A PTAS for Planar Group Steiner Tree via Spanner Bootstrapping and Prize Collecting

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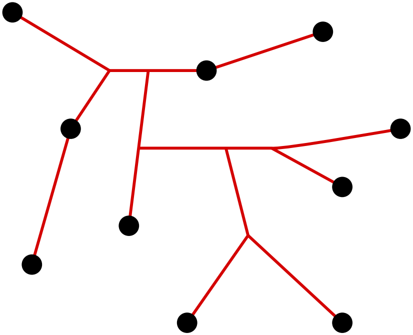
Steiner Tree

Steiner Tree

Given an edge-weighted graph G and set $T \subseteq V(G)$ of terminals, find a minimum-weight tree in G containing every vertex of T.



Steiner Tree

Known results:

- APX-hard on general graphs.
- 1.386-approximation on general graphs [Byrka et al. 2010].
- PTAS on planar graphs [Borradaile et al. 2009].

Generalizations:

- Steiner Forest:
  connect given pairs (s_i, t_i).
- Directed Steiner Tree:
  connection from the root to every terminal.
- Strongly Connected Steiner Subgraph:
  connect $t_i \to t_j$ for every i, j.
- Group Steiner Tree:
  reach one vertex of each group (this talk)

Group Steiner Tree

Group Steiner Tree

Given an edge-weighted graph G and sets $T_1, \ldots, T_k \subseteq V(G)$ of terminals, find a minimum-weight tree in G containing at least one vertex from each T_i.

- Best approximation for general graphs: $O(\log^3 n)$ [Garg et al. 2000]
- Best approximation for trees: $O(\log^2 n)$ [Garg et al. 2000]
- No $O(\log^{2-\epsilon})$-approximation for trees, unless NP admits quasipolynomial-time Las Vegas algorithms [Halperin and Krauthgamer 2003].

Group Steiner Tree

Group Steiner Tree
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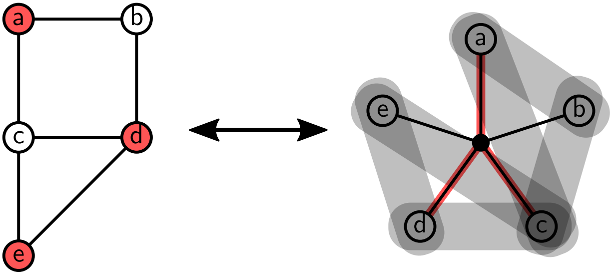
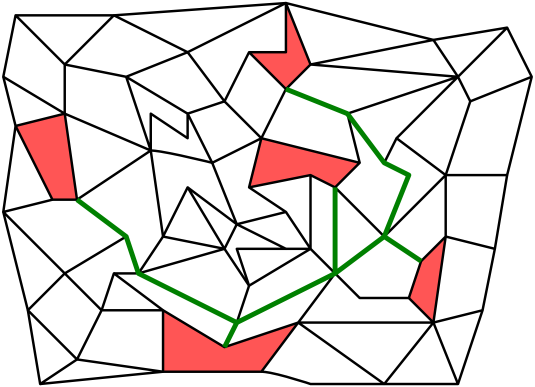
Problem is APX-hard even on trees:



Vertex Cover | Group Steiner Tree

Planar Group Steiner Tree

Variant where each group corresponds to the vertices of one face:



Main result

Planar Group Steiner Tree admits an EPTAS:
a $(1+\epsilon)$-approximation can be obtained in time $f(1/\epsilon)n^{O(1)}$.

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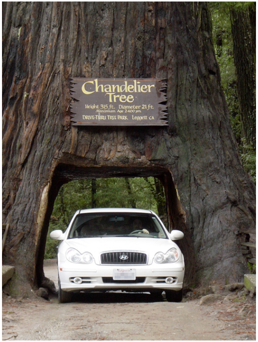
a $(1+\epsilon)$-approximation can be obtained in time $f(1/\epsilon)n^{O(1)}$.

