# Recent Advances on the Complexity of Parameterized Counting Problems

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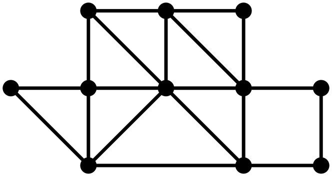

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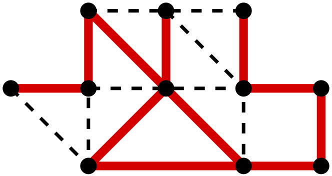

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# Finding vs. counting

**Finding** a perfect matching in a bipartite graph is polynomial-time solvable.

[Ford and Fulkerson 1956]

vs.

**Counting** the number of perfect matchings in a bipartite graph is #P-hard.

[Valiant 1979]

# This talk

Counting problems in the area of parameterized algorithms.

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- Connection between counting homomorphisms and subgraphs.
- Algorithmic applications.
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**Main message**

Parameterized subgraph counting problems can be understood via homomorphism counting problems.

...and this connection gives both algorithmic and complexity results!

# Parameterized problems

## Main idea

Instead of expressing the running time as a function $T(n)$ of $n$, we express it as a function $T(n,k)$ of the input size $n$ and some parameter $k$ of the input.

In other words: we do not want to be efficient on all inputs of size $n$, only for those where $k$ is small.

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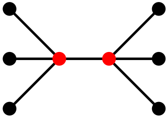
In other words: we do not want to be efficient on all inputs of size $n$, only for those where $k$ is small.

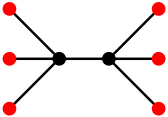
What can be the parameter $k$?

- The size $k$ of the solution we are looking for.
- The maximum degree of the input graph.
- The dimension of the point set in the input.
- The length of the strings in the input.
- The length of clauses in the input Boolean formula.
- ...

# Parameterized complexity

| | | |
|---|---|---|
| **Problem:** | VERTEX COVER | INDEPENDENT SET |
| **Input:** | Graph $G$, integer $k$ | Graph $G$, integer $k$ |
| **Question:** | Is it possible to cover the edges with $k$ vertices? | Is it possible to find $k$ independent vertices? |





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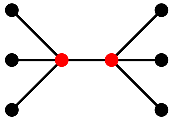
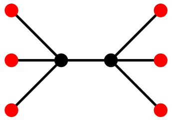
| |  |  |
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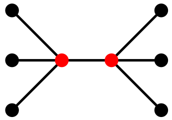

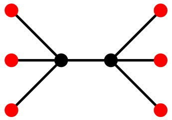

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$O(2^k n^2)$ **algorithm exists** ☺

**No** $n^{o(k)}$ **algorithm known** ☹

# Example: Bounded search tree method

Algorithm for VERTEX COVER:

$e_1 = u_1 v_1$

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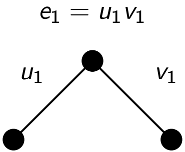

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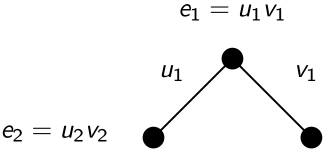

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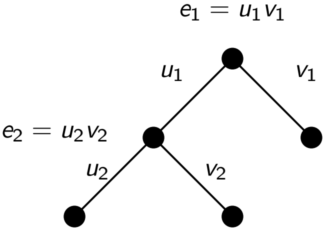

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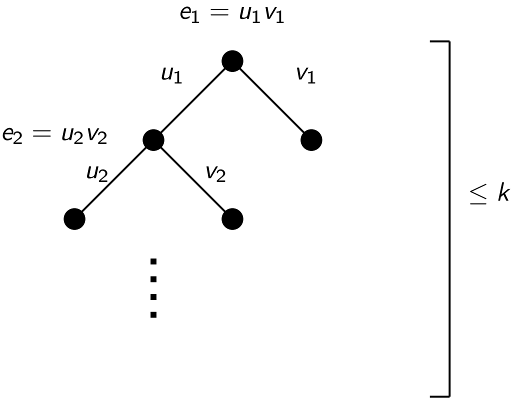


Height of the search tree $\leq k$ $\Rightarrow$ at most $2^k$ leaves $\Rightarrow$ $2^k \cdot n^{O(1)}$ time algorithm.

# Fixed-parameter tractability

**Main definition**

A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant $c$.

