_1, \ldots, \bar{u}_7$ can be colored as well, since each of them is connected to only one vertex of the core. Let $\phi(\bar{r}_j) = \psi(\bar{r}_j)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Now the coloring can be extended to the multi-implication gadget $\bar{F}$, since its outputs have the same colors as in ψ (Statement 2 of Lemma 7).

For every $1 \leq j \leq m$, the coloring is extended as follows:

- assign to vertex $\bar{a}_j$ a color different from $\phi(v_2)$ and $\phi(v_5)$,
- assign to vertex $\bar{b}_j$ a color different from $\phi(v_2)$ and $\phi(v_7)$,
- assign to vertex $\bar{t}_j$ a color different from $\phi(\bar{a}_j)$ and $\phi(\bar{b}_j)$,
- assign to vertex $\bar{s}_j$ a color different from $\phi(\bar{t}_j)$ and $\phi(\bar{r}_j)$, and
- assign to vertex $\bar{x}_j$ a color different from $\phi(\bar{s}_j)$.

This completes the description of the extension of ϕ to the upper side. The way we extend ϕ to the lower side depends on which of the three possibilities in Claim 10 holds. Assume first that ϕ is different from $\alpha_1, \ldots, \alpha_7$, which means that the coloring of $u_1, \ldots, u_7$ turns off the gadget F. For every $1 \leq i \leq n$, assign to vertex a_i a color different from $\phi(v_4)$ and $\phi(v_7)$; assign to vertex b_i a color different from $\phi(v_4)$ and $\phi(v_5)$; assign to s_i a color different from x_i and t_i; and assign to r_i a color different from s_i. Since gadget F is turned off, this last assignment can be extended to the gadget F.

Assume next that $\phi(v_4) = \phi(v_7)$ holds. In this case, for every $1 \leq i \leq n$, set $\phi(r_i) = \psi(r_i)$ and assign to vertex s_i a color different from $\phi(r_i)$ and from $\phi(x_i)$. As $\phi(v_4) = \phi(v_7)$, the coloring of the core does not force a color on vertex b_i, hence vertex a_i can be assigned a color different from $\phi(v_4)$ and $\phi(v_7)$; vertex t_i can be assigned a color different from $\phi(a_i)$ and $\phi(s_i)$; and finally vertex b_i can be assigned a color different from $\phi(v_4) = \phi(v_5)$ and $\phi(t_i)$. In the last case of Claim 10, when $\phi(v_4) = \phi(v_7)$, the situation is similar, but we use the fact that the color of vertex a_i is not forced by the coloring of the core.

If we assign to vertex u a color different from $\psi(u)$, then the multi-implication gadget H is turned off, and coloring ϕ can be extended to the vertices of H as well. Therefore, coloring ϕ can be extended to the whole gadget, proving Statement 2(a). Statement 2(b) can be proved by a symmetrical argument: one can show that coloring $\bar{\phi}$ of the core can be extended to a required coloring. This completes the proof of Lemma 9.

7 The reduction

In this section we present a polynomial-time many-one reduction from ∃∃!-SAT to unique $(2,3)$-list coloring, thereby proving the Σ_2^p-completeness of the latter problem. In Lemma 3 we have shown that for every $k \geq 3$, unique $(2,3)$-list coloring can be reduced to unique k-list coloring, hence it follows that UkLC is Σ_2^p-complete for every $k \geq 3$.

Theorem 11 *Unique* $(2,3)$*-list coloring is* Σ_2^p*-complete.*

PROOF. We have seen in Section 2 (Proposition 2) that the problem is in Σ_2^p. By a reduction from ∃∃!-SAT (Section 3), we prove that the problem is Σ_2^p-hard.