Rest of the talk:

- Quick overview of the main conceptual steps of a framework that has been used for various planar PTASs, including STEINER TREE.
- Highlighting the two new conceptual steps that we introduce for PLANAR GROUP STEINER TREE.

Bounded treewidth

- Using standard dynamic programming techniques, an optimal solution for Steiner Tree on graphs of treewidth w can be found in time $2^{O(w \log w)} \cdot n^{O(1)}$.
- Recent advances improved the running time to $2^{O(w)} \cdot n^{O(1)}$ [Cygan et al. 2011] [Bodlaender et al. 2013] [Fomin et al. 2014]
- Can be extended to Planar Group Steiner Tree.



We need that the input graph G has bounded treewidth!

Shifting strategy

Theorem [Klein 2008]

Given an edge-weighted planar graph G and an $\epsilon > 0$, we can find in polynomial time a set F of edges such that $w(F) \leq \epsilon w(G)$ and G/F has treewidth $O(1/\epsilon)$.

- We "buy" F by contracting it in G and putting it into a solution.
- We can solve the problem on the graph G/F of treewidth $O(1/\epsilon)$ optimally $\Rightarrow$ $\mathrm{OPT} + \epsilon w(G)$ solution
- Gives an additive $\epsilon w(G)$ approximation in time $2^{O(1/\epsilon)} \cdot n^{O(1)}$.

We need that the input graph G itself
is a constant-factor approximation of the optimum!

Spanner construction

Main part of the STEINER TREE PTAS of [Borradaile et al. 2009]:

Spanner construction

Given an initial solution L, we can extend it to L' such that

1. $w(L') \leq f(1/\epsilon) \cdot w(L)$ and
2. If there is a tree X of G containing some terminals T on L, then there is a tree $X' \subseteq L'$ also containing T with $w(X') \leq w(X) + \epsilon w(L)$.

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If L is a constant-factor approximation of the solution, then

1. Graph L' is also a constant-factor approximation.
2. Restriction to L' introduces only $O(\epsilon \mathrm{OPT})$ additive error.

We need that the initial solution L
is a constant-factor approximation!

$O(1)$ Constant-factor approximation

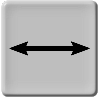
How to get a constant-factor approximation for Steiner Tree?

- Easy 2-approximation: use a minimum spanning tree.
- 1.386-approximation on general graphs [Byrka et al. 2010].

PTAS for Steiner Tree

$O(1)$ Constant-factor approximation in polynomial time.


Construction of the spanner.


Reduction to bounded treewidth
with the shifting strategy


Solving the bounded-treewidth instances
using dynamic programming.

PTAS for Planar Group Steiner Tree

Spanner bootstrapping

Reaching the relevant terminals using prize collecting.

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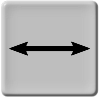

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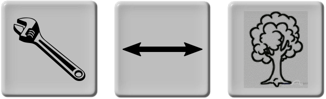
Spanner bootstrapping

First problem:

No $O(1)$-approximation is known for PLANAR GROUP STEINER.



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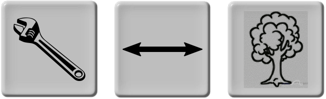

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Bootstrapping:

- Given a c-approximation L for large c, we get a solution with approximation ratio $(1 + \epsilon c) \le c/2$.
- Given a $c/2$-approximation $\Rightarrow$ we get ratio $c/4$.
- Given a $c/4$-approximation $\Rightarrow$ we get ratio $c/8$.
- ...

Starting from a trivial $O(n)$-approximation, we need to repeat this $O(\log n)$ times to get a $O(1)$-approximation!