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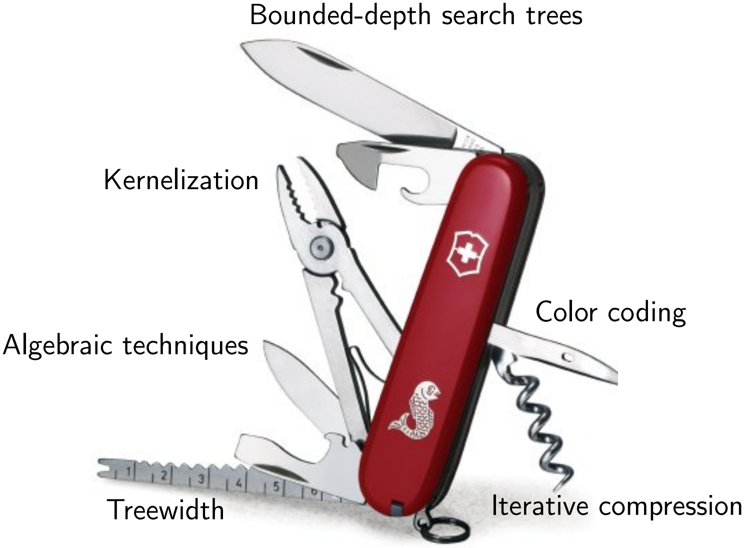
A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant $c$.

Examples of NP-hard problems that are FPT:

- Finding a vertex cover of size $k$.
- Finding a path of length $k$.
- Finding $k$ disjoint triangles.
- Drawing the graph in the plane with $k$ edge crossings.
- Finding disjoint paths that connect $k$ pairs of points.
- ...

# FPT techniques



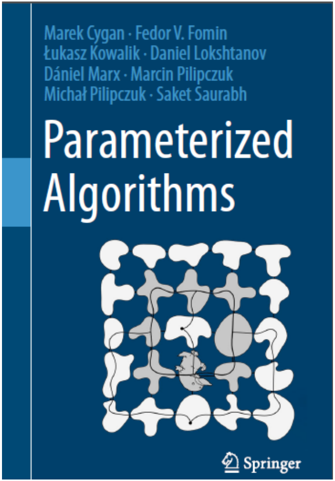
# W[1]-hardness

Negative evidence similar to NP-completeness. If a problem is **W[1]-hard,** then the problem is not FPT unless FPT=W[1].

Some W[1]-hard problems:

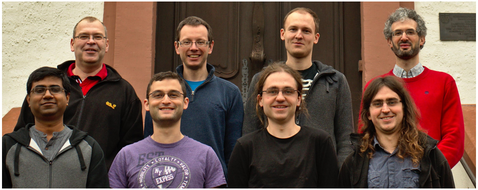
- Finding a clique/independent set of size $k$.
- Finding a dominating set of size $k$.
- Finding $k$ pairwise disjoint sets.
- ...

# Parameterized Algorithms




Marek Cygan, Fedor V. Fomin,
Lukasz Kowalik, Daniel Lokshtanov,
Dániel Marx, Marcin Pilipczuk,
Michał Pilipczuk, Saket Saurabh

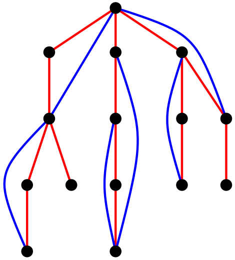
Springer 2015



# Example: Win/Win for $k$-Path

Simple $2^{O(k)} \cdot n^{O(1)}$ time algorithm for finding a path of length $k$.

1. Compute a DFS tree.



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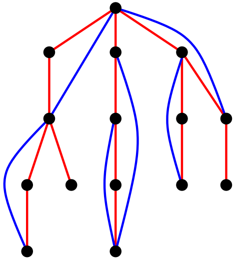
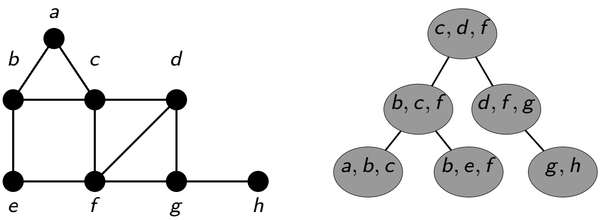


2. If DFS tree has height $> k$:
   we can find a $k$-path.
3. If DFS tree has height $\leq k$:
   treewidth is $\leq k \Rightarrow$ Use an algorithm with running time $2^{O(\text{tw})} \cdot n^{O(1)}$ for finding the longest path.

# Treewidth — a measure of "tree-likeness"

**Tree decomposition:** Vertices are arranged in a tree structure satisfying the following properties:

1. If $u$ and $v$ are neighbors, then there is a bag containing both of them.
2. For every $v$, the bags containing $v$ form a connected subtree.