Constructing the instance. The reduction uses the gadgets defined in Sections 4–6. By Theorem 5, it can be assumed that $\phi(\mathbf{x}, \mathbf{y})$ is a 3-CNF formula with variables $x_1, \dots, x_n, y_1, \dots, y_m$, and clauses $C_1, \dots, C_r$. For $1 \leq i \leq n$, denote by $o(x_i)$ the number of occurrences of variable x_i in ϕ; denote by $o^+(x_i)$ (resp., $o^-(x_i)$) the number of positive (resp., negated) occurrences of x_i. Similar definitions apply for $o(y_j)$, $o^+(y_j)$, $o^-(y_j)$ for $1 \leq j \leq m$. The constructed graph contains one L-variable gadget for each variable x_i, while there is both an L-variable and a C-variable gadget for each variable y_i. The vertices of the gadgets are named as follows:

- The L-variable gadget $L[x_i]$ has $o(x_i) + o(x_i)$ output vertices: the left side is $x_{i,1}, \dots, x_{i,o^+(x_i)}, \bar{x}^*_{i,1}, \dots, \bar{x}^*_{i,o^-(x_i)}$, the right side is $\bar{x}_{i,1}, \dots, \bar{x}_{i,o^-(x_i)}, x^*_{i,1}, \dots, x^*_{i,o^+(x_i)}$, $(1 \leq i \leq n)$.
- The L-variable gadget $L[y_j]$ has $o^+(y_j) + o^-(y_j)$ output vertices $y_{j,1}, \dots, y_{j,o^+(y_j)}$ and $\bar{y}_{j,1}, \dots, \bar{y}_{j,o^-(y_j)}$ $(1 \leq j \leq m)$.
- The C-variable gadget $C[y_j]$ has $o^+(y_j) + o^-(y_j)$ output vertices $y^*_{j,1}, \dots, y^*_{j,o^+(y_j)}$ and $\bar{y}^*_{j,1}, \dots, \bar{y}^*_{j,o^-(y_j)}$; the control vertex is u_j $(1 \leq j \leq m)$.

For each clause C_k $(1 \leq k \leq r)$, we add a vertex p_k with list size 2 to the graph. The 3 neighbors of p_k correspond to the 3 literals in C_k: if clause C_k contains the ℓ-th positive (resp., negated) occurrence of variable x_i, then connect p_k and vertex $x_{i,\ell}$ (resp., $\bar{x}_{i,\ell}$). Similarly, if the clause contains the ℓ-th occurrence of literal y_j or $\bar{y}_j$, then p_k is connected to vertex $y_{j,\ell}$ or $\bar{y}_{j,\ell}$. Moreover, for each clause C_k we also add four vertices $p^*_k, p^*_{k,1}, p^*_{k,2}, p^*_{k,3}$, where p^*_k is connected to the other three vertices. The list size of vertex p^*_k is 3, and it is 2 for $p^*_{k,1}, p^*_{k,2}, p^*_{k,3}$. For $d = 1, 2, 3$, vertex $p^*_{k,d}$ is connected to a vertex $x^*_{i,\ell}$, $\bar{x}^*_{i,\ell}$, $y^*_{j,\ell}$, or $\bar{y}^*_{j,\ell}$ that represents the d-th literal of clause C_k. Notice that each output vertex of every gadget $L[x_i]$, $L[y_j]$, and $C[y_j]$ is connected to exactly one vertex outside the gadget.

Finally, we add a multi-implication gadget whose $7m$ inputs are the core vertices of the C-variable gadgets $C[y_j]$ ($1 \le j \le m$) and whose outputs are the following vertices:

- vertex u_j for $1 \le j \le m$,
- vertices p^*_k, $p^*_{k,1}$, $p^*_{k,2}$, $p^*_{k,3}$ for $1 \le j \le m$,
- the output vertices $x^*_{i,1}, \dots, x^*_{i,o^+(x_i)}, \bar{x}^*_{i,1}, \dots, \bar{x}^*_{i,o^-(x_i)}$ of gadget $L[x_i]$ for every $1 \le i \le n$,
- the $o^+(y_j) + o^-(y_j)$ output vertices $y^*_{j,1}, \dots, y^*_{j,o^+(y_j)}, \bar{y}^*_{j,1}, \dots, \bar{y}^*_{j,o^-(y_j)}$ of gadget $C[y_j]$ for every $1 \le j \le m$.

