Everything you always wanted to know about the parameterized complexity of SUBGRAPH ISOMORPHISM

(but were afraid to ask)

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SUBGRAPH ISOMORPHISM

Input: Graphs H and G.

Decide: Is H isomorphic to a subgraph of G?

- Hard in general: HAMILTONIAN CYCLE is a special case.
- Hard even for planar graphs and 3-regular graphs.

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Is SUBGRAPH ISOMORPHISM easy on bounded-treewidth graphs?

DP and bounded treewidth

Standard dynamic programming on a tree decomposition yields the following result:

Fact

SUBGRAPH ISOMORPHISM can be solved in time $f(|V(H)|, \mathrm{tw}(G)) \cdot n$ for some computable function f.

This algorithm needs that

- H is **small** and
- G has **bounded treewidth.**

Color coding and bounded treewidth

The color coding technique of [Alon, Yuster, Zwick 1994] gives the following algorithm:

Fact

Subgraph Isomorphism can be solved in time $2^{O(|V(H)|)} \cdot n^{O(\mathrm{tw}(H))}$.

This algorithm needs that

- H is **small**,
- H has **bounded treewidth**,

but the treewidth of G can be arbitrary.

Another DP

A dynamic programming algorithm of [Matoušek and Thomas 1992] gives the following result:

Fact

Subgraph Isomorphism for connected H can be solved in time $f(|\Delta(H)|) \cdot n^{O(\mathrm{tw}(G))}$ for some computable function f.

This algorithm needs that

- H has **bounded degree**,
- H is **connected**, and
- G has **bounded treewidth**,

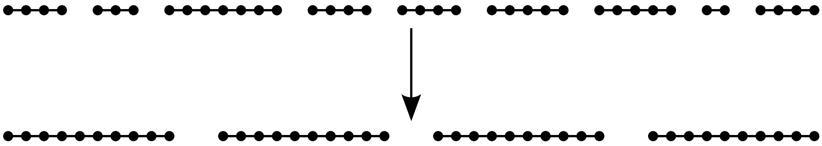
but the size of H can be arbitrary.

Subgraph Isomorphism and Bin Packing

We can reduce Bin Packing (with polynomially bounded sizes) to Subgraph Isomorphism with both H and G being a set of paths:

Fact

Subgraph Isomorphism is NP-hard even if both H and G are sets of paths.



The requirement that H is connected is essential in the [Matoušek and Thomas 1992] result!

Parameters of Subgraph Isomorphism

We have seen that the complexity of Subgraph Isomorphism is influenced by the following parameters of H and G:

- number of vertices,
- treewidth,
- maximum degree,
- number of components (connectedness)

. . . and these parameters can appear either in the exponent of n or as a multiplier of the running time.

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What other natural parameters influence the complexity of Subgraph Isomorphism?

Cliquewidth and Subgraph Isomorphism

Dynamic programming on a tree decomposition:

Fact

Subgraph Isomorphism can be solved in time $f(|V(H)|, \mathrm{tw}(G)) \cdot n$ for some computable function f.

Can be generalized to clique width [Courcelle, Makowsky, Rotics 2000]:

Fact

Subgraph Isomorphism can be solved in time $f(|V(H)|, \mathrm{cw}(G)) \cdot n$ for some computable function f.

Cliquewidth and Subgraph Isomorphism

Color coding:

Fact

Subgraph Isomorphism can be solved in time $2^{O(|V(H)|)} \cdot n^{O(\mathrm{tw}(H))}$.

Cannot be generalized to cliquewidth: cliques have cliquewidth 2 and k-Clique is W[1]-hard.

Fact

Subgraph Isomorphism cannot be solved in time $f(|V(H)|) \cdot n^{O(\mathrm{cw}(H))}$ for any computable function f, unless $\mathrm{W[1]} = \mathrm{FPT}$.

Planarity and Subgraph Isomorphism

Fact

Subgraph Isomorphism can be solved in time $f(|V(H)|, \mathrm{tw}(G)) \cdot n$ for some computable function f.

The result can be generalized to bounded local treewidth and implies the following result for planar G:

Fact

Subgraph Isomorphism for planar G can be solved in time $f(|V(H)|) \cdot n$ for some computable function f.

Planarity and Subgraph Isomorphism

Fact

Subgraph Isomorphism can be solved in time $f(|V(H)|, \mathrm{tw}(G)) \cdot n$ for some computable function f.

Bounded local treewidth can be further generalized:

Fact

Subgraph Isomorphism can be solved in time $f(|V(H)|, \mathrm{x}(G)) \cdot n$ for some computable function f, where

- $\mathrm{x}(G)$ is the genus of G,
- $\mathrm{x}(G)$ is the size of the largest clique minor,
- $\mathrm{x}(G)$ is the size of the largest topological clique minor.

Follows from the linear-time solvability of first-order model checking on graphs of bounded expansion [Dvořak, Král, Thomas 2010], [Grohe, Kreutzer, Siebertz 2013].

Treewidth and feedback vertex set number

Fact [Bodlaender 1990], [Ponomarenko 1988]

Graph Isomorphism can be solved in time $n^{O(\mathrm{tw}(G))}$.