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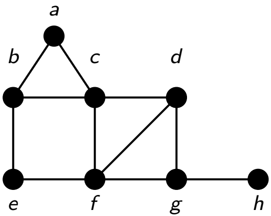


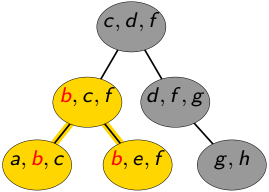

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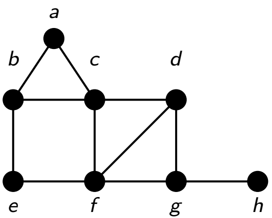


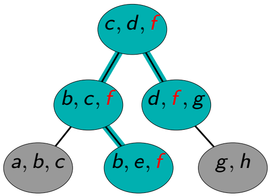

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**Width of the decomposition:** largest bag size $-1$.

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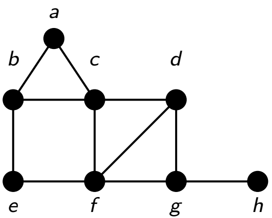


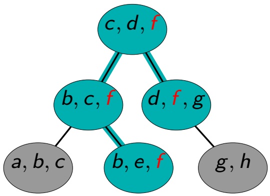

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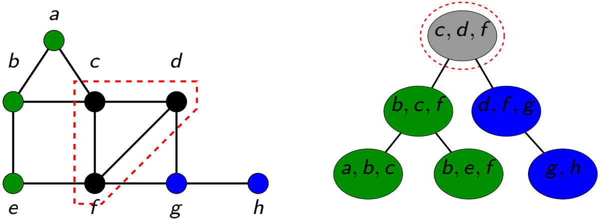


Each bag is a separator.

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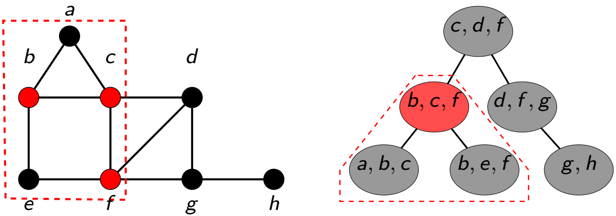


A subtree communicates with the outside world only via the root of the subtree.

# Counting complexity

- W[1]-hardness: "as hard as finding a $k$-clique"
- #W[1]-hardness: "as hard as counting $k$-cliques"

What can happen to the counting version of an FPT problem?

1. **(easy)** The same algorithmic technique shows that the counting problem is FPT.
2. **(easy, but different)** New algorithmic techniques are needed to show that the counting version is FPT.
3. **(hard)** New lower bound techniques are needed to show that the counting version is #W[1]-hard.

# Finding vs. counting

Generalization to counting:

- **WORKS** for the VERTEX COVER branching algorithm

  $\Rightarrow$ #VERTEX COVER is FPT.

- **DOES NOT WORK** for the #$k$-PATH win/win algorithm

  What is the parameterized complexity of #$k$-PATH?

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  What is the parameterized complexity of #$k$-PATH?

Even more troubling question:

What is the parameterized complexity of
the (even simpler) #$k$-MATCHING?

#$k$-MATCHING $\Rightarrow$ #$k$-PATH

# Counting $k$-paths and $k$-matchings

Colorful history:

- $\#k$-Path is #W[1]-hard
  [Flum and Grohe, FOCS 2002]

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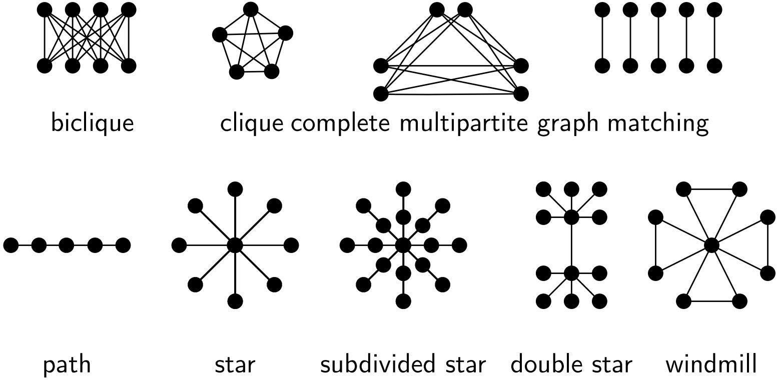
  [Curticapean, Dell, and M, STOC 2017] — *tells the real story.*

# Counting patterns

## Main question

Which type of subgraph patterns are easy to count?