This completes the description of the reduction, it is clear that the graph can be constructed in polynomial time.

Unique list coloring $\Rightarrow$ $\exists\exists$!-SAT. First we prove that if the constructed graph has a uniquely colorable list assignment L_0, then there is an assignment $\mathbf{x}_0$ such that there is a unique $\mathbf{y}_0$ that satisfies $\phi(\mathbf{x}_0, \mathbf{y}_0)$. Statement 2(a) or 2(b) of Lemma 8 holds for each gadget $L[x_i]$ and $L[y_j]$. Let x_i be true in the variable assignment $\mathbf{x}_0$ if Statement 2(a) of Lemma 8 holds for gadget $L[x_i]$, and let x_i be false if Statement 2(b) holds (if both 2(a) and 2(b) hold, then choose arbitrarily). Similarly, let y_j be true in $\mathbf{y}_0$ if Statement 2(a) of Lemma 8 holds for gadget $L[y_j]$.

We claim that $\phi(\mathbf{x}_0, \mathbf{y}_0)$ is true. Assume that, on the contrary, some clause C_k of ϕ is not satisfied in $\mathbf{x}_0$, $\mathbf{y}_0$. Let ψ be the unique coloring of L_0, we arrive to a contradiction by showing that L_0 has a coloring different from ψ. Change the color of vertex p_k to be different from $\psi(p_k)$. If this causes a conflict between p_k and a neighbor z, then change the color of z to be different from the color assigned to p_k. Notice that vertex z (which is of the form $x_{i,\ell}$, $\bar{x}_{i,\ell}$, $y_{j,\ell}$, or $\bar{y}_{j,\ell}$) corresponds to a false literal in $\mathbf{x}_0$, $\mathbf{y}_0$. By the way $\mathbf{x}_0$ and $\mathbf{y}_0$ were defined, this means that z is on the side of the L-variable gadget whose color is not forced by the list assignment (Statement 2(a) or 2(b) of Lemma 8). Therefore, the gadget containing z can be recolored to accommodate the new color of z. Repeating this for each conflicting neighbor of p_k yields a proper list coloring of L_0, contradicting the assumption that ψ is the unique coloring of L_0.

What remains to be shown is that for assignment $\mathbf{x}_0$ there is exactly one assignment $\mathbf{y}$ such that $\phi(\mathbf{x}_0, \mathbf{y})$ is true. Assume that, on the contrary, there are two such assignments $\mathbf{y}_1$ and $\mathbf{y}_2$ (one of these two assignments may be the assignment $\mathbf{y}_0$ defined above, but this will not be important). For each C-variable gadget $C[y_j]$, Statement 2 of Lemma 9 defines two colorings ϕ and $\bar{\phi}$ of the core. Consider the variable assignment where y_j is true if ψ restricted to the core of $C[y_j]$ is different from $\bar{\phi}$, and false if it is different from ϕ (if it is different from both, then choose arbitrarily). As $\phi \ne \bar{\phi}$, it cannot happen that the restriction of ψ is different from neither ϕ nor $\bar{\phi}$. At least one of $\mathbf{y}_1$

and $\mathbf{y}_2$, say $\mathbf{y}_1$, is different from this assignment. Based on $\mathbf{y}_1$, we construct a coloring ψ' of L_0 that is different from ψ, a contradiction.