Reaching relevant terminals

Second problem:

The spanner construction of [Borradaile et al. 2009] considers only terminals that are already on the initial solution:

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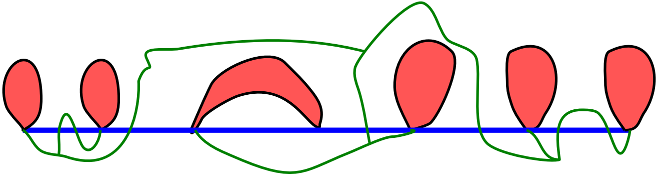

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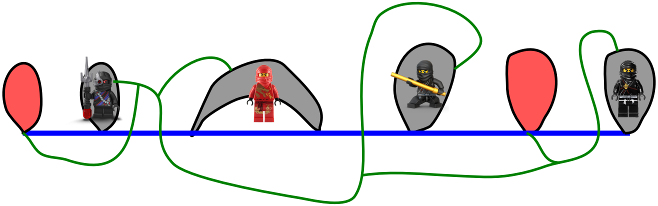
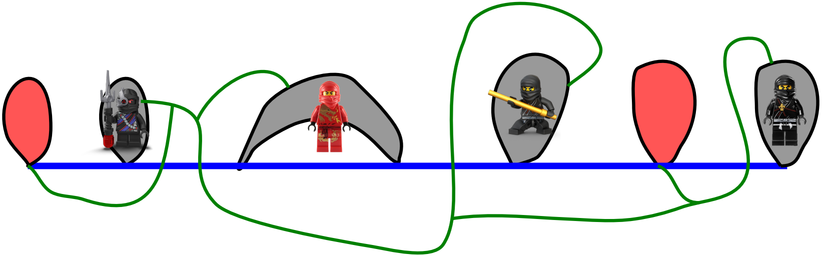
The solution may reach terminals that are not on spanner:





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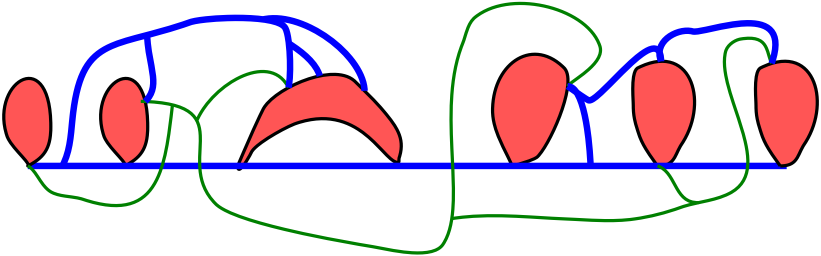

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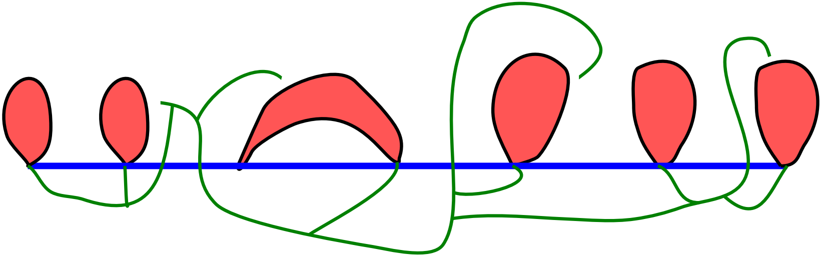
1. **Extend** the spanner to reach the terminals used by the solution.



There are two potential ways to solve the problem:

1. **Extend** the spanner to reach the terminals used by the solution.
2. **Fix** the groups solution by connecting the solution to their terminals on the spanner.

Neither solution is likely to work in general...

Prize collecting

Our goal is to extend the initial solution L to L' such that the terminals on L' are sufficient for a $(1+\epsilon)$-approximate solution.

- Assign a potential $\pi(g)$ to each group g such that
  - total potential is $O(w(L))$ and
  - if X is a solution that reaches the "bad" terminals of the groups in S, then the groups in S can be **fixed** at cost $\pi(S)$.