Major open question if there is a $f(\mathrm{tw}(G)) \cdot n^{O(1)}$ time algorithm.

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Feedback vertex set number fvs(G): minimum number of vertices whose deletion makes the graph a forest.

Easy: $\mathrm{tw}(G) \leq \mathrm{fvs}(G) + 1$.

Fact [Kratsch and Schweitzer 2010]

Graph Isomorphism can be solved in time $f(\mathrm{fvs}(G)) \cdot n^{O(1)}$.

Is feedback vertex set number a relevant parameter also for Subgraph Isomorphism?

Parameters

We consider the following 10 parameters for H and G:

1. Number of vertices $|V(\cdot)|$.
2. Number of connected components $\mathrm{cc}(\cdot)$.
3. Maximum degree $\Delta(\cdot)$.
4. Treewidth $\mathrm{tw}(\cdot)$.
5. Pathwidth $\mathrm{pw}(\cdot)$.
6. Feedback vertex set number $\mathrm{fvs}(\cdot)$.
7. Clique width $\mathrm{cw}(\cdot)$.
8. Genus $\mathrm{genus}(\cdot)$.
9. Hadwiger number (largest clique minor) $\mathrm{hadw}(\cdot)$.
10. Topological Hadwiger number (largest topological clique minor) $\mathrm{hadw}_T(\cdot)$.

Main result

Goal

Determine for every combination of these parameters whether there is an algorithm with running time

$$f_1(p_1, p_2, \dots, p_\ell) \cdot n^{f_2(p_{\ell+1}, \dots, p_t)}.$$

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5 possible additional restrictions on H or G:

1. Genus is 0 (i.e., planar).
2. Number of components is 1 (i.e., connected).
3. Treewidth is at most 1 (i.e., graph is a forest).
4. Maximum degree at most 2 (i.e., paths and cycles).
5. Maximum degree at most 3.

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Determine for every combination of these parameters whether there is an algorithm with running time

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For any combination of the 2×10 parameters in the multiplier and the exponent and for any combinations of the 5 additional restrictions, we either show an algorithm or prove that no such algorithm exists (under standard complexity assumption).

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For any combination of the 2×10 parameters in the multiplier and the exponent and for any combinations of the 5 additional restrictions, we either show an algorithm or prove that no such algorithm exists (under standard complexity assumption).

Every question in this framework is completly answered by

- 11 positive results and
- 17 negative results.

Results

Short Description	Thm	*H*										*G*								
		$\lvert V(\cdot)\rvert$	cc	Δ	fvs	pw	tw	cw	genus	hadw	hadw$_T$	cc	Δ	fvs	pw	tw	cw	genus	hadw	hadw$_T$
FO model checking	Thm P.1 (page 17)	M															M			
	Thm P.2 (page 17)	M																		M
Color coding	Thm P.3 (page 17)	M					E													
Matoušek-Thomas	Thm P.4 (page 18)		M	M												E				
Paths&Cycles → Paths&Cycles	Thm P.5 (page 18)											E	2							
Dynamic Programming	Thm P.6 (page 19)		E	2												M				
	Thm P.7 (page 21)		E	2													E			
	Thm P.8⋆ (page 23)		M													1				
FVS and CSPs	Thm P.9⋆ (page 28)		M	2									M	M						
	Thm P.10⋆ (page 36)		E										M	M				E		
	Thm P.11⋆ (page 46)		E	E									M	M					E	
Bin Packing	Thm N.1 (page 46)											M	2			1				
	Thm N.2 (page 47)		1				1							E	E			0		
	Thm N.3 (page 47)			2								1	3		E	1				
Planar cubic HamPath	Thm N.4 (page 48)		1	2			1						3					0		
Clique	Thm N.5 (page 48)	M	1					E												
HamPath in bounded **cw**	Thm N.6 (page 48)		1	2			1										M			
Grid Tiling, 1-in-n gadgets	Thm N.7⋆ (page 57)		M			E	1					1	3	M	M			0		
	Thm N.8⋆ (page 61)		1			E	1						M	M	M			M	E	
	Thm N.9⋆ (page 61)		1			E	1						3	M	M			M		
	Thm N.10⋆ (page 63)		1	3		E	1						M	M	M		E	M		
Grid Tiling, moustache gadgets	Thm N.11⋆ (page 64)		1	3		E	1							M	M			0		
	Thm N.12⋆ (page 66)		1			E	1						3		M			0		
Small planar graph	Thm N.13⋆ (page 67)	M	1	3					0											
Exact Planar Arc Supply	Thm N.14⋆ (page 74)		M	2			1					1		M	M			0		
	Thm N.15⋆ (page 77)		M	2			1					1	3		M			0		
	Thm N.16⋆ (page 79)		M	2			1					1		M	M		E	M		
	Thm N.17⋆ (page 79)		M	2			1					1	M		M		E	M		

Figure 1: Positive and negative results in the paper. Results marked with ⋆ are new findings that were not known before.

Comparing specifications

Finding an algorithm satisfying a specification does not become any easier if we

- remove a parameter,
- move a parameter from the exponent to the multiplier,
- remove a constraint,
- adding a parameter to the multiplier or the exponent whose value is already bounded by a constraint, or
- adding a parameter to the multiplier (resp., exponent) whose value can be bounded by a computable function of the parameters already in the multiplier (resp., exponent) on instances where all the constraints in the description hold.