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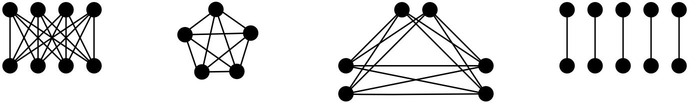
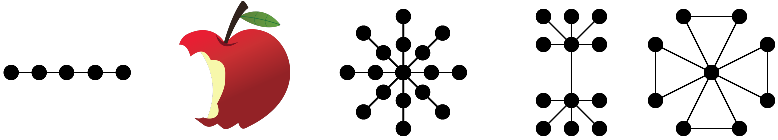


biclique clique complete multipartite graph matching



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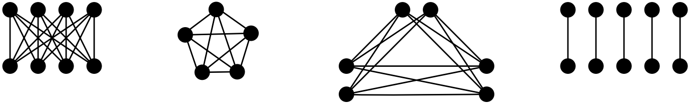

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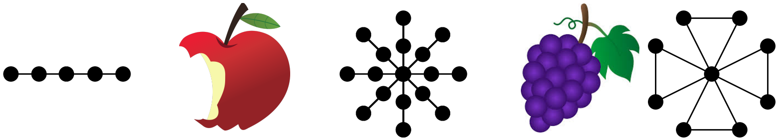

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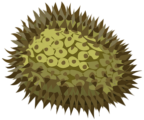
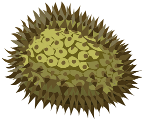
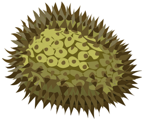
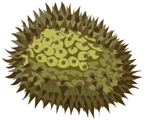

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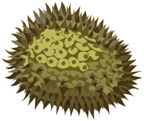

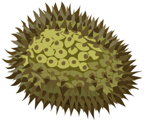
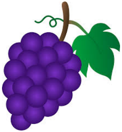
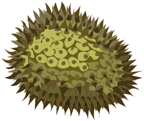

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# Counting patterns

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**Theorem** [multiple references]

$\#\textsc{Sub}(H)$ can be solved in time $n^{\mathsf{vc}(H)+O(1)}$.

But what about patterns with large vertex cover number?

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**Theorem** [multiple references]

$\#\textsc{Sub}(H)$ can be solved in time $n^{\mathrm{vc}(H)+O(1)}$.

But what about patterns with large vertex cover number?

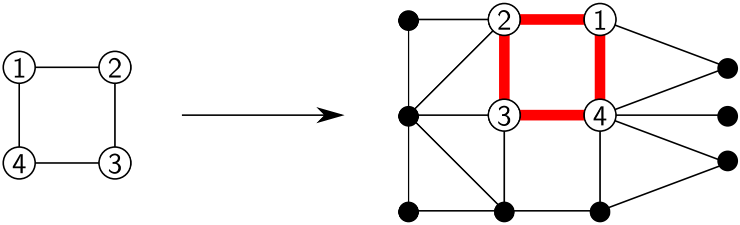
We will understand the complexity of counting any class of patterns, not just paths or matchings!

**Main message**

Parameterized subgraph counting problems can be understood via homomorphism counting problems.

# Homomorphisms

A **homomorphism** from $H$ to $G$ is a mapping $f\colon V(H) \to V(G)$ such that if $ab$ is an edge of $H$, then $f(a)f(b)$ is an edge of $G$.



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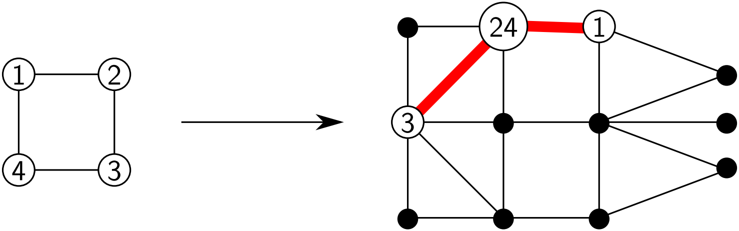

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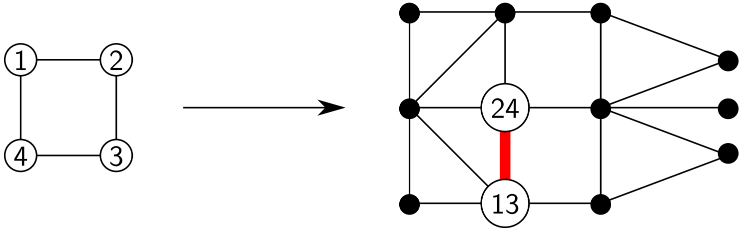

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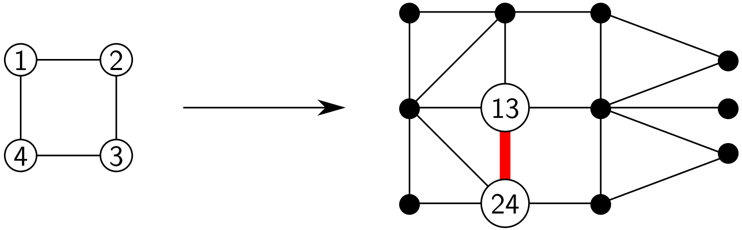