Coloring ψ' is the same as ψ on the gadgets $L[y_j]$, and on the vertices p_k. In the gadget $L[x_i]$ ($1 \le i \le n$), colorings ψ' and ψ are also identical on the vertices $x_{i,1}, \ldots, x_{i,o^+(x_i)}, \bar{x}_{i,1}, \ldots, \bar{x}_{i,o^-(x_i)}$. Moreover, if x_i is true in $\mathbf{x}_0$, then ψ' assigns the same color as ψ to the vertices $\bar{x}^*_{i,1}, \ldots, \bar{x}^*_{i,o^-(x_i)}$; if x_i is false, then ψ and ψ' are the same on vertices $x^*_{i,1}, \ldots, x^*_{i,o^+(x_i)}$. If y_j is true in $\mathbf{y}_1$, then color the core and vertices $\bar{y}^*_{j,1}, \ldots, \bar{y}^*_{j,\ell}$ of $C[y_j]$ using coloring ϕ defined by Statement 2(a) of Lemma 9. If y_j is false in $\mathbf{y}_1$, then color the core and vertices $y^*_{j,1}, \ldots, y^*_{j,\ell}$ of $C[y_j]$ using coloring $\bar{\phi}$ defined by Statement 2(b) of Lemma 9. Notice that all the vertices $x_{i,\ell}$, $\bar{x}_{i,\ell}$, $y_{j,\ell}$, $\bar{y}_{j,\ell}$ are colored the same way as in ψ, and they do not conflict with the vertices p_k. Moreover, among the vertices $x^*_{i,\ell}$, $\bar{x}^*_{i,\ell}$, $y^*_{j,\ell}$, $\bar{y}^*_{j,\ell}$, so far exactly those vertices received a color that correspond to false literals in $\mathbf{x}_0$ and $\mathbf{y}_1$. By the definition of $\mathbf{y}_1$, it is clear that the core of $C[y_j]$ was recolored for at least one $1 \le j \le m$, hence the multi-implication gadget M is turned off in ψ'.

For each $1 \le k \le r$, vertices p^*_k, $p^*_{k,1}$, $p^*_{k,1}$, $p^*_{k,1}$ are colored as follows. For $d = 1, 2, 3$, let $z_{k,d}$ be the neighbor of $p^*_{k,d}$ different from p^*_k. Since clause C_k is satisfied by the assignment $\mathbf{x}_0$, $\mathbf{y}_1$, the clause contains at least one true literal. This means that at least one $z_{k,d}$, say $z_{k,1}$ does not have a color yet. Assign to $p^*_{k,2}$ a color different from $\psi'(z_{k,2})$, assign to $p^*_{k,3}$ a color different from $\psi'(z_{k,3})$, and assign to p^*_k a color different from the colors assigned to $p^*_{k,2}$ and $p^*_{k,3}$ (this can be done, since the list of p^*_k contains 3 colors). Let $\psi'(p^*_{k,1}) \in L_0(p^*_{k,1})$ be a color different from $\psi'(p^*_k)$, and let $\psi'(z_{k,1}) \in L_0(z_{k,1})$ be a color different from $\psi'(p^*_{k,1})$.

We assigned a color to each output vertex of gadget $L[x_i]$. The assignments were done in such a way that Statement 2(a) or 2(b) of Lemma 8 ensures that the coloring can be extended to the whole gadget. That is, if Statement 2(a) holds (implying that x_i true in $\mathbf{x}_0$), then ψ and ψ' are the same on the left side $x_{i,1}, \ldots, x_{i,o^+(x_i)}, \bar{x}^*_{i,1}, \ldots, \bar{x}^*_{i,o^-(x_i)}$; if Statement 2(b) holds (implying that x_i is false in $\mathbf{x}_0$), then ψ and ψ' are the same on the right side $\bar{x}_{i,1}, \ldots, \bar{x}_{i,o^-(x_i)}, x^*_{i,1}, \ldots, x^*_{i,o^+(x_i)}$. Similarly, the coloring ψ' on the core and output vertices of $C[y_j]$ can be extended to the whole gadget. This follows from Statement 2(a) or 2(b) of Lemma 9: after coloring the core as ϕ or $\bar{\phi}$, the side corresponding to the true literals can have arbitrary colors. Finally, as noted earlier, the coloring on the cores of the gadgets $C[y_j]$ turns off the multi-implication gadget M, hence coloring ψ' can be extended to M as well. Thus we obtain a coloring ψ' of L_0 that is different from ψ, a contradiction.

