Interval Deletion is fixed-parameter tractable

Yixin Cao   Dániel Marx

Institute for Computer Science and Control
Hungarian Academy of Sciences (MTA SZTAKI)
Budapest, Hungary

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Problems on graph classes

For various classes $\mathcal{G}$ of graphs (planar, chordal, interval, etc.), there is a large literature on

- how to recognize if a graph is a member of $\mathcal{G}$ and
- how to solve certain problems on $\mathcal{G}$ more efficiently than on general graphs.

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- how to solve certain problems on $\mathcal{G}$ more efficiently than on general graphs.

Can we ask the same questions about graphs that "almost" belong to $\mathcal{G}$?

Graph modification problems

For every class $\mathcal{G}$ of graphs, we can study the following type of problems:

$\mathcal{G}$-graph modification problem

Input: a graph G of size n and a nonnegative integer k
Task: find $\leq k$ modifications that transform G into a graph in $\mathcal{G}$

Allowed typical modification operations:

- removing edges,
- adding edges,
- removing vertices.

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In other words, the question is if G belongs to the class

- $\mathcal{G} + ke$: a graph from $\mathcal{G}$ with k extra edges;
- $\mathcal{G} - ke$: a graph from $\mathcal{G}$ with k missing edges;
- $\mathcal{G} + kv$: a graph from $\mathcal{G}$ with k extra vertices.

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Theorem [Lewis and Yannakakis 1980]

If the graph class $\mathcal{G}$ is **nontrivial** and **hereditary**, then it is NP-hard to decide if a graph is in $\mathcal{G} + kv$.

Examples

If $\mathcal{G}$ is polynomial-time recognizable, we can test in time $n^{O(k)}$ whether G is in $\mathcal{G} + kv$.

But can we solve it in time $f(k) \cdot n^{O(1)}$, i.e., is it FPT?

$\mathcal{F}$	Problems	Complexity
dis co n n ec ted graphs	Vertex Connectivity	$\in$ P
independent	Vertex Cover	$1.31^k \cdot n^{O(1)}$
acyclic trees graphs	Feedback Vertex Set	$3.83^k \cdot n^{O(1)}$
chordal graphs	Chordal Deletion	$2^{\mathcal{O}(k \log k)} \cdot n^{O(1)}$
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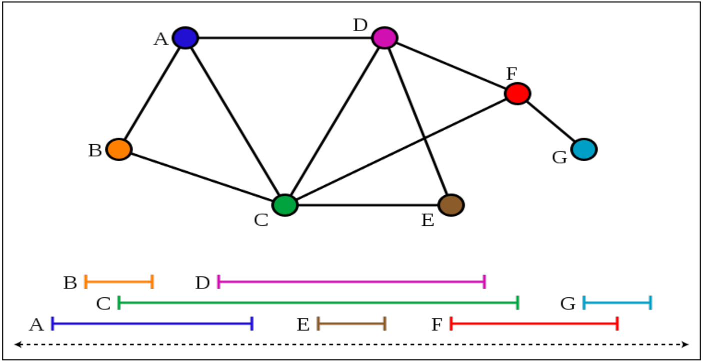
INTERVAL GRAPHS	Interval Deletion	$10^k \cdot n^{O(1)}$

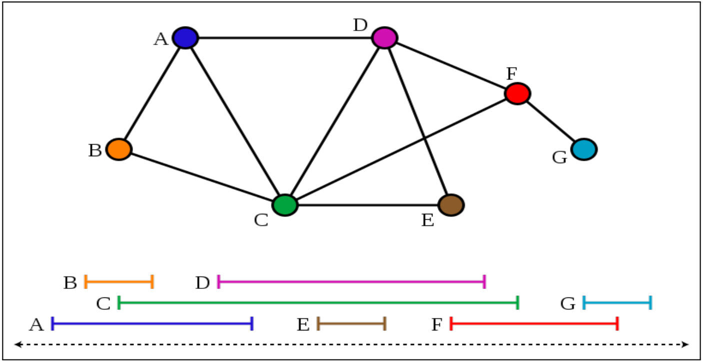
Interval graphs



Definition

There are a set of intervals $\mathcal{I}$ in the real line and $\phi : V \to \mathcal{I}$ such that $uv \in E(G)$ iff $\phi(u)$ intersects $\phi(v)$.