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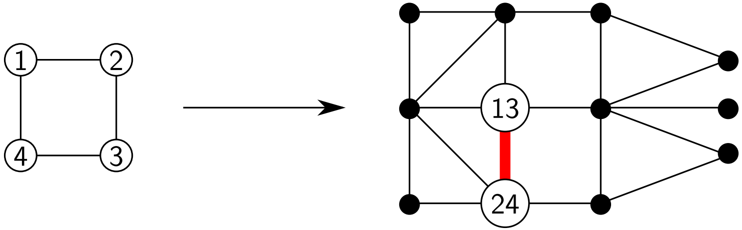


Which pattern graphs $H$ are easy for counting homomorphisms?

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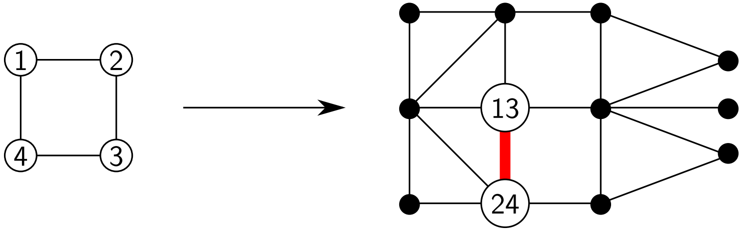


Which pattern graphs $H$ are easy for counting homomorphisms?

**Theorem (trivial)**

For every fixed $H$, the problem $\#\textsc{Hom}(H)$ (count homomorphisms from $H$ to the given graph $G$) is polynomial-time solvable.

... because we can try all $|V(G)|^{|V(H)|}$ possible mappings $f\colon V(H) \to V(G)$.

# Counting homomorphisms

**Better questions:**

- Which classes $\mathcal{H}$ (e.g., paths, stars, matchings) of patterns can be counted in polynomial time?
- What is the best exponent for $\textsc{Hom}(H)$?

**Fact**

$\#\textsc{Hom}(H)$ can be solved in time $O(n^{\mathrm{tw}(H)+1})$.

# Counting homomorphisms

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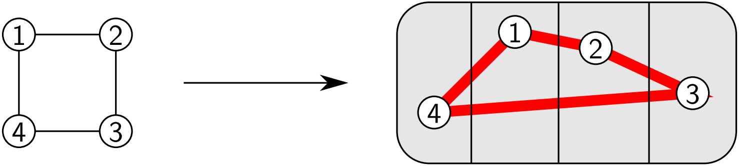
Difference between finding and counting:

| | | |
|---|---|---|
| **Finding:** $\textsc{Hom}(K_{k,k})$ is trivial | vs. | **Counting:** $\#\textsc{Hom}(K_{k,k})$ is W[1]-hard |

# Partitioned homomorphism

**PARTITIONED HOMOMORPHISM**
**Input:** $H$, $G$, and partition $\Pi$ of $V(G)$ into $|V(H)|$ classes.
**Task:** Find a homomorphism $\phi$ that maps the vertices of $H$ to different classes.




**Theorem** [M 2010]

There is a universal constant $\gamma > 0$ such that if for some $H$ there is an $O(n^{\gamma \cdot \mathrm{tw}(H)/\log \mathrm{tw}(H)})$ time algorithm for PARTHOM($H$), then ETH fails.

# Counting partitioned homomorphisms

$\#\textsc{PartHom}(H)$ $\Rightarrow$ $\#\textsc{Hom}(H)$

$G_P$ for $P \subseteq \mathrm{class}(\Pi)$: subgraph of $G$ induced by the classes $P$.

Simple application of the inclusion-exclusion principle:

$$\#\text{part-hom}(H, G) = \sum_{P \subseteq \mathrm{class}(\Pi)} (-1)^{|\mathrm{class}(P)| - |P|} \cdot \#\text{hom}(H, G_P)$$

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**Theorem** [Dalmau and Jonsson 2004][M 2010]

There is a universal constant $\gamma > 0$ such that if for some $H$ there is an $O(n^{\gamma \cdot \mathrm{tw}(H)/\log \mathrm{tw}(H)})$ time algorithm for $\#\textsc{Hom}(H)$, then ETH fails.

# Counting homomorphisms — summary

Treewidth of $H$ determines the complexity of the problem:

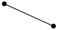
- $O(n^{\mathrm{tw}(H)+1})$ upper bound.
- $\Omega(n^{\gamma \cdot \mathrm{tw}(H)/\log \mathrm{tw}(H)})$ lower bound (assuming ETH).

