W[1]-hardness

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Recent Advances in Parameterized Complexity
Tel Aviv, Israel, December 3, 2017
Lower bounds

So far we have seen positive results: basic algorithmic techniques for fixed-parameter tractability.

What kind of negative results we have?

- Can we show that a problem (e.g., \texttt{Clique}) is \textbf{not} FPT?
- Can we show that a problem (e.g., \texttt{Vertex Cover}) has no algorithm with running time, say, \(2^{o(k)} \cdot n^{O(1)}\)?

This would require showing that P \(\neq\) NP: if P = NP, then, e.g., \(k\)-\texttt{Clique} is polynomial-time solvable, hence FPT.

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This would require showing that \(P \neq \text{NP}\): if \(P = \text{NP}\), then, e.g., \(k\text{-Clique}\) is polynomial-time solvable, hence FPT.

Can we give some evidence for negative results?
Goals of this talk

Two goals:

1. Explain the theory behind parameterized intractability.
2. Show examples of parameterized reductions.
Classical complexity

Nondeterministic Turing Machine (NTM): single tape, finite alphabet, finite state, head can move left/right only one cell. In each step, the machine can branch into an arbitrary number of directions. Run is successful if at least one branch is successful.

NP: The class of all languages that can be recognized by a polynomial-time NTM.

Polynomial-time reduction from problem $P$ to problem $Q$: a function $\phi$ with the following properties:
- $\phi(x)$ is a yes-instance of $Q$ $\iff$ $x$ is a yes-instance of $P$,
- $\phi(x)$ can be computed in time $|x|^{O(1)}$.

Definition: Problem $Q$ is NP-hard if any problem in NP can be reduced to $Q$.

If an NP-hard problem can be solved in polynomial time, then every problem in NP can be solved in polynomial time (i.e., $P = NP$).
Parameterized complexity

To build a complexity theory for parameterized problems, we need two concepts:

- An appropriate notion of reduction.
- An appropriate hypothesis.

Polynomial-time reductions are not good for our purposes.
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Polynomial-time reductions are not good for our purposes.

**Example:** Graph $G$ has an independent set $k$ if and only if it has a vertex cover of size $n - k$.

$\Rightarrow$ Transforming an **Independent Set** instance $(G, k)$ into a **Vertex Cover** instance $(G, n - k)$ is a correct polynomial-time reduction.

However, **Vertex Cover** is FPT, but **Independent Set** is not known to be FPT.
Parameterized reduction

**Definition**

Parameterized reduction from problem $P$ to problem $Q$: a function $\phi$ with the following properties:

- $\phi(x)$ is a yes-instance of $Q$ $\iff$ $x$ is a yes-instance of $P$,
- $\phi(x)$ can be computed in time $f(k) \cdot |x|^{O(1)}$, where $k$ is the parameter of $x$,
- If $k$ is the parameter of $x$ and $k'$ is the parameter of $\phi(x)$, then $k' \leq g(k)$ for some function $g$.

**Fact:** If there is a parameterized reduction from problem $P$ to problem $Q$ and $Q$ is FPT, then $P$ is also FPT.
**Parameterized reduction**

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**Parameterized reduction** from problem $P$ to problem $Q$: a function $\phi$ with the following properties:

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**Fact:** If there is a parameterized reduction from problem $P$ to problem $Q$ and $Q$ is FPT, then $P$ is also FPT.

**Non-example:** Transforming an **Independent Set** instance $(G, k)$ into a **Vertex Cover** instance $(G, n - k)$ is not a parameterized reduction.

**Example:** Transforming an **Independent Set** instance $(G, k)$ into a **Clique** instance $(\overline{G}, k)$ is a parameterized reduction.
Multicolored Clique

A useful variant of Clique:

**Multicolored Clique:** The vertices of the input graph $G$ are colored with $k$ colors and we have to find a clique containing one vertex from each color.

(or **Partitioned Clique**)

**Theorem**

There is a parameterized reduction from **Clique** to **Multicolored Clique**.
Multicolored Clique

Theorem

There is a parameterized reduction from Clique to Multicolored Clique.

Create $G'$ by replacing each vertex $v$ with $k$ vertices, one in each color class. If $u$ and $v$ are adjacent in the original graph, connect all copies of $u$ with all copies of $v$.

$k$-clique in $G$ $\iff$ multicolored $k$-clique in $G'$. 
**Multicolored Clique**

**Theorem**

There is a parameterized reduction from *Clique* to *Multicolored Clique*.

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**Similarly:** reduction to *Multicolored Independent Set*. 

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Theorem

There is a parameterized reduction from **Multicolored Independent Set** to **Dominating Set**.