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Claim

For any specification, the 11 positive and 17 negative results imply a positive or negative answer using these rules.

Can be verified by a computer program.

Finding a tree in a tree

Fact

SUBGRAPH ISOMORPHISM is polynomial-time solvable if both H and G are trees.

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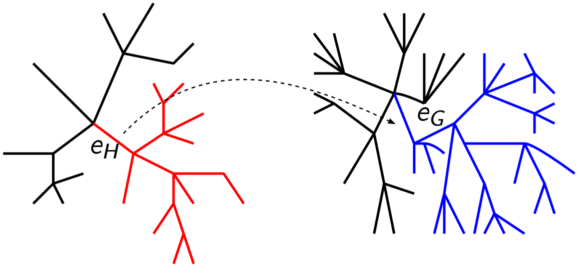
Dynamic programming: for every edge-defined subtree $T_H \subseteq H$ and $T_G \subseteq G$, we let $x(T_H, T_G) = \text{true}$ if there is a subgraph isomorphism from T_H to T_G matching the root edges.



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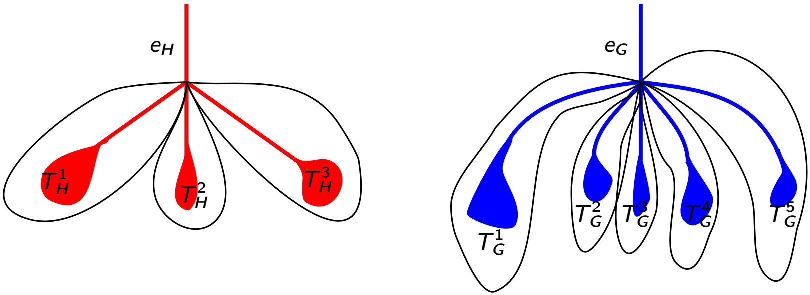
We need to match the children trees of T_H to the children trees of T_G: we need to solve a bipartite matching problem.



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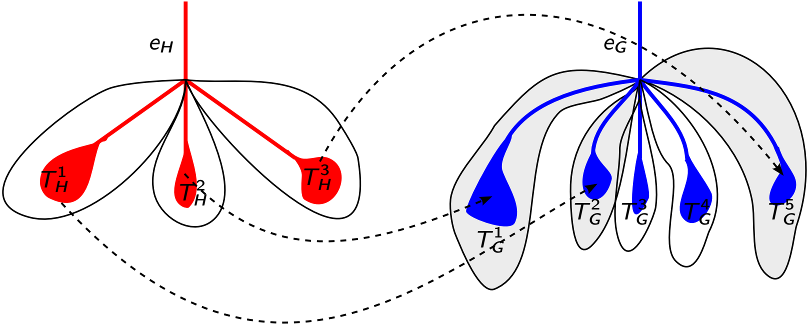

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Positive result P.8

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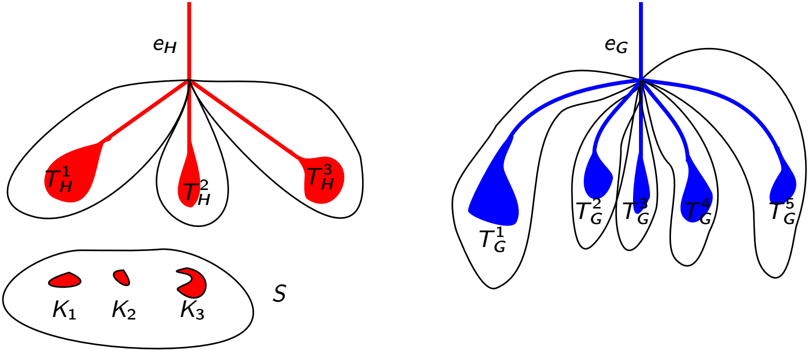

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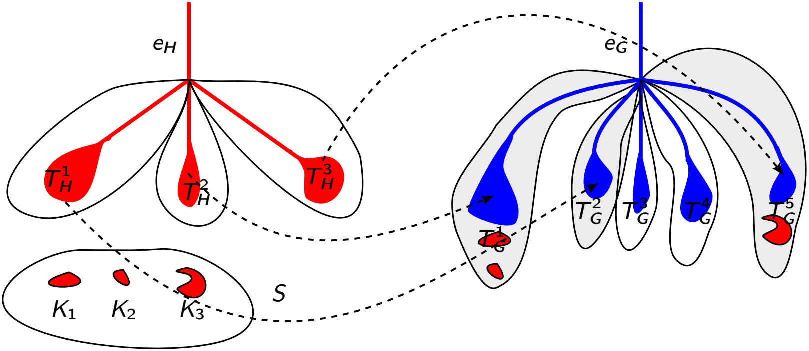

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Fact [Mulmuley, Vazirani, Vazirani 1987]

Given a bipartite (multi)graph B with nonnegative integer weights and a target weight w, there is a randomized algorithm for finding a perfect matching of weight **exactly** w in time polynomial in $|B|$ and w.