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- **Extend** L by "cheap" trees: the cost of the tree is not much larger than the potential of the groups it reaches.

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- **Extend** L by "cheap" trees: the cost of the tree is not much larger than the potential of the groups it reaches.
- Suppose that the solution has a subtree F that reaches the "bad" terminals of the groups S.
  - If $w(F)$ is small compared to $\pi(S)$: argue that F is cheap, we should have **extended** L with it.
  - If $w(F)$ is large compared to $\pi(S)$: **fix** S at cost $\pi(S)$ and charge it on F.

Prize collecting — a special case

Special case: each group has at most 2 terminals.

- We can define a **submodular** potential function π with total potential $w(L) \cdot O(\log n)$.
- We can use the submodular prize-collecting procedure of [Bateni et al. SODA 2011] to collect cheap trees.

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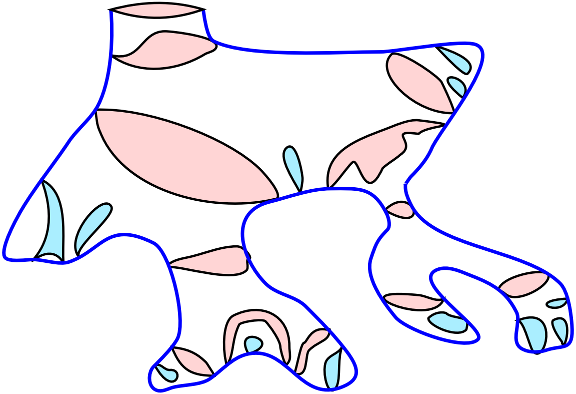
We get a spanner L' of weight $f(1/\epsilon) \cdot \log n \cdot \mathrm{OPT}$
$\Rightarrow$ we want additive error $\epsilon' w(L')$ for $\epsilon' := \epsilon/(f(1/\epsilon)\log n)$
$\Rightarrow$ running time is $2^{O(1/\epsilon')} \cdot n^{O(1)} = n^{f(1/\epsilon)}$.

Theorem

Planar Group Steiner Tree admits a $n^{f(1/\epsilon)}$ time PTAS if every group has only two terminals.

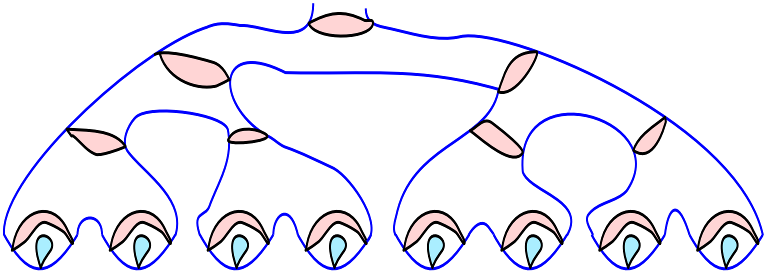
Minimal vs. nonminimal

- We may assume that L is path
  (standard trick of cutting open a tree).
- Two types of groups: **minimal** and **nonminimal**.



Which type is easier to handle?

Potential for nonminimal groups



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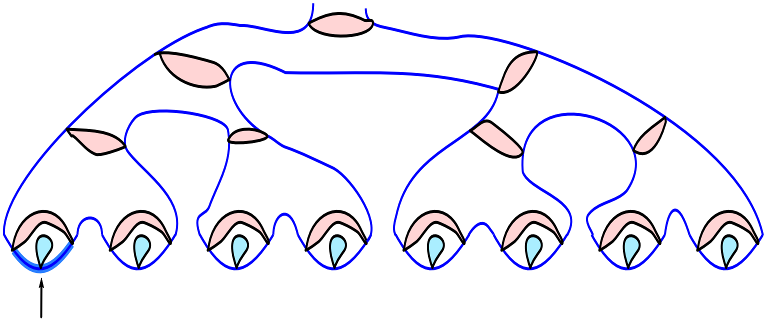