**Proof:** Let $G$ be a graph with color classes $V_1, \ldots, V_k$. We construct a graph $H$ such that $G$ has a multicolored $k$-clique iff $H$ has a dominating set of size $k$.

The dominating set has to contain one vertex from each of the $k$ cliques $V_1, \ldots, V_k$ to dominate every $x_i$ and $y_i$. 
Dominating Set

Theorem

There is a parameterized reduction from Multicolored Independent Set to Dominating Set.

Proof: Let $G$ be a graph with color classes $V_1, \ldots, V_k$. We construct a graph $H$ such that $G$ has a multicolored $k$-clique iff $H$ has a dominating set of size $k$.

- The dominating set has to contain one vertex from each of the $k$ cliques $V_1, \ldots, V_k$ to dominate every $x_i$ and $y_i$.
- For every edge $e = uv$, an additional vertex $w_e$ ensures that these selections describe an independent set.
Variants of **Dominating Set**

- **Dominating Set**: Given a graph, find $k$ vertices that dominate every vertex.

- **Red-Blue Dominating Set**: Given a bipartite graph, find $k$ vertices on the red side that dominate the blue side.

- **Set Cover**: Given a set system, find $k$ sets whose union covers the universe.

- **Hitting Set**: Given a set system, find $k$ elements that intersect every set in the system.

All of these problems are equivalent under parameterized reductions, hence at least as hard as **Clique**.
Basic hypotheses

It seems that parameterized complexity theory cannot be built on assuming $P \neq NP$ – we have to assume something stronger.

Let us choose a basic hypothesis:

<table>
<thead>
<tr>
<th>Engineers’ Hypothesis</th>
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$k$-Step Halting Problem (is there a path of the given NTM that stops in $k$ steps?) cannot be solved in time $f(k) \cdot n^{O(1)}$. 
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**Exponential Time Hypothesis (ETH)**

$n$-variable 3SAT cannot be solved in time $2^{o(n)}$.

Which hypothesis is the most plausible?
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Summary

1. **Independent Set** and **$k$-Step Halting Problem** can be reduced to each other $\Rightarrow$ Engineers’ Hypothesis and Theorists’ Hypothesis are equivalent!

2. **Independent Set** and **$k$-Step Halting Problem** can be reduced to **Dominating Set**.

Moreover, there is no parameterized reduction from **Dominating Set** to **Independent Set**. Unlike in NP-completeness, where most problems are equivalent, here we have a hierarchy of hard problems. **Independent Set** is W[1]-complete and **Dominating Set** is W[2]-complete. Does not matter if we only care about whether a problem is FPT or not!
Summary

- **Independent Set** and **k-Step Halting Problem** can be reduced to each other ⇒ Engineers’ Hypothesis and Theorists’ Hypothesis are equivalent!

- **Independent Set** and **k-Step Halting Problem** can be reduced to **Dominating Set**.

- Is there a parameterized reduction from **Dominating Set** to **Independent Set**?
  - Probably not. Unlike in **NP**-completeness, where most problems are equivalent, here we have a hierarchy of hard problems.
    - **Independent Set** is **W[1]**-complete.
    - **Dominating Set** is **W[2]**-complete.

- Does not matter if we only care about whether a problem is FPT or not!
A **Boolean circuit** consists of input gates, negation gates, AND gates, OR gates, and a single output gate.

**Circuit Satisfiability**: Given a Boolean circuit $C$, decide if there is an assignment on the inputs of $C$ making the output true.
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**Circuit Satisfiability**: Given a Boolean circuit $C$, decide if there is an assignment on the inputs of $C$ making the output true.

**Weight of an assignment**: number of true values.

**Weighted Circuit Satisfiability**: Given a Boolean circuit $C$ and an integer $k$, decide if there is an assignment of weight $k$ making the output true.
Weighted Circuit Satisfiability

Independent Set can be reduced to Weighted Circuit Satisfiability:

Dominating Set can be reduced to Weighted Circuit Satisfiability:
**Weighted Circuit Satisfiability**

**Independent Set** can be reduced to **Weighted Circuit Satisfiability**:

![Diagram of Independent Set reduction](attachment:independent_set_reduction.png)

**Dominating Set** can be reduced to **Weighted Circuit Satisfiability**:

![Diagram of Dominating Set reduction](attachment:dominating_set_reduction.png)

To express **Dominating Set**, we need more complicated circuits.
Depth and weft

The **depth** of a circuit is the maximum length of a path from an input to the output.

A gate is **large** if it has more than 2 inputs. The **weft** of a circuit is the maximum number of large gates on a path from an input to the output.