$\exists\exists!$-SAT $\Rightarrow$ **Unique list coloring.** To prove the other direction of the equivalence, we show that if there is a variable assignment $\mathbf{x}_0$ such that exactly one assignment $\mathbf{y}_0$ makes $\phi(\mathbf{x}_0, \mathbf{y}_0)$ true, then the constructed graph is uniquely

(2, 3)-list colorable. Assume that $\mathbf{x}_0$ is such an assignment, and $\mathbf{y}_0$ is the unique assignment with $\phi(\mathbf{x}_0, \mathbf{y}_0)$ true. We construct a uniquely colorable list assignment L_0 of the graph as follows. Lemma 8 defines two different list assignments L_1 and L_2 on each L-variable gadget. Use the list assignment L_1 on gadget $L[x_i]$ if variable x_i is true in variable assignment $\mathbf{x}_0$, use list assignment L_2 on the gadget if variable x_i is false in the assignment. Similarly, use list assignment L_1 (resp., L_2) on gadget $L[y_j]$ if variable y_j is true (resp., false) in variable assignment $\mathbf{y}_0$. Lemma 9 defines two list assignments L and $\bar{L}$ for each C-variable gadget $C[y_j]$. Use list assignment L (resp., $\bar{L}$) on gadget $C[y_j]$ if variable y_j is true (resp., false) in assignment $\mathbf{y}_0$. This defines L_0 on the variable gadgets. Furthermore, for $1 \le k \le r$, let $L_0(p_k) = \{1, 2\}$, $L_0(p_k^*) = \{\alpha, \beta, \gamma\}$, $L_0(p_{k,1}^*) = \{1, \alpha\}$, $L_0(p_{k,2}^*) = \{1, \beta\}$, $L_0(p_{k,3}^*) = \{1, \gamma\}$.

Let us forget the multi-implication gadget for a moment. We show that the graph has a proper coloring ψ with the list assignment defined above. Set

- $\psi(x_{i,\ell}) = 1$ $(1 \le i \le n)$, $1 \le \ell \le o^+(x_i)$,
- $\psi(\bar{x}_{i,\ell}) = 1$ $(1 \le i \le n)$, $1 \le \ell \le o^-(x_i)$,
- $\psi(y_{j,\ell}) = 1$ $(1 \le j \le m)$, $1 \le \ell \le o^+(y_j)$,
- $\psi(\bar{y}_{j,\ell}) = 1$ $(1 \le j \le m)$, $1 \le \ell \le o^-(y_j)$,
- $\psi(x_{i,\ell}^*) = 2$ (resp., 1) if x_i is true (resp., false) in $\mathbf{x}_0$ $(1 \le i \le n$, $1 \le \ell \le o^+(x_i))$,
- $\psi(\bar{x}_{i,\ell}^*) = 1$ (resp., 2) if x_i is true (resp., false) in $\mathbf{x}_0$ $(1 \le i \le n$, $1 \le \ell \le o^-(x_i))$,
- $\psi(y_{j,\ell}^*) = 2$ (resp., 1) if y_j is true (resp., false) in $\mathbf{y}_0$ $(1 \le j \le m$, $1 \le \ell \le o^+(y_j))$,
- $\psi(\bar{y}_{j,\ell}^*) = 1$ (resp., 2) if y_j is true (resp., false) in $\mathbf{y}_0$ $(1 \le j \le m$, $1 \le \ell \le o^+(y_j))$,
- $\psi(u_j) = 1$ for $(1 \le j \le m)$, and
- $\psi(p_k) = 2$ for $(1 \le k \le r)$.

Notice that every neighbor of p_k receives color 1, hence no conflict arises with these assignments. For the vertices $x_{i,\ell}^*$, $\bar{x}_{i,\ell}^*$, $y_{j,\ell}^*$, $\bar{y}_{j,\ell}^*$, the color of the vertex is 2 if and only if the value of the corresponding literal is true. All the vertices $x_{i,\ell}$, $\bar{x}_{i,\ell}$, $y_{j,\ell}$, $\bar{y}_{j,\ell}$ have color 1. Moreover, we will use the fact that if a vertex $x_{i,\ell}$, $\bar{x}_{i,\ell}$, $y_{j,\ell}$, or $\bar{y}_{j,\ell}$ corresponds to a true literal in $\mathbf{x}_0$ and $\mathbf{y}_0$, then this vertex receives color 1 in *every* coloring of L_0. This follows from the way L_0 was defined on the L-variable gadgets.