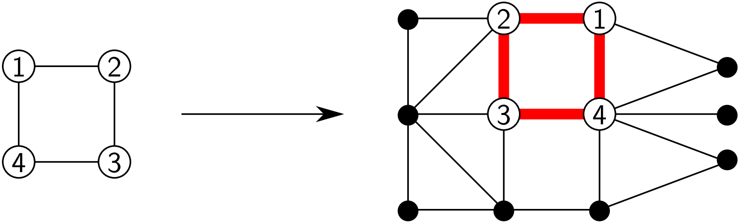
If we restrict the problem to a class $\mathcal{H}$ of patterns:

- If $\mathcal{H}$ has bounded treewidth (e.g, stars, paths, ...), then the problem is polynomial-time solvable.
- If $\mathcal{H}$ has unbounded treewidth (e.g, cliques, bicliques, grids, ...), then the problem is **not** polynomial-time solvable (assuming ETH).

# Subgraphs $\Leftrightarrow$ homomorphisms

**Easy to check:**

hom( , $G$) $=$ 8sub( , $G$) $+$ 4sub( , $G$) $+$ 2sub( , $G$)



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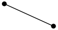
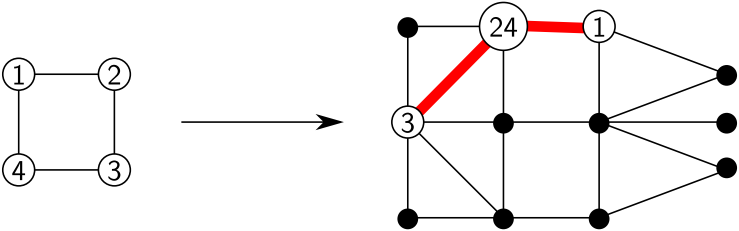
hom(  , $G$) $=$ 8sub(  , $G$) $+$ 4sub(  , $G$) $+$ 2sub(  , $G$)



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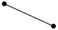
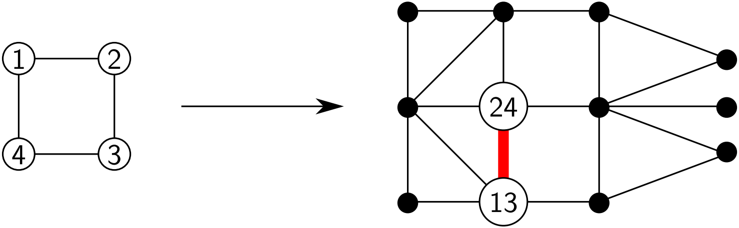
hom(  , $G$) $=$ 8sub(  , $G$) $+$ 4sub(  , $G$) $+$ 2sub(  , $G$)



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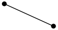
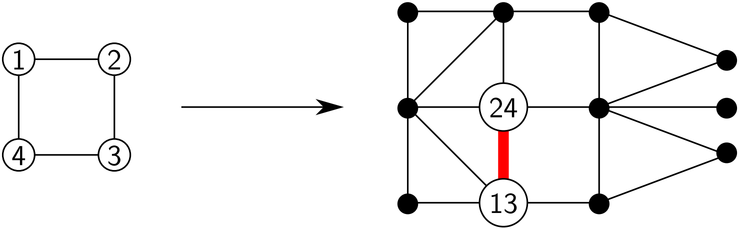
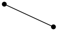
hom(  , $G$) $=$ 8sub(  , $G$) $+$ 4sub(  , $G$) $+$ 2sub(  , $G$)




**Not completely obvious:**

The formula can be reversed by inclusion-exclusion.

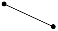
sub(  , $G$) $= \frac{1}{8}$hom(  , $G$) $- \frac{1}{4}$hom(  , $G$) $+ \frac{1}{8}$hom(  , $G$)

# Subgraphs ⇔ homomorphisms

**Definition**

surj($H$, $G$): number of surjective homomorphisms from $H$ to $G$ (every vertex and edge of $G$ appears in the image).

Homomorphisms can be counted by classifying according to the image $F$:

hom(  , $G$) $=$ 8sub(  , $G$) $+$ 4sub(  , $G$) $+$ 2sub(  , $G$)

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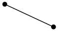
hom(  , $G$) = 8sub(  , $G$) + 4sub(  , $G$) + 2sub(  , $G$)

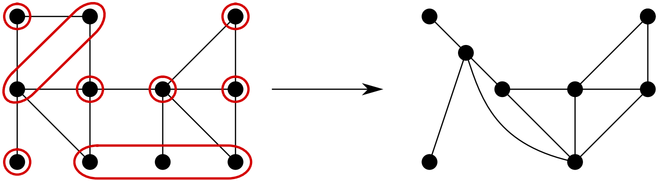
$$\Downarrow$$

$$\hom(H, G) = \sum_F \mathrm{surj}(H, F)\mathrm{sub}(F, G)$$

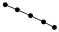
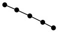
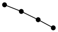
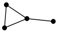
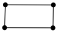
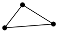
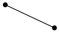
Which of the terms can be nonzero?