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We can solve our colored matching problem in time $f(\ell) \cdot n^{O(1)}$:

- Replace each edge of label i with two parallel edges of weight 0 and $2^{i-1} + 2^{2\ell-i}$.
- Find a perfect matching of weight exactly $w = \sum_{i=1}^{\ell}(2^{i-1} + 2^{2\ell-i})$.

— intermission —

Constraint satisfaction problems

We define a Constraint Satisfaction Problem (CSP) by

- a domain D of values,
- a set V of variables, and
- a set of constraints, where a constraint is a binary relation on two variables.

Examples:

- 3-Coloring of G is a CSP with $|D| = 3$ and $V = V(G)$,
- k-Coloring of G is a CSP with $D = V(G)$ and $|V| = k$.

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Primal graph: vertex set is V, there is an edge between u and v if there is a constraint on u and v.

Fact [Freuder 1990]

A CSP instance with primal graph G can be solved in time $n^{O(\mathrm{tw}(G))}$.

Projections

Projection graph: vertex set is V, there is an edge $\overrightarrow{uv}$ if the constraint on u and v is a projection from u to v.

- a **projection source** makes the problem easy to solve.
- a **projection sink** is useless in general.

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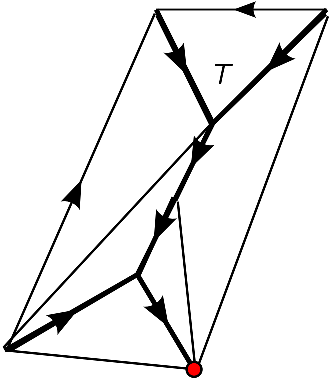
Fact

We can solve in polynomial time a CSP instance if its primal graph is planar and has a projection sink.

Projections

- There is a spanning in-tree T rooted at the projection sink.



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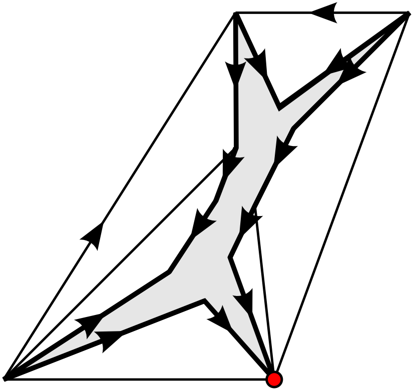

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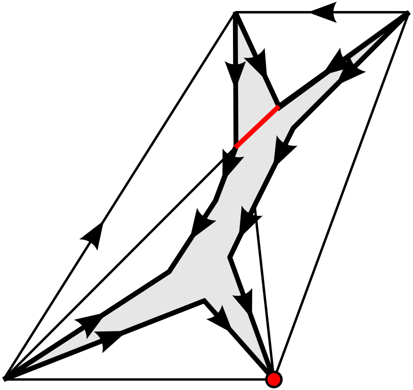

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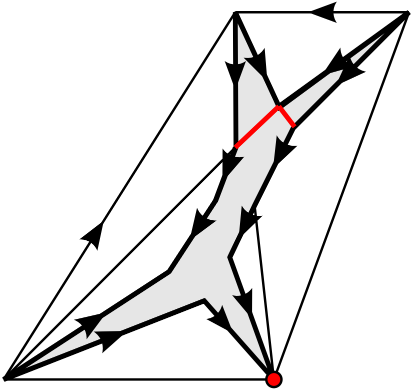

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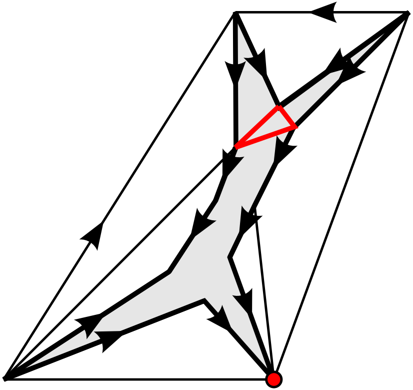

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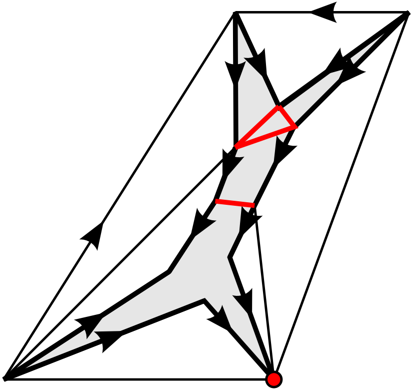

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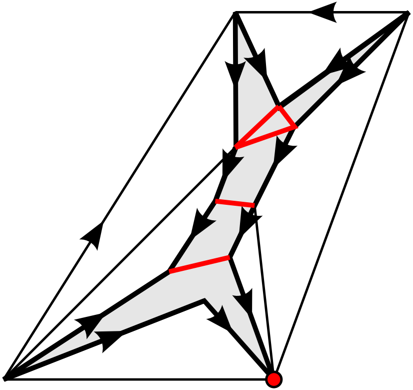