**Independent Set:** weft 1, depth 3

```
X1  X2  X3  X4  X6  X7
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
\downarrow  \downarrow  \downarrow  \downarrow  \downarrow  \downarrow
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
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\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
```

**Dominating Set:** weft 2, depth 2

```
X1  X2  X3  X4  X6  X7
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
\downarrow  \downarrow  \downarrow  \downarrow  \downarrow  \downarrow
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
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\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
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```

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The W-hierarchy

Let \( C[t, d] \) be the set of all circuits having weft at most \( t \) and depth at most \( d \).

**Definition**

A problem \( P \) is in the class \( W[t] \) if there is a constant \( d \) and a parameterized reduction from \( P \) to Weighted Circuit Satisfiability of \( C[t, d] \).

We have seen that *Independent Set* is in \( W[1] \) and *Dominating Set* is in \( W[2] \).

**Fact:** *Independent Set* is \( W[1] \)-complete.

**Fact:** *Dominating Set* is \( W[2] \)-complete.
The W-hierarchy

Let \( C[t, d] \) be the set of all circuits having weft at most \( t \) and depth at most \( d \).

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We have seen that Independent Set is in \( W[1] \) and Dominating Set is in \( W[2] \).

**Fact:** Independent Set is \( W[1] \)-complete.

**Fact:** Dominating Set is \( W[2] \)-complete.

If any \( W[1] \)-complete problem is FPT, then \( \text{FPT} = W[1] \) and every problem in \( W[1] \) is FPT.


\( \Rightarrow \) If there is a parameterized reduction from Dominating Set to Independent Set, then \( W[1] = W[2] \).
**Weft** is a term related to weaving cloth: it is the thread that runs from side to side in the fabric.
Parameterized reductions

Typical **NP**-hardness proofs: reduction from e.g., **CLIQUE** or **3SAT**, representing each vertex/edge/variable/clause with a gadget.

Usually does not work for parameterized reductions: cannot afford the parameter increase.
Parameterized reductions

Typical NP-hardness proofs: reduction from e.g., **Clique** or **3SAT**, representing each vertex/edge/variable/clause with a gadget.

Usually does not work for parameterized reductions: cannot afford the parameter increase.

Types of parameterized reductions:
- Reductions keeping the structure of the graph.
  - **Clique** \(\Rightarrow\) **Independent Set**
- Reductions with vertex representations.
  - **Multicolored Independent Set** \(\Rightarrow\) **Dominating Set**
- Reductions with vertex and edge representations.
List Coloring is a generalization of ordinary vertex coloring: given a
- graph \( G \),
- a set of colors \( C \), and
- a list \( L(v) \subseteq C \) for each vertex \( v \),
the task is to find a coloring \( c \) where \( c(v) \in L(v) \) for every \( v \).

**Theorem**

**Vertex Coloring** is FPT parameterized by treewidth.

However, list coloring is more difficult:

**Theorem**

**List Coloring** is \( W[1] \)-hard parameterized by treewidth.
List Coloring

Theorem

List Coloring is $W[1]$-hard parameterized by treewidth.

Proof: By reduction from Multicolored Independent Set.

- Let $G$ be a graph with color classes $V_1, \ldots, V_k$.
- Set $C$ of colors: the set of vertices of $G$.
- The colors appearing on vertices $u_1, \ldots, u_k$ correspond to the $k$ vertices of the clique, hence we set $L(u_i) = V_i$. 

```
    u_2 : V_2
    u_1 : V_1
    u_3 : V_3
    u_5 : V_5
    u_4 : V_4
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- Set \(C\) of colors: the set of vertices of \(G\).
- The colors appearing on vertices \(u_1, \ldots, u_k\) correspond to the \(k\) vertices of the clique, hence we set \(L(u_i) = V_i\).
- If \(x \in V_i\) and \(y \in V_j\) are adjacent in \(G\), then we need to ensure that \(c(u_i) = x\) and \(c(u_j) = y\) are not true at the same time \(\Rightarrow\) we add a vertex adjacent to \(u_i\) and \(u_j\) whose list is \(\{x, y\}\).
Vertex representation

**Key idea**

- Represent the $k$ vertices of the solution with $k$ gadgets.
- Connect the gadgets in a way that ensures that the represented values are **compatible**.
Odd Set

**Odd Set**: Given a set system \( \mathcal{F} \) over a universe \( U \) and an integer \( k \), find a set \( S \) of at most \( k \) elements such that \( |S \cap F| \) is odd for every \( F \in \mathcal{F} \).