# Spasm

- $\text{Part}_0(H)$: set of partitions of $V(H)$ where each class is an independent set.
- For $\Pi \in \text{Part}_0(H)$, $H_{|\Pi}$ is obtained by contracting each class of $\Pi$ to a single vertex.



- $\text{Spasm} = \{H_{|\Pi} \mid \Pi \in \text{Part}_0(H)\}$

Spasm(  ) = { , , , , , , ,  }

# Subgraphs $\Leftrightarrow$ homomorphisms

**From subgraphs to homomorphisms:**

$$\mathrm{hom}(H, G) = \sum_F \mathrm{surj}(H, F)\mathrm{sub}(F, G)$$

where $\mathrm{surj}(H, F) \neq 0$ if and only if $F \in \mathrm{Spasm}(H)$.

**From homomorphisms to subgraphs:** [Lovász 1967]

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Extremely useful for applications in algorithms and complexity!

# Algorithmic applications

$$\mathrm{sub}(H, G) = \sum_{F \in \mathrm{Spasm}(H)} \beta_F \cdot \mathrm{hom}(F, G)$$

$$\Downarrow$$

Max. treewidth in $\mathrm{Spasm}(H)$ gives an upper bound on complexity:

**Corollary**

If every graph in $\mathrm{Spasm}(H)$ has treewidth at most $c$, then $\mathrm{sub}(H, G)$ can be computed in time $O(n^{c+1})$.

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**Observe:** If $H$ has $k$ edges, then every graph in $\mathrm{Spasm}(H)$ has at most $k$ edges.

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## Theorem [Scott and Sorkin 2007]

Every graph with $\leq k$ edges has treewidth at most $0.174k + O(1)$.

## Corollary

If $H$ has $k$ edges, then $\mathrm{sub}(H, G)$ can be computed in time $n^{0.174k+O(1)}$.

# Counting $k$-paths

**Corollary**

If $H$ has $k$ edges, then $\mathrm{sub}(H, G)$ can be computed in time $n^{0.174k+O(1)}$.

**Example:** Counting $k$-paths

- Brute force: $O(n^k)$.
- Meet in the middle: $O(n^{0.5k})$
  [Björklund et al., ESA 2009],[Koutis and Williams, ICALP 2009]
- [Björklund et al., SODA 2014]: $n^{0.455k+O(1)}$.
- New! by counting homomorphisms in the spasm: $n^{0.174k+O(1)}$.

# Counting small cycles

Counting triangles using matrix multiplication:

$$\mathrm{sub}(C_3, G) = \frac{1}{6}\mathrm{tr}\ \mathrm{Adj}^3(G)$$

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**Theorem** [Alon, Yuster, and Zwick, ESA 1994]

For $k \leq 7$, we can compute $\text{sub}(C_k, G)$ in time $n^\omega$ (where $\omega < 2.373$ is the matrix-multiplication exponent).

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We can recover this result:

- Check: if $k \leq 7$, then every graph in $\mathrm{Spasm}(C_k, G)$ has treewidth at most $2$.
- For treewidth 2, the $O(n^{2+1})$ homomorphism algorithm can be improved to $O(n^\omega)$ with fast matrix multiplication.
- $\Rightarrow O(n^\omega)$ algorithm for $\mathrm{sub}(C_k, G)$ if $k \leq 7$.

# Complexity applications

$$\mathrm{sub}(H, G) = \sum_{F \in \mathrm{Spasm}(H)} \beta_F \cdot \mathrm{hom}(F, G)$$

**Note:** Every $\beta_F$ is nonzero.

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Complexity of $\mathrm{hom}(F, G)$ for any $F \in \mathrm{Spasm}(H)$ is a **lower bound** on the complexity of $\mathrm{sub}(H, G)$.

# Matrices

Fix an enumeration of graphs with $\leq k$ edges with nondecreasing number of edges.

- Hom matrix: row $i$, column $j$ is $\hom(H_i, H_j)$.
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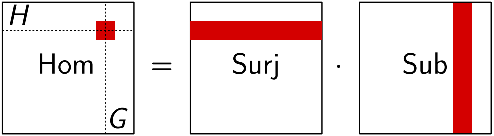

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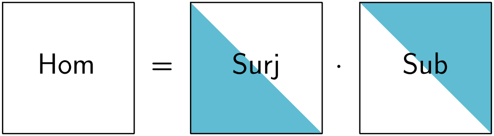


The Hom matrix is invertible!