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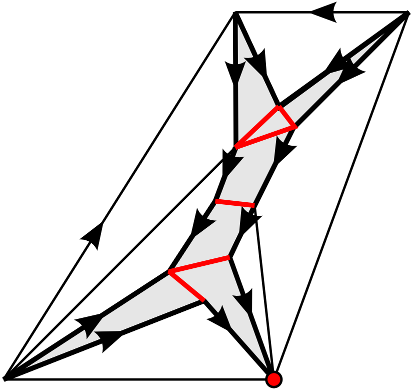

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- There is a spanning in-tree T rooted at the projection sink.
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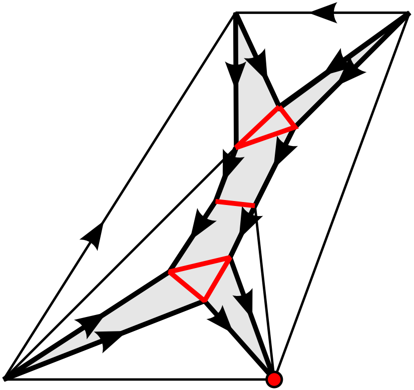
- Resulting primal graph is outerplanar $\Rightarrow$ has treewidth $\leq 3 \Rightarrow$ polynomial-time solvable.



— end of intermission —

Feedback vertex set number

Positive result

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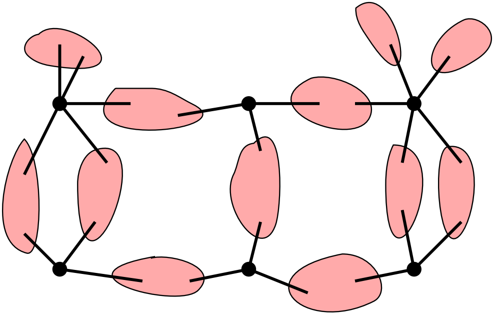
Fact

There is a set Z of $O(\Delta^2(G) \cdot \mathrm{fvs}(G))$ vertices such that every component of $G \setminus Z$ is a tree and has at most two edges to Z.



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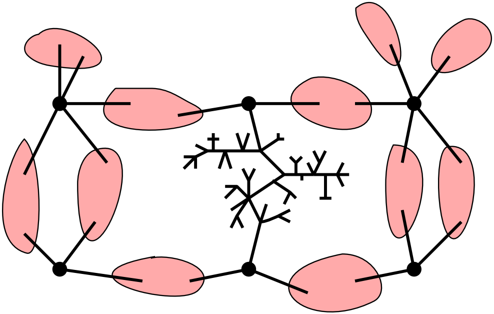
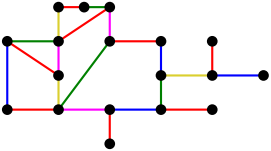
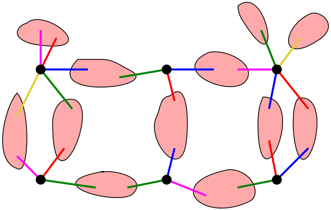


Feedback vertex set number

- Let us guess which subset of edges indicident to Z is used by the solution.
- Let us fix an edge coloring of H and let us guess the correct edge coloring of the edges incident to Z.



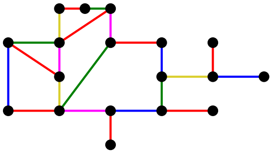
H



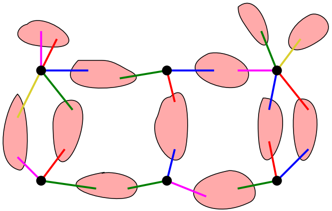
G

Feedback vertex set number

- We formulate finding the subgraph isomorphism $\phi : V(H) \to V(G)$ as a CSP problem: the variables correspond to the vertices in Z and the value of a $u \in Z$ is the preimage $\phi^{-1}(u)$.



H



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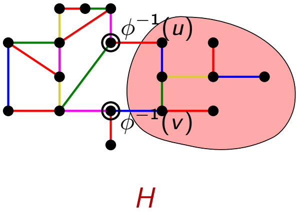
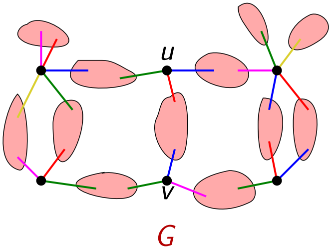
- If $G \setminus Z$ has a component adjacent to $u, v \in Z$, then this forces a constraint on $\phi^{-1}(u)$ and $\phi^{-1}(v)$.





Feedback vertex set number

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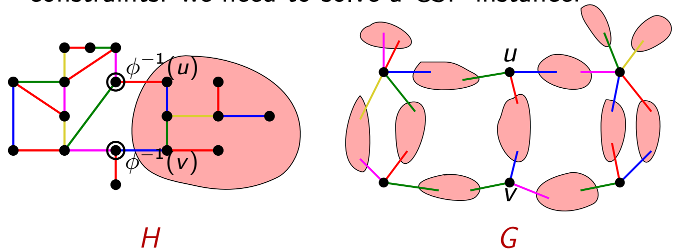
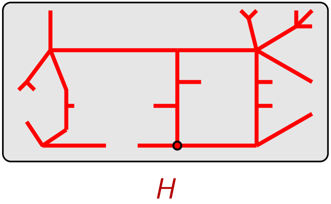
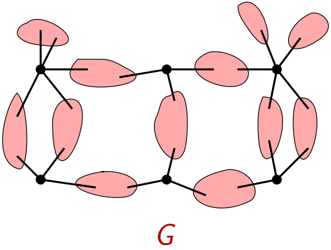
- If $G \setminus Z$ has a component adjacent to $u, v \in Z$, then this forces a constraint on $\phi^{-1}(u)$ and $\phi^{-1}(v)$.
- After taking care of some technicalities, these are the only constraints: we need to solve a CSP instance.



