A subexponential parameterized algorithm for Subset TSP on planar graphs

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http://xkcd.com/399/

SODA 2014
January 7, 2014
Portland, OR
**TSP**

**Input:** A set $T$ of cities and a distance function $d$ on $T$

**Output:** A tour on $T$ with minimum total distance

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**Theorem [Held and Karp 1962]**

TSP with $n$ cities can be solved in time $2^n \cdot n^2 \cdot \log D$, where $D$ is the maximum (integer) distance.

**Dynamic programming:**
Let $x(v, T')$ be the minimum length of path from $v_{\text{start}}$ to $v$ visiting all the cities $T' \subseteq T$. 
**c-change TSP**

- *c*-change operation: removing *c* steps of the tour and connecting the resulting *c* paths in some other way.
- A solution is *c*-OPT if no *c*-change can improve it.
- We can find a *c*-OPT solution in \( n^{O(c)} \cdot D \) time, where *D* is the maximum (integer) distance.
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TSP on planar graphs

Assume that the cities correspond to the set of all vertices of a (weighted) planar graph and distance is measured in this (weighted) planar graph.
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- Can be solved in time $n^{O(\sqrt{n})}$.
- Admits a PTAS.
**Subset TSP on planar graphs**

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- Can be solved in time $n^{O(\sqrt{n})}$.
- Can be solved in time $2^k \cdot n^{O(1)}$.
- **Question:** Can we restrict the exponential dependence to $k$ and exploit planarity?
**Subset TSP on planar graphs**

Assume that the cities correspond to a subset $T$ of vertices of a planar graph and distance is measured in this planar graph.

**Theorem**

*Subset TSP* for $k$ cities in a unit-weight planar graph can be solved in time $2^{O(\sqrt{k \log k})} \cdot n^{O(1)}$. 
**Subset TSP on planar graphs**

Assume that the cities correspond to a subset $T$ of vertices of a planar graph and distance is measured in this planar graph.

*Theorem*

**Subset TSP** for $k$ cities in a weighted planar graph can be solved in time $\left(2^{O(\sqrt{k \log k})} + W\right) \cdot n^{O(1)}$ if the weights are integers not more than $W$. 
Partial solutions

**General idea:** build larger and larger partial solutions.

**Held-Karp algorithm:** the partial solutions are $v_{\text{start}} - v$ paths visiting a subset $T'$ of cities.
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**Held-Karp algorithm:** the partial solutions are $v_{\text{start}} - v$ paths visiting a subset $T'$ of cities.

![Diagram showing partial solutions]

**Generalization:** a partial solution is a set of at most $d$ pairwise disjoint paths with specified cities as endpoints.

The **type** of a partial solution can be described by
- the set of endpoints of the paths,
- a matching between the endpoints, and
- the subset $T'$ of visited cities.
Merging partial solutions

Two compatible partial solutions can be merged in an obvious way:
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Algorithm

- Start with an initial set of trivial partial solutions.
- Combine two partial solutions as long as possible.
- Keep at most one partial solution from each type: the best one encountered so far.
- Return the best partial solution that consists of a single path (cycle) visiting all vertices.
Running time

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With careful implementation, the running time is dominated by the number of types, whose number has two factors:

$$k^2$$ possibilities, describing the subset $$T'$$ of visited cities

We can increase $$d$$ up to $$O(\sqrt{k})$$, but we need to reduce somehow the number of possible subsets of cities!
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With careful implementation, the running time is dominated by the number of types, whose number has two factors:

- endpoints described by at most $d$ pairs of vertices
  \[ \Rightarrow k^{2d} \text{ possibilities}, \]
- describing the subset $T'$ of visited cities
  \[ \Rightarrow 2^k \text{ possibilities}. \]

We can increase $d$ up to $O(\sqrt{k})$, but we need to reduce somehow the number of possible subsets of cities!
Restricting the subset of cities

We restrict attention to a collection $\mathcal{T}$ of subsets of cities and consider only partial solutions that visit a subset in $\mathcal{T}$.

We need: a collection $\mathcal{T}$ of size $k^{O(\sqrt{k})}$ that guarantees finding an optimum solution.
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**Definition of $\mathcal{T}$:**
- Find a 4-OPT tour.
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- A subset is in $\mathcal{T}$ if and only if it induces $O(\sqrt{k})$ consecutive intervals on the 4-OPT tour.
Main result

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- A subset is in $\mathcal{T}$ if and only if it induces $O(\sqrt{k})$ consecutive intervals on the 4-OPT tour.