Interval graphs



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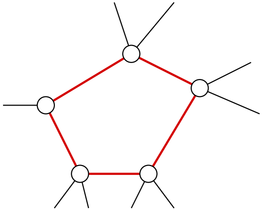
- supersets: chordal graphs, and circular-arc graphs;
- subsets: unit interval graphs.

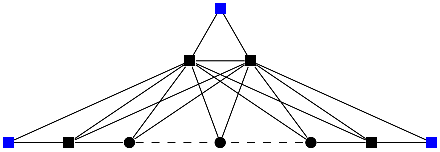
Characterization by forbidden induced subgraphs

Theorem [Lekkerkerker and Boland 1962]

G is an interval graph iff it contains no holes or asteroidal triples (ATs).

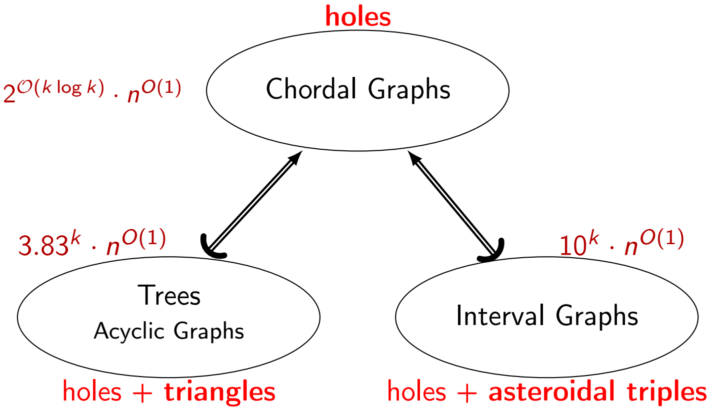


Hole: a chordless cycle of length ≥ 4



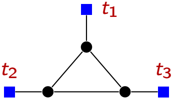
Asteroidal triple: three vertices such that each pair of them is connected by a path avoiding neighbors of the third one.

Holes and others

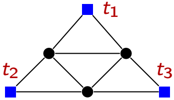


Minimal chordal asteroidal triples

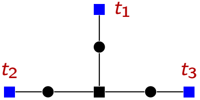
Completely described by [Lekkerkerker and Boland 1962]:



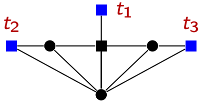
(a) net



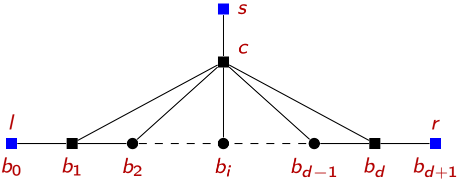
(b) tent



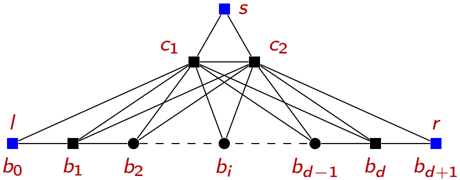
(c) long claw



(d) whipping top



(e) $\dagger_d$ $(s : c, c : l, B, r)$ $(d = |B| \geq 3)$



(f) $\ddagger_d$ $(s : c_1, c_2 : l, B, r)$ $(d = |B| \geq 3)$

Reduction 1: small forbidden subgraphs

Standard technique: if the graph class $\mathcal{G}$ can be characterized by forbidden subgraphs of bounded size, then the problem can be solved by branching.

Same approach for the small forbidden subgraphs:

Given an instance (G, k) and a forbidden subgraph X of no more than 10 vertices, we branch into $|X|$ instances, $(G - v, k - 1)$ for each $v \in X$.

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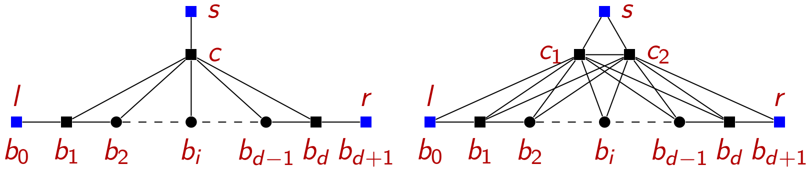
Same approach for the small forbidden subgraphs:

Given an instance (G, k) and a forbidden subgraph X of no more than 10 vertices, we branch into $|X|$ instances, $(G - v, k - 1)$ for each $v \in X$.