**Theorem**

Odd Set is \( W[1] \)-hard parameterized by \( k \).
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**First try:** Reduction from **Multicolored Independent Set**. Let $U = V_1 \cup \ldots \cup V_k$ and introduce each set $V_i$ into $\mathcal{F}$.

$\Rightarrow$ The solution has to contain exactly one element from each $V_i$.

If $xy \in E(G)$, how can we express that $x \in V_i$ and $y \in V_j$ cannot be selected simultaneously?
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If $xy \in E(G)$, how can we express that $x \in V_i$ and $y \in V_j$ cannot be selected simultaneously? Seems difficult:

- introducing $\{x, y\}$ into $F$ forces that exactly one of $x$ and $y$ appears in the solution,
**Odd Set**

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If $xy \in E(G)$, how can we express that $x \in V_i$ and $y \in V_j$ cannot be selected simultaneously? Seems difficult:

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If $xy \in E(G)$, how can we express that $x \in V_i$ and $y \in V_j$ cannot be selected simultaneously? Seems difficult:

- introducing $\{x, y\}$ into $\mathcal{F}$ forces that exactly one of $x$ and $y$ appears in the solution,
- introducing $\{x\} \cup (V_j \setminus \{y\})$ into $\mathcal{F}$ forces that either both $x$ and $y$ or none of $x$ and $y$ appear in the solution.
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Odd Set

Reduction from Multicolored Clique.

- $U := \bigcup_{i=1}^{k} V_i \cup \bigcup_{1 \leq i < j \leq k} E_{i,j}$.
- $k' := k + \binom{k}{2}$.
- Let $\mathcal{F}$ contain $V_i$ ($1 \leq i \leq k$) and $E_{i,j}$ ($1 \leq i < j \leq k$).
Odd Set

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- Let \( \mathcal{F} \) contain \( V_i \) (\( 1 \leq i \leq k \)) and \( E_{i,j} \) (\( 1 \leq i < j \leq k \)).
- For every \( v \in V_i \) and \( x \neq i \), we introduce the sets:
  - \( (V_i \setminus \{v\}) \cup \{\text{every edge from } E_{i,x} \text{ with endpoint } v\} \)
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![Diagram](image.png)
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- $v \in V_i$ selected $\iff$ edges with endpoint $v$ are selected from $E_{i,x}$ and $E_{x,i}$
Odd Set

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- $v \in V_i$ selected $\iff$ edges with endpoint $v$ are selected from $E_{i,x}$ and $E_{x,i}$

- $v_i \in V_i$ selected $\iff$ edge $v_i v_j$ is selected in $E_{i,x}$

- $v_j \in V_j$ selected

\[ V_1 \quad V_2 \quad V_3 \quad V_4 \]

\[ E_{1,2} \quad E_{1,3} \quad E_{1,4} \quad E_{2,3} \quad E_{2,4} \quad E_{3,4} \]
Vertex and edge representation

Key idea

- Represent the vertices of the clique by $k$ gadgets.
- Represent the edges of the clique by $\binom{k}{2}$ gadgets.
- Connect edge gadget $E_{i,j}$ to vertex gadgets $V_i$ and $V_j$ such that if $E_{i,j}$ represents the edge between $x \in V_i$ and $y \in V_j$, then it forces $V_i$ to $x$ and $V_j$ to $y$. 
Variants of Odd Set

The following problems are $W[1]$-hard:

- **Odd Set**
- **Exact Odd Set** (find a set of size exactly $k$ ... )
- **Exact Even Set**
- **Unique Hitting Set**
  (at most $k$ elements that hit each set exactly once)
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Open question:

**Even Set**: Given a set system $\mathcal{F}$ and an integer $k$, find a nonempty set $S$ of at most $k$ elements such that $|F \cap S|$ is even for every $F \in \mathcal{F}$. 
**Grid Tiling**

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$k = 3, D = 5$

**Input:** A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

**Find:** A pair $s_{i,j} \in S_{i,j}$ for each cell such that
- Vertical neighbors agree in the 1st coordinate.
- Horizontal neighbors agree in the 2nd coordinate.
Grid Tiling

**Input:** A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

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$k = 3, \ D = 5$
Grid Tiling

**GRID TILING**

**Input:** A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

A pair $s_{i,j} \in S_{i,j}$ for each cell such that

**Find:**
- Vertical neighbors agree in the 1st coordinate.
- Horizontal neighbors agree in the 2nd coordinate.

**Simple proof:**

**Fact**

There is a parameterized reduction from $k$-CLIQUE to $k \times k$ GRID TILING.
Grid Tiling is W[1]-hard

Reduction from \( k\text{-CLIQUE} \)