**Theorem**

After setting $\mathcal{T}$ as above and $d = O(\sqrt{k})$, the Algorithm finds an optimum solution for Subset TSP on planar graphs.

**Corollary**

Subset TSP for $k$ cities in a planar graph can be solved in time $(2^{O(\sqrt{k} \log k)} + W) \cdot n^{O(1)}$ if the weights are integers at most $W$. 
The treewidth bound

Consider the union of an optimum solution and a 4-OPT solution as a graph on $k$ vertices:

Lemma

For every 4-OPT solution, there is an optimum solution such that their union has treewidth $O(\sqrt{k})$. 
The treewidth bound

**Lemma**

For every 4-OPT solution, there is an optimum solution such that their union has treewidth $O(\sqrt{k})$.

- The union has separators of size $O(\sqrt{k})$.
- In each component, the set of cities visited by the optimum solution is nice: it is the same as what $O(\sqrt{k})$ segments of the 4-OPT tour visited.
- We can use this tree decomposition to prove that the Algorithm finds an optimum solution.
Proof of the treewidth bound

Consider the closed walk corresponding to the 4-OPT solution and pick an optimum solution and a closed walk representing that.

The union is a planar graph (we ignore degree-2 vertices now):

Select the optimum solution and the closed walk such that the two tours cross each other the minimum number of times.
Proof of the treewidth bound

Consider the closed walk corresponding to the 4-OPT solution and pick an optimum solution and a closed walk representing that. The union is a planar graph (we ignore degree-2 vertices now):

We give an $O(\sqrt{k})$ bound on the treewidth of this planar graph

$\Downarrow$

A $O(\sqrt{k})$ bound follows for the $k$-vertex graph, as it is a minor of this graph after duplicating the vertices.
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Proof of the treewidth bound

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The union is a planar graph (we ignore degree-2 vertices now):

We prove that every 3-connected component of the planar graph has $O(k)$ vertices of degree $> 2$

$\Downarrow$

$O(\sqrt{k})$ treewidth bound on the 3-connected components

$\Downarrow$

same bound for the whole graph.
Grids

A grid is a 16-vertex subgraph of the union of the 4-OPT solution and the optimum solution:
**Grids**

A **grid** is a 16-vertex subgraph of the union of the 4-OPT solution and the optimum solution:

![Grid Example](image)

**Lemma**

If a 3-connected component of the union has size $\Omega(k)$, then there is a grid.

**Proof idea:** 4-regular and $O(k)$ faces have length $< 4$

$\Rightarrow$ Euler’s formula implies that most of the faces have length 4

$\Rightarrow$ a 4-face surrounded by 4-faces should be a grid.
Grids

Suppose that the grid is used like this by two tours:

Let us exchange these two sets of edges between the two tours. The 4-OPT tour cannot improve. The optimum tour cannot improve. We get another optimum tour that has fewer crossings with the 4-OPT tour.
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Let us exchange these two sets of edges between the two tours.
- The 4-OPT tour cannot improve.
- The optimum tour cannot improve.
- We get another optimum tour that has fewer crossings with the 4-OPT tour.
Grids — other cases:

C type + S type:
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S type + S type:
Grids — other cases:

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Grids — other cases:

S type + S type:
Grids — other cases:

S type + inverted S type:
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Overview

- **Algorithm:**
  - Find a 4-OPT tour.
  - Partial solutions: $O(\sqrt{k})$ disjoint paths, visiting $O(\sqrt{k})$ consecutive intervals on the 4-OPT tour.
  - Merge partial solutions until the optimum solution is found.
- **Treewidth bound:** the union of the 4-OPT tour and some optimum tour is a $k$-vertex graph with treewidth $O(\sqrt{k})$.
  - Study the union in the planar graph.
  - Every 3-connected component has $O(k)$ vertices of degree $> 2$, otherwise there is a grid and an exchange argument could be used.
  - Union in the planar graph has treewidth $O(\sqrt{k}) \Rightarrow$ the $k$-vertex graph has treewidth $O(\sqrt{k})$. 