We are left with long holes (at least 11 vertices) and



Modules

M is a **module** if every vertex in M has the same neighborhood outside M: $u, v \in M$ and $x \notin M$, $u \sim x$ iff $v \sim x$.

Trivial modules: $\{v\}$ and $V(G)$.

Proposition

If M is a module and U induces a minimal forbidden subgraph of size at least 11, then either $U \subseteq M$, or $|M \cap U| \leq 1$.

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Theorem

Let M be a module in a 4-hole-free graph G and Q be a minimum interval deletion set. Either $M \subset Q$, or $Q \cap M$ is a minimum interval deletion set to $G[M]$.

Reduction 2: nontrivial modules

Instance (G, k) where G is 4-hole-free, and nontrivial module M

1. If every MFS U is contained in M, then return $(G[M], k)$.
2. If no MFS is in M, then insert edges to make $G[M]$ a clique.
3. Otherwise, we branch into two cases:
    1. include M in the solution: $I_1 = (G - M, k - |M|)$;
    2. at least one vertex of M is not deleted:
       $I_2 = (G[M], k - 1)$ and $I_3 = (G', k - 1)$,
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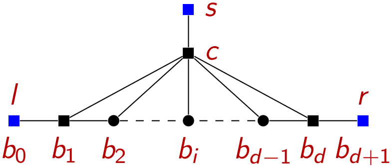
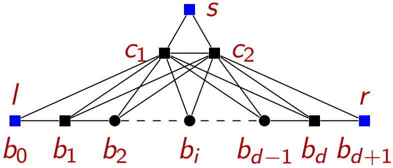
Applying the two reductions exhaustively, we get a **reduced** graph where

1. each MFS contains at least 11 vertices; and
2. each non-trivial module is a clique.

Shallow terminals

Shallow terminal: the terminal s of the AT "close" to the $l - r$ path.





Theorem (Main theorem I)

In a reduced graph, every shallow terminal is simplicial (i.e., its neighborhood induces a clique).

Congenial holes

Definition

Two holes H_1 and H_2 are called **congenial** (to each other) if $H_1 \subseteq N[H_2]$ and $H_2 \subseteq N[H_1]$.

Theorem (Main theorem II)

All holes are pairwise congenial in a reduced graph.

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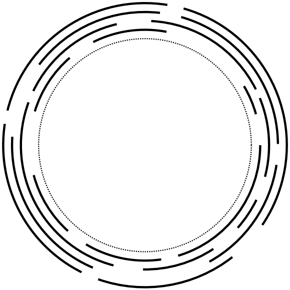
All holes are pairwise congenial in a reduced graph.

Example: All holes are congenial in a circular arc graph.



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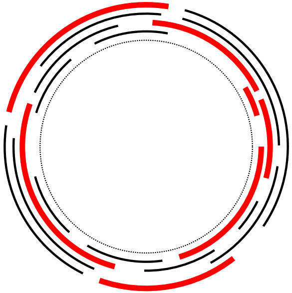
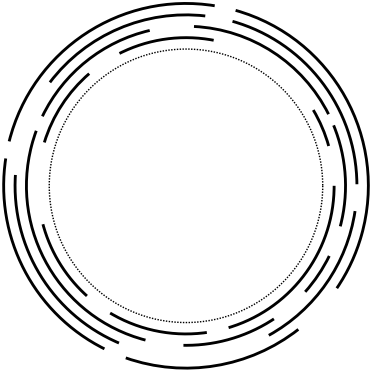


Hole covers

How to break holes in a circular arc graph?



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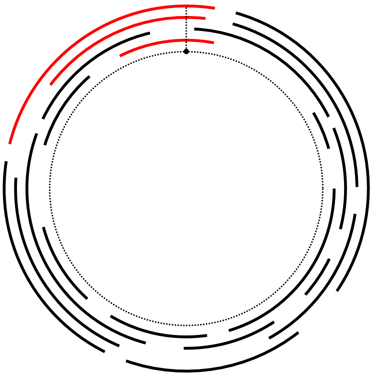

Intuitively, it seem to be a good idea to remove all arcs containing a certain point of the circle.