# Categorical product

One of the standard graph products:

**Definition**

$G_1 \times G_2$ has vertex set $V(G_1) \times V(G_2)$ and $(v_1, v_2)$ and $(v_1', v_2')$ adjacent in $G_1 \times G_2 \iff v_1 v_1' \in E(G_1)$ and $v_2 v_2' \in E(G_2)$.

[missing figure]

Exercise:

$$\hom(H, G_1 \times G_2) = \hom(H, G_1) \cdot \hom(H, G_2)$$

# Extracting a term

**Lemma**

Given an algorithm for $\mathrm{sub}(H, G) = \sum_{F \in \mathrm{Spasm}(H)} \beta_F \cdot \mathrm{hom}(F, G)$ (with $\beta_F \neq 0$), we can compute $\mathrm{hom}(F, G)$ for any $F \in \mathrm{Spasm}(H)$.

Use the algorithm on $Z \times G$ for every $Z$ with $\leq k = |E(H)|$ edges.

$$\mathrm{sub}(H, Z \times G) = b_Z$$

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$$\sum_{F \in \mathrm{Spasm}(H)} \beta_F \cdot \mathrm{hom}(F, Z \times G) = b_Z$$

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The Hom matrix is invertible, so we can solve this system of equations!

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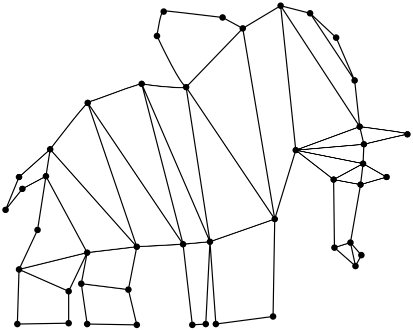
**Bottom line:**

| complexity of $\#\textsc{Sub}(H)$ | $=$ | hardest $\#\textsc{Hom}(F)$ for $F \in \mathsf{Spasm}(F)$ |
|---|---|---|

Complexity depends on the maximum treewidth in $\mathsf{Spasm}(H)$!

# Complexity of counting patterns

What is the best exponent for counting occurrences of this 46-vertex graph $H$?



**Answer:** Compute Spasm($H$) and find the best exponent for each of the resulting homomorphism problems!

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**More precise bound:**

$\text{Spasm}(M_k)$ contains every graph with $k$ edges

$\Downarrow$

$\text{Spasm}(M_k)$ contains graphs with treewidth $\Omega(k)$

$\Downarrow$

no $f(k)n^{o(k/\log k)}$ time algorithm for $\#k$-Matching, assuming ETH.

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What property of $H$ determines the max. treewidth in $\mathrm{Spasm}(H)$?

The vertex cover number of $H$:

- **Upper bound:**
  For every $F \in \mathrm{Spasm}(H)$, we have $\mathrm{tw}(F) \leq \mathrm{vc}(F) \leq \mathrm{vc}(H)$.
- **Lower bound:**
  $H$ contains a matching of size $\mathrm{vc}(H)/2$. We can show that for any $F$ with at most $\mathrm{vc}(H)/2$ edges, there is a graph in $\mathrm{Spasm}(H)$ that contains $F$ as a minor $\Rightarrow$ there is a graph in $\mathrm{Spasm}(H)$ with treewidth $\Omega(\mathrm{vc}(H))$.

## Counting subgraphs — summary

Vertex cover number of $H$ determines the complexity of $\textsc{Sub}(H)$:

- $n^{\mathrm{vc}(H)+O(1)}$ upper bound.
- $\Omega(n^{\gamma\cdot \mathrm{vc}(H)/\log \mathrm{vc}(H)})$ lower bound.

If we restrict the problem to a class $\mathcal{H}$ of patterns:

- If $\mathcal{H}$ has bounded vertex cover number (e.g, stars, double stars, ...), then the problem is polynomial-time solvable.
- If $\mathcal{H}$ has unbounded vertex cover number (e.g, cliques, paths, matchings, disjoint triangles, ...), then the problem is **not** polynomial-time solvable (assuming ETH).

# Conclusions

**Main message**

Parameterized subgraph counting problems can be understood via homomorphism counting problems.

. . .and this connection gives both algorithmic and complexity results!

Working on counting problems is fun:

- You can revisit fundamental, “well-understood” problems.
- Requires a new set of lower bound techniques.
- Requires new algorithmic techniques.