Feedback vertex set number

- Fix a spanning tree in H and guess how the tree goes via the components of $G \setminus Z$ (which components are traversed).





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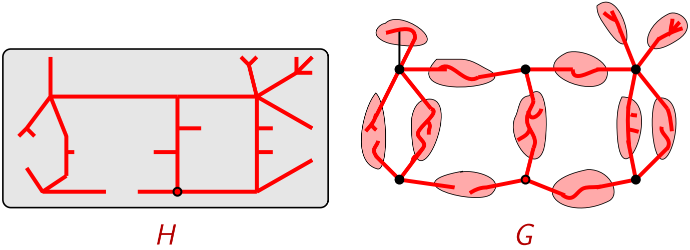
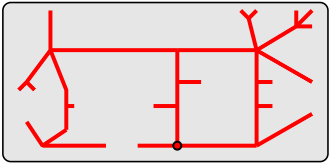
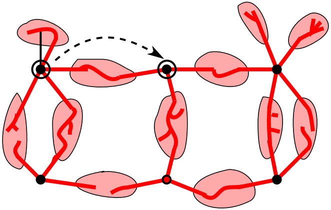


Feedback vertex set number

- Fix a spanning tree in H and guess how the tree goes via the components of $G \setminus Z$ (which components are traversed).
- This forces some constraints to be projections, in fact, creates a projection sink.
- As G is planar, the primal graph is planar and the CSP instance can be solved in polynomial time.



H



G

Feedback vertex set number

Positive result

SUBGRAPH ISOMORPHISM can be solved in time $f(\mathrm{fvs}(G), \Delta(G)) \cdot n^{O(1)}$ if G is planar and H is connected.

Feedback vertex set number

Positive result

Subgraph Isomorphism can be solved in time $f(\mathrm{fvs}(G), \Delta(G)) \cdot n^{O(1)}$ if G is planar and H is connected.

Can be generalized:

Positive result P.10

Subgraph Isomorphism can be solved in time $f_1(\mathrm{fvs}(G), \Delta(G)) \cdot n^{f_2(\mathrm{genus}(G), \mathrm{cc}(H))}$.

Positive result P.11

Subgraph Isomorphism can be solved in time $f_1(\mathrm{fvs}(G), \Delta(G)) \cdot n^{f_2(\mathrm{hadw}(G), \mathrm{cc}(H), \Delta(H))}$.

Graph Structure Theorem

Decomposing H-minor-free graphs into almost embeddable parts:

Theorem [Robertson-Seymour]

For every graph H, there is an integer k and a surface Σ such that every H-minor-free graph

- can be built by clique sums from graphs that are k-almost embeddable in Σ,

(or equivalently)

- has a tree decomposition where every torso is k-almost embeddable in Σ.

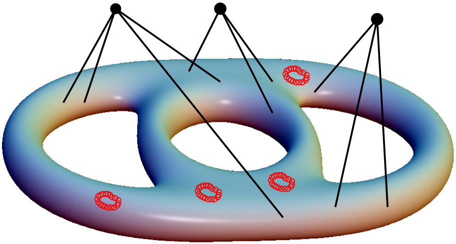
Originally stated only combinatorially, algorithmic versions are known.

k-almost embeddable

Definition

Graph G is k**-almost embeddable** in surface Σ if

- there is a set X of at most k apex vertices and
- a graph G_0 embedded in Σ, such that
- $G \setminus X$ can be obtained from G_0 by attaching vortices of width k on disjoint disks $D_1, \ldots, D_k$.



Projection sinks

Fact

We can solve in polynomial time a CSP instance if its primal graph G is planar and has a projection sink.

Straightforward generalization:

Fact

We can solve in time $n^{f(\text{genus}(G))}$ a CSP instance if its primal graph G has a projection sink.

Projection sinks

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Straightforward generalization:

Fact

We can solve in time $n^{f(\text{genus}(G))}$ a CSP instance if its primal graph G has a projection sink.

By using the Robertson-Seymour structure theorem and carefully handling vortices and cliques sums, we get the following generalization:

Fact

We can solve in time $n^{f(\text{hadw}(G))}$ a CSP instance if its primal graph G has a projection sink.

Hardness proofs

To prove that there is no algorithm for Subgraph Isomorphism with running time

$$n^{f(p_1,\ldots,p_t)},$$

we should show that Subgraph Isomorphism is NP-hard even for instances where each of $p_1, \ldots, p_t$ is bounded by a universal constant.