Consider a vertex $p_{k,d}^*$ for some $1 \le k \le r$ and $d = 1, 2, 3$. This vertex has a neighbor $z_{k,d}$ in a gadget $L[x_i]$ or $C[y_i]$. If $z_{k,d}$ was assigned color 2 in ψ, then let $\psi(p_{k,d}^*) = 1$, otherwise let $\psi(p_{k,d}^*) = L_0(p_{k,d}^*) \setminus 1$, which must be one of α, β, or γ. For a given k, it cannot happen that all three of $\psi(p_{k,1}^*) = \alpha$, $\psi(p_{k,2}^*) = \beta$, $\psi(p_{k,3}^*) = \gamma$ hold: that would mean that all three literals of C_k are false in the assignment $\mathbf{x}_0$, $\mathbf{y}_0$, contradicting the assumption that $\phi(\mathbf{x}_0, \mathbf{y}_0)$

is true. Therefore, at least one of the three vertices $p^*_{k,1}$, $p^*_{k,2}$, $p^*_{k,3}$ has color 1, hence vertex p^*_k can receive one of the colors α, β, γ.

Coloring ψ defined above can be extended to each gadget in a unique way. For each gadget $L[x_i]$ ($1 \le i \le n$), one side of the output is colored with color 1, hence by Statement 1(a) or 1(b) of Lemma 8, the coloring can be uniquely extended to the gadget, regardless of the colors on the other side of the output. The situation is similar for the gadgets $L[y_j]$ ($1 \le j \le m$). For a gadget $C[y_j]$, the color of the control vertex u_j is 1, one side of the output has color 1, the other side has color 2. Since the list assignment of the gadget is either L or $\bar{L}$ defined by Lemma 9, it follows that ψ can be uniquely extended to the vertices of the gadget.

Now we define the list assignment L_0 on the vertices of the multi-implication gadget M. The coloring ψ defined above gives some color to each input and each output vertex of M. Extend L_0 to M in such a way that this particular combination of colors on the input vertices forces the colors assigned by ψ to the output vertices (Statements 1(a)–(c) of Lemma 7). It is clear that coloring ψ can be extended to the vertices of M, hence list assignment L_0 admits at least one proper coloring.

We claim that ψ is the unique coloring of L_0, hence the graph is uniquely list colorable. Assume that ψ' is a coloring of L_0 different from ψ. First we show that $\psi'(p_k) = 2$ for every $1 \le k \le r$. Since $\phi(\mathbf{x}_0, \mathbf{y}_0)$ is true, there is at least one true literal in clause C_k. Consider the vertex $x_{i,\ell}$, $\bar{x}_{i,\ell}$, $y_{j,\ell}$, or $\bar{y}_{j,\ell}$ corresponding to this true literal. We have noted that such a vertex corresponding to a true literal has to receive color 1 in every coloring of L_0, and this forces color 2 on vertex p_k. Moreover, color 2 on vertex p_k forces color 1 on each of its three neighbors, hence there has to be color 1 on every $x_{i,\ell}$, $\bar{x}_{i,\ell}$, $y_{j,\ell}$, and $\bar{y}_{j,\ell}$. This means that for every $1 \le j \le m$, the output vertices of gadget $L[y_j]$ has color 1, therefore the coloring of $L[y_j]$ is the same in ψ' as in ψ.