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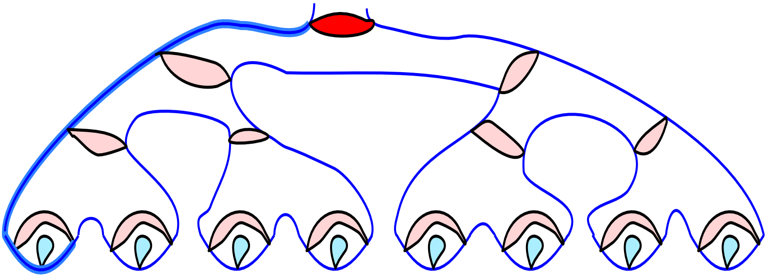

Key observation: a subpath of the initial solution L can be used to fix a nonminimal group.

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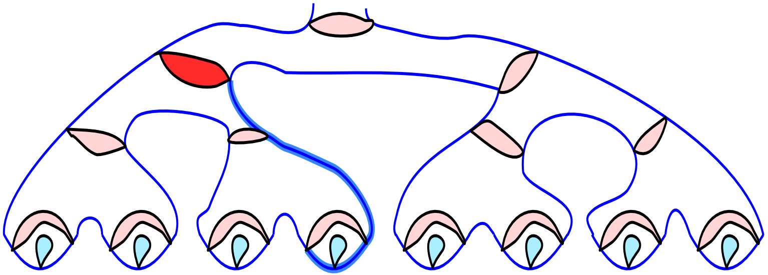

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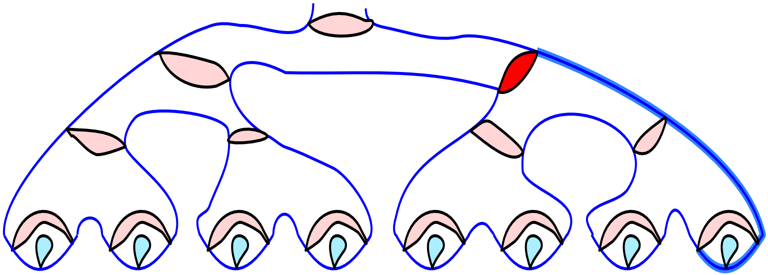

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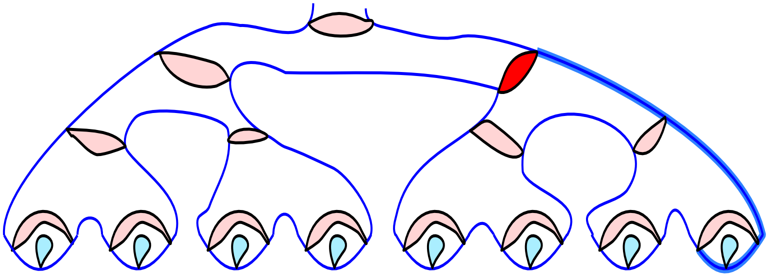

Key observation: a subpath of the initial solution L can be used to fix a nonminimal group.

We can assign a potential $\pi(g)$ to the nonminimal groups such that

- total potential is $O(w(L))$
- each nonminimal group g can be **fixed** with cost $\pi(g)$.

This is the beginning of a long journey...

PTAS for Planar Group Steiner Tree

Spanner bootstrapping

Reaching the relevant terminals using prize collecting.

Construction of the spanner.

Reduction to bounded treewidth
with the shifting strategy

Solving the bounded-treewidth instances
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Summary

Main result

PLANAR GROUP STEINER TREE admits an EPTAS:
a $(1+\epsilon)$-approximation can be obtained in time $f(1/\epsilon)n^{O(1)}$.

Open: PTAS for the following two generalizations?

- GROUP STEINER TREE on planar graphs where the groups are connected and disjoint.
- DIRECTED STEINER TREE on planar graphs.