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To prove that there is no algorithm for Subgraph Isomorphism with running time

$$f_1(p_1, p_2, \ldots, p_\ell) \cdot n^{f_2(p_{\ell+1},\ldots,p_t)},$$

we use the fact that there is no $f(k) \cdot n^{O(1)}$ algorithm for k-Clique unless $\mathrm{FPT} = \mathrm{W}[1]$. Then we need a reduction from k-Clique to Subgraph Isomorphism such that

- each of $p_1, \ldots, p_\ell$ is bounded by a function of k, and
- each of $p_{\ell+1}, \ldots, p_t$ is bounded by a universal constant.

Grid Tiling

Grid Tiling

Input: A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

Find: A pair $s_{i,j} \in S_{i,j}$ for each cell such that

- Horizontal neighbors agree in the first component.
- Vertical neighbors agree in the second component.

(1,1) (1,3) (4,2)	(1,5) (4,1) (3,5)	(1,1) (4,2) (3,3)
(2,2) (4,1)	(1,3) (2,1)	(2,2) (3,2)
(3,1) (3,2) (3,3)	(1,1) (3,1)	(3,2) (3,5)

$k = 3, \ D = 5$

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$k = 3,\ D = 5$

Grid Tiling

Reduction from k-Clique

Definition of the sets:

- For $i = j$: $(x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $(x, y) \in S_{i,j} \iff x$ and y are adjacent.

	(v_i, v_i)			

Each diagonal cell defines a value v_i . . .

Grid Tiling

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	$(., v_i)$			
$(v_i, .)$	(v_i, v_i)	$(v_i, .)$	$(v_i, .)$	$(v_i, .)$
	$(., v_i)$			
	$(., v_i)$			
	$(., v_i)$			

... which appears on a "cross"

Grid Tiling

Reduction from k-CLIQUE

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	$(., v_i)$			
$(v_i, .)$	(v_i, v_i)	$(v_i, .)$	$(v_i, .)$	$(v_i, .)$
	$(., v_i)$			
	$(., v_i)$		(v_j, v_j)	
	$(., v_i)$			

v_i and v_j are adjacent for every $1 \leq i < j \leq k$.

Grid Tiling

Reduction from k-Clique

Definition of the sets:

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	$(., v_i)$		$(., v_j)$	
$(v_i, .)$	(v_i, v_i)	$(v_i, .)$	(v_i, v_j)	$(v_i, .)$
	$(., v_i)$		$(., v_j)$	
$(v_j, .)$	(v_j, v_i)	$(v_j, .)$	(v_j, v_j)	$(v_j, .)$
	$(., v_i)$		$(., v_j)$	

v_i and v_j are adjacent for every $1 \leq i < j \leq k$.

Hardness proofs

Negative result N.7

Unless $\mathrm{FPT} = \mathrm{W}[1]$, there is no algorithm for SUBGRAPH ISOMORPHISM with running time

$$f_1(\mathrm{cc}(H), \mathrm{pw}(G), \mathrm{fvs}(G)) \cdot n^{f_2(\mathrm{pw}(H))},$$

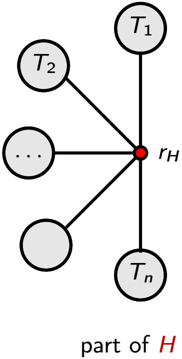
even if H is a forest and G is a connected planar graph with maximum degree 3.

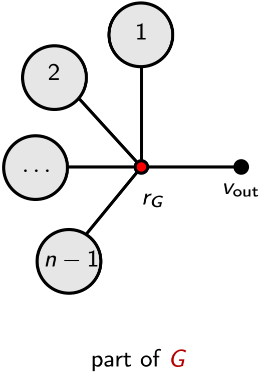
We need to reduce $k \times k$ GRID TILING to an instance of SUBGRAPH ISOMORPHISM where

- $\mathrm{cc}(H)$, $\mathrm{pw}(G)$, $\mathrm{fvs}(G)$ is bounded by a function of k,
- $\mathrm{pw}(H)$ is bounded by a universal constant,
- H is a forest,
- G is a connected planar graph with maximum degree 3.

Gadget construction

Consider the following subgraphs in H and G:





If a subgraph isomorphism maps special vertex r_H to r_G, then one of the tree T_i protrudes out at vertex v_{out}.

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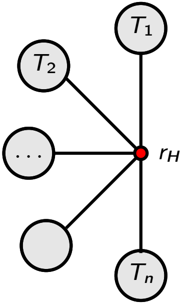
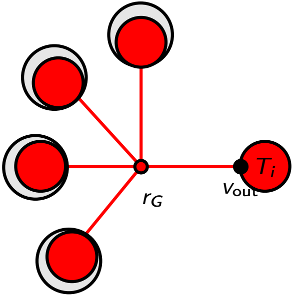


part of H



part of G

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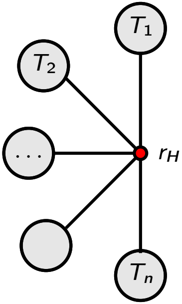
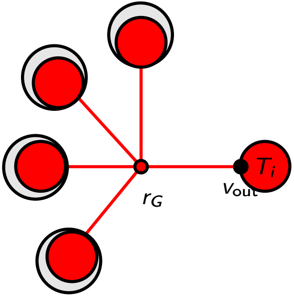


