Known Algorithms on Graphs of Bounded Treewidth are Probably Optimal

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Joint work with

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Treewidth

Treewidth: A measure of how "tree-like" the graph is.
(Introduced by Robertson and Seymour in the Graph Minors project.)

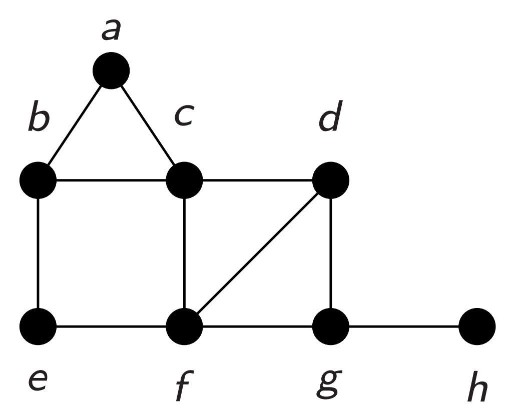
Significance:

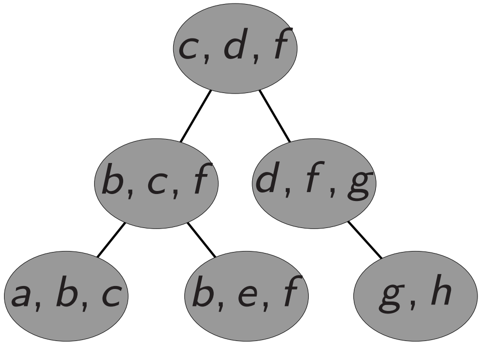
- Appears naturally in graph structure theory.
- Polynomial or even linear algorithms for NP-hard problems on bounded treewidth graphs.
- Crucial tool for planar approximation schemes.
- Useful for fixed-parameter tractability results.

Treewidth

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. If u and v are neighbors, then there is a bag containing both of them.
2. For every vertex v, the bags containing v form a connected subtree.





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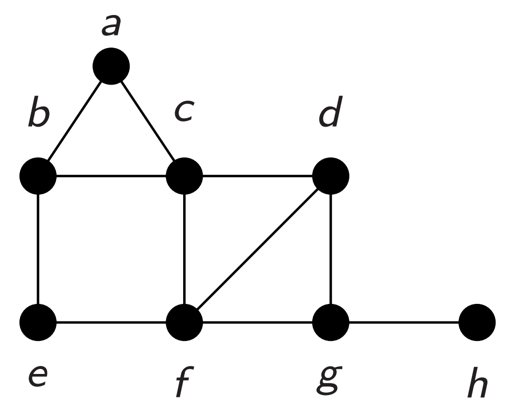

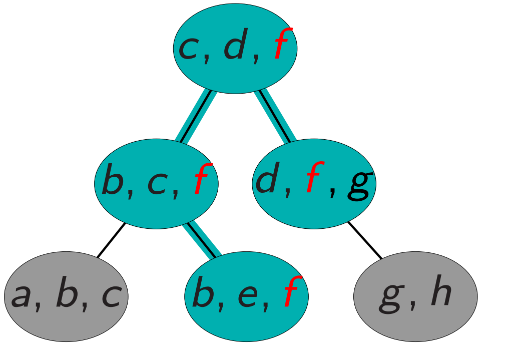

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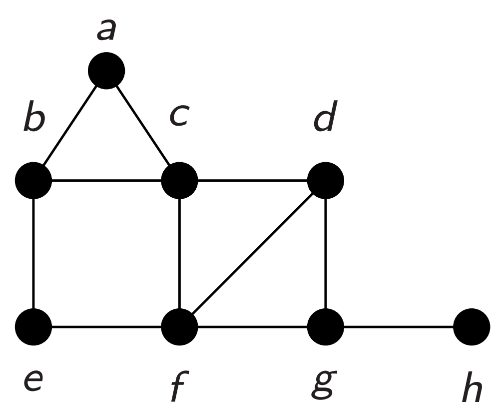
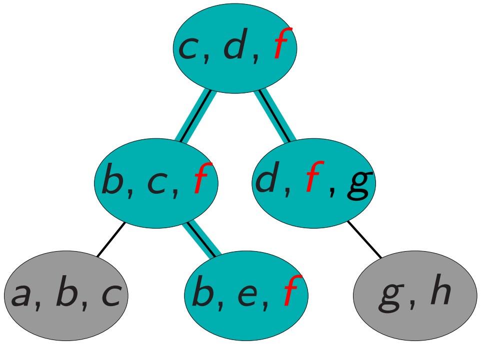
Width of decomposition: largest bag size -1.

treewidth: width of the best decomposition.

Fact: treewidth $= 1 \iff$ graph is a forest





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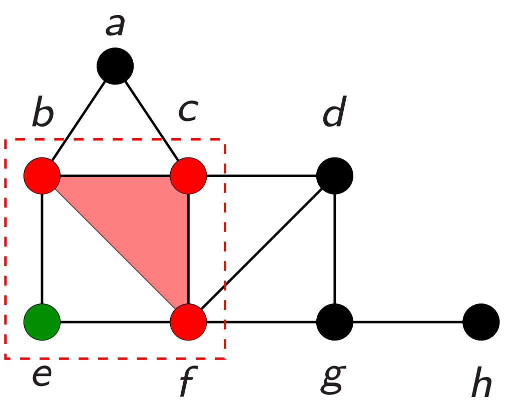

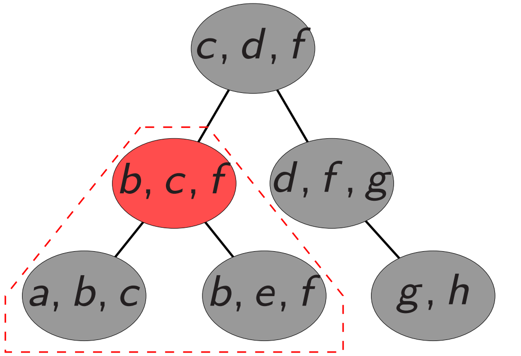

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