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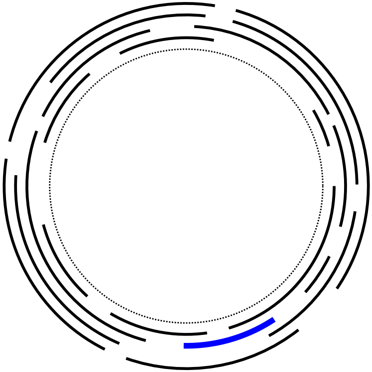

A different way to express this: pick a vertex v, consider the interval graph $G \setminus N[v]$ and remove a minimal separator.

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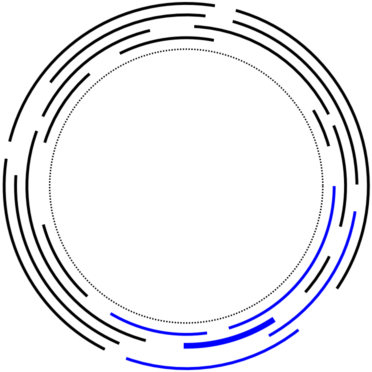

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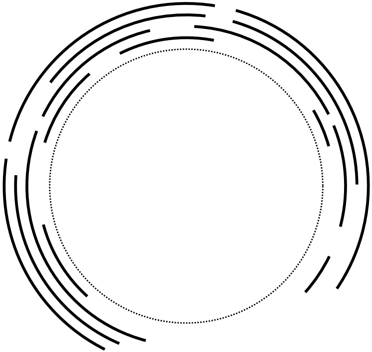

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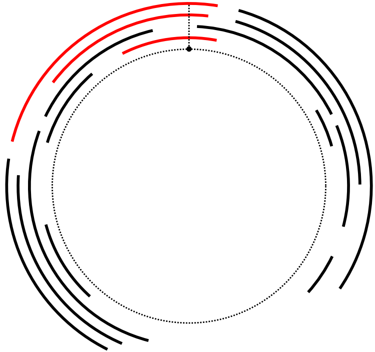

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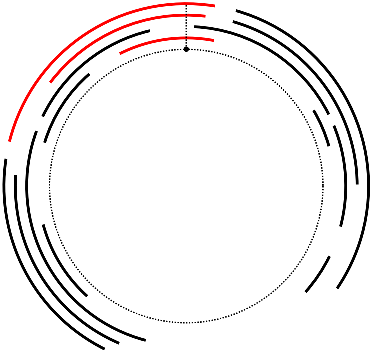


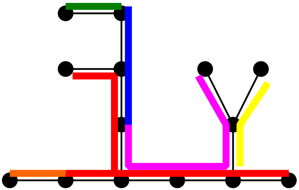
Works also for reduced graphs: in a similar way, we can enumerate $O(n^2)$ sets such that every hole cover fully contains at least one of these sets $\Rightarrow$ branch.

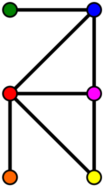
Caterpillar decomposition

At this point

- The graph has no holes, i.e., it is chordal.
- The graph has no small ATs.
- The shallow terminal of each large AT is simplicial.

Chordal graphs can be characterized as the intersection graphs of subtrees of a tree.





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Theorem

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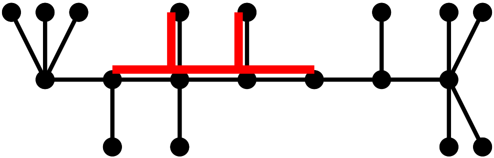

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Proof:

- $G - ST$ is an interval graph, where ST is the set of shallow terminals.
- $G - ST$ has a clique path decomposition.
- to which we can add the simplicial ST back.

Branching rule

Analyzing the way the ATs can appear in the caterpillar decomposition, we obtain the following branching rule.

Theorem

Take the leftmost minimal AT T with shortest base. The minimal interval deletion set to G contains either one of

$$\{s, c_1, c_2, l, r, b_{d-3}, b_{d-2}, b_{d-1}, b_d\},$$

or the minimum separator of l and b_{d-3}.

Therefore, by branching into 10 directions, we can identify at least one vertex of the solution.

Summary

- A $10^k \cdot n^{O(1)}$ algorithm for Interval Deletion.
- Main steps:
  1. Simple reduction rule: branching on small forbidden sets.
  2. Reduction rule using modules.
  3. Theorem I: Shallow terminals are simplicial.
  4. Theorem II: All holes are congenial.
  5. $O(n^2)$ minimal hole covers.
  6. Branching on the leftmost minimal AT in a caterpillar decomposition.