part of H



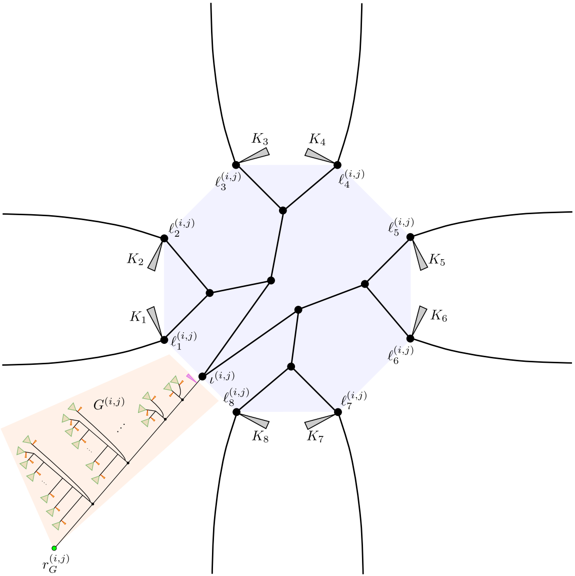
part of G

If a subgraph isomorphism maps special vertex r_H to r_G, then one of the tree T_i protrudes out at vertex v_{out}.

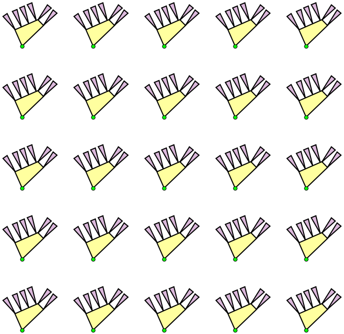
Slightly more challenging: construct such gadgets where H has bounded degree and bounded pathwidth.

Hardness proofs

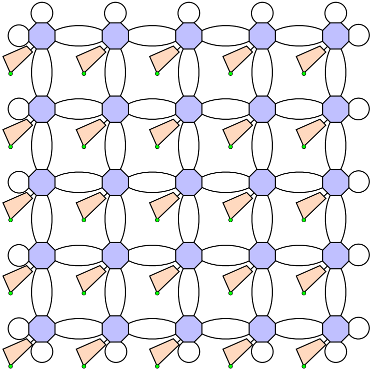
Generalizing it to a gadget where 8 paths of certain lengths protrude out:



Hardness proofs



Graph *H*



Graph *G*

Hardness proofs

Negative result N.7

Unless $\mathsf{FPT} = \mathsf{W}[1]$, there is no algorithm for Subgraph Isomorphism with running time

$$f_1(\mathrm{cc}(H), \mathrm{pw}(G), \mathrm{fvs}(G)) \cdot n^{f_2(\mathrm{pw}(H))},$$

even if H is a forest and G is a connected planar graph with maximum degree 3.

Hardness proofs

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A variant of the result:

Negative result N.8

Unless FPT = W[1], there is no algorithm for Subgraph Isomorphism with running time

$$f_1(\cancel{\mathrm{cc}(H)}, \mathrm{pw}(G), \mathrm{fvs}(G), \mathrm{genus}(G), \Delta(G)) \cdot n^{f_2(\mathrm{pw}(H), \mathrm{hadw}(G))},$$

even if H is a ~~forest~~ tree and ~~G is a connected planar graph with maximum degree 3~~.

Essentially: add a new vertex to H and G to make them connected.

Hardness proofs

Negative result N.7

Unless $\mathsf{FPT} = \mathsf{W}[1]$, there is no algorithm for SUBGRAPH ISOMORPHISM with running time

$$f_1(\mathrm{cc}(H), \mathrm{pw}(G), \mathrm{fvs}(G)) \cdot n^{f_2(\mathrm{pw}(H))},$$

even if H is a forest and G is a connected planar graph with maximum degree 3.

Hardness proofs

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A variant of the result:

Negative result N.9

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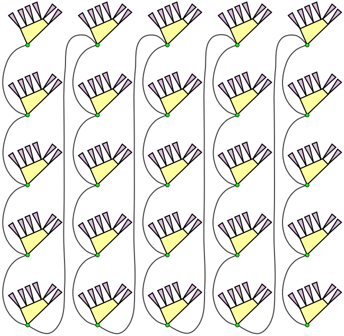
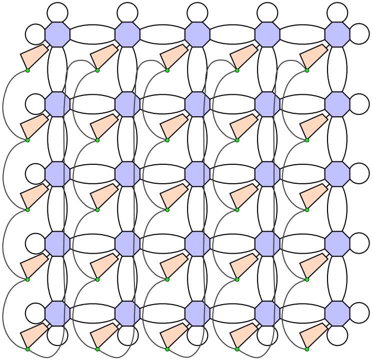
even if H is a ~~forest~~ tree and G is a connected ~~planar~~ graph with maximum degree 3.

Essentially: connecting the gadgets in a path like manner.

Hardness proofs



Graph H



Graph G

Summary

- We formulated a framework with 2×10 parameters and 5 constraints.
- We showed that 11 positive results and 17 negative results (some known, some new) answer every question in this framework.
- We developed a computer program to check for complete coverage and to find the questions that are not yet explained by the results.
- Some interesting new positive results and very careful and nontrivial hardness proofs.
- Full paper and program on arxiv.