**Definition of the sets:**

- For \( i = j \): \((x, y) \in S_{i,j} \iff x = y\)
- For \( i \neq j \): \((x, y) \in S_{i,j} \iff x \text{ and } y \text{ are adjacent.}\)

Each diagonal cell defines a value \( v_i \ldots \)
Grid Tiling is W[1]-hard

Reduction from $k$-CLIQUE

Definition of the sets:

- For $i = j$: $(x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $(x, y) \in S_{i,j} \iff x$ and $y$ are adjacent.

\[
\begin{array}{cccc}
(v_i, .) & & & \\
(., v_i) & (v_i, v_i) & (., v_i) & (., v_i) \\
(., v_i) & (v_i, .) & & \\
(v_i, .) & (v_i, .) & & \\
(., v_i) & (v_i, .) & & \\
(., v_i) & & & \\
(., v_i) & & & \\
(., v_i) & & & \\
(., v_i) & & & \\
\end{array}
\]

\ldots which appears on a “cross”
## Grid Tiling is W[1]-hard

### Reduction from \( k \)-\textsc{CLIQUE} 

**Definition of the sets:**

- For \( i = j \): \((x, y) \in S_{i,j} \iff x = y\)
- For \( i \neq j \): \((x, y) \in S_{i,j} \iff x \text{ and } y \text{ are adjacent.}\)

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<th>((v_i, .))</th>
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\(v_i\) and \(v_j\) are adjacent for every \(1 \leq i < j \leq k\).
Grid Tiling is W[1]-hard

Reduction from $k$-CLIQUE

Definition of the sets:

- For $i = j$: $(x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $(x, y) \in S_{i,j} \iff x$ and $y$ are adjacent.

$v_i$ and $v_j$ are adjacent for every $1 \leq i < j \leq k$. 
Grid Tiling and planar problems

**Theorem**

$k \times k$ Grid Tiling is W[1]-hard and, assuming ETH, cannot be solved in time $f(k)n^{o(k)}$ for any function $f$.

This lower bound is the key for proving hardness results for planar graphs.

**Examples:**

- **Multiway Cut** on planar graphs with $k$ terminals
- **Independent Set** for unit disks
- **Strongly Connected Steiner Subgraph** on planar graphs
- **Scattered Set** on planar graphs
Input: A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

A pair $s_{i,j} \in S_{i,j}$ for each cell such that

Find:

- 1st coordinate of $s_{i,j} \leq$ 1st coordinate of $s_{i+1,j}$.
- 2nd coordinate of $s_{i,j} \leq$ 2nd coordinate of $s_{i,j+1}$.

$k = 3, \ D = 5$
Grid Tiling with $\leq$

**Grid Tiling with $\leq$**

**Input:** A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

A pair $s_{i,j} \in S_{i,j}$ for each cell such that

- 1st coordinate of $s_{i,j} \leq$ 1st coordinate of $s_{i+1,j}$.
- 2nd coordinate of $s_{i,j} \leq$ 2nd coordinate of $s_{i,j+1}$.

**Find:**

Variant of the previous proof:

**Theorem**

There is a parameterized reduction from $k \times k$-Grid Tiling to $O(k) \times O(k)$ Grid Tiling with $\leq$.

Very useful starting point for geometric (and also some planar) problems!
Reduction to unit disks

**Theorem**

**Independent Set** for unit disks is $W[1]$-hard.

\[
\begin{array}{|c|c|c|}
\hline
(5,1) & (4,3) & (2,3) \\
(1,2) & (3,2) & (2,5) \\
(3,3) & (2,5) & \\
\hline
(2,1) & (4,2) & (5,1) \\
(5,5) & (5,3) & (3,2) \\
(3,5) & & \\
\hline
(5,1) & (2,1) & (3,1) \\
(2,2) & (4,2) & (3,2) \\
(5,3) & & (3,3) \\
\hline
\end{array}
\]

Every pair is represented by a unit disk in the plane.

$\leq$ relation between coordinates $\iff$ disks do not intersect.
Reduction to unit disks

Theorem

**Independent Set** for unit disks is \( W[1] \)-hard.

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Theorem

**Independent Set** for unit disks is $W[1]$-hard.

Every pair is represented by a unit disk in the plane. 
$\leq$ relation between coordinates $\iff$ disks do not intersect.
Summary

- By parameterized reductions, we can show that lots of parameterized problems are at least as hard as \texttt{Clique}, hence unlikely to be fixed-parameter tractable.

- Connection with Turing machines gives some supporting evidence for hardness (only of theoretical interest).

- The \textbf{W}-hierarchy classifies the problems according to hardness (only of theoretical interest).

- Important trick in $\textbf{W}[1]$-hardness proofs: vertex and edge representations.