We consider two cases. First assume that the coloring of the core of $C[y_j]$ is the same in ψ and ψ' for every $1 \le j \le m$. This means that in ψ' the multi-implication gadget M is turned on, implying that ψ' and ψ are the same on the output vertices of M. In particular, $\psi'(p^*_k) = \psi(p^*_k)$ and $\psi'(p^*_{k,d}) = \psi(p^*_{k,d})$ for every $1 \le j \le m$ and $d = 1, 2, 3$. Moreover, M ensures that the output vertices and the control vertex of each gadget $C[y_j]$ have the same color in ψ' and ψ. Therefore, by Statement 1(b) or 1(e) of Lemma 9, the whole gadget has the same coloring in ψ and ψ'. Similarly, in coloring ψ', gadget M forces output vertices $x^*_{i,1}, \ldots, x^*_{i,o^+(x_i)}, \bar{x}^*_{i,1}, \ldots, \bar{x}^*_{i,o^-(x_i)}$ of gadget $L[x_i]$ to the same color as in ψ. We have already seen that ψ and ψ' are the same on output vertices $x_{i,1}, \ldots, x_{i,o^+(x_i)}, \bar{x}_{i,1}, \ldots, \bar{x}_{i,o^-(x_i)}$ of gadget $L[x_i]$. Therefore, by Statement 1(a) or 1(b) of Lemma 8, the output vertices of gadget $L[x_i]$ forces the gadget to have the same coloring in ψ and ψ'. Thus we have shown that ψ' is the same as ψ on every vertex of the graph, a contradiction.

Now assume that the coloring of the core of $C[y_{j'}]$ is different in ψ and ψ' for some $1 \le j' \le m$; as the core of $C[y_{j'}]$ has only two colorings, this also means that vertex v_1 of $C[y_{j'}]$ has different colors in ψ and ψ'. Define the variable assignment $\mathbf{y}_1$ by setting variable y_j to true if and only if vertex v_1 of $C[y_k]$ has color 2 in ψ'. It is clear that $\mathbf{y}_1$ is different from $\mathbf{y}_0$; therefore, $\phi(\mathbf{x}_0, \mathbf{y}_1)$ is false, as we assumed that $\mathbf{y}_0$ is the only assignment that satisfies $\phi(\mathbf{x}_0, \mathbf{y}_0)$. Hence for some $1 \le k \le r$, assignment $\mathbf{x}_0$, $\mathbf{y}_1$ does not satisfy clause C_k.

If variable x_i is true in $\mathbf{x}_0$, then by Statement 1(a) of Lemma 8, the list assignment used on gadget $L[x_i]$ ensures that vertices $\bar{x}^*_{i,1}, \dots, \bar{x}^*_{i,o^-(x_i)}$ receive color 1 in ψ'. Similarly, by Statement 1(b) of Lemma 8, if x_i is false, then $x^*_{i,1}, \dots, x^*_{i,o^+(x_i)}$ receive color 1 in ψ'. If y_j is true in $\mathbf{y}_1$, then core vertex v_1 of $C[y_j]$ has color 2, hence by Statement 1(a) or 1(d) of Lemma 9, vertices $\bar{y}^*_{j,1}, \dots, \bar{y}^*_{j,o^-(y_j)}$ are forced to color 1 in ψ'. Similarly, if y_j is false in $\mathbf{y}_1$, then $y^*_{j,1}, \dots, y^*_{j,o^+(y_j)}$ are forced to color 1. Therefore, we can conclude that those vertices $x^*_{i,\ell}$, $\bar{x}^*_{i,\ell}$, $y^*_{j,\ell}$, $\bar{y}^*_{i,\ell}$ that correspond to a false literal in assignment $\mathbf{x}_0$, $\mathbf{y}_1$ have to receive color 1 in ψ'. We assumed that clause C_k is not satisfied in $\mathbf{x}_0$, $\mathbf{y}_1$, hence each of the three vertices $p^*_{k,1}$, $p^*_{k,2}$, $p^*_{k,3}$ has a neighbor that receives color 1 in ψ'. Thus $\psi'(p^*_{k,1}) = \alpha$, $\psi'(p^*_{k,1}) = \beta$, $\psi'(p^*_{k,1}) = \gamma$, and there remains no color for vertex p^*_k, a contradiction.

Using Lemma 3, the hardness result can be extended for arbitrary list sizes:

Corollary 12 *Unique k-list colorability is Σ_2^p-complete for every $k \ge 3$.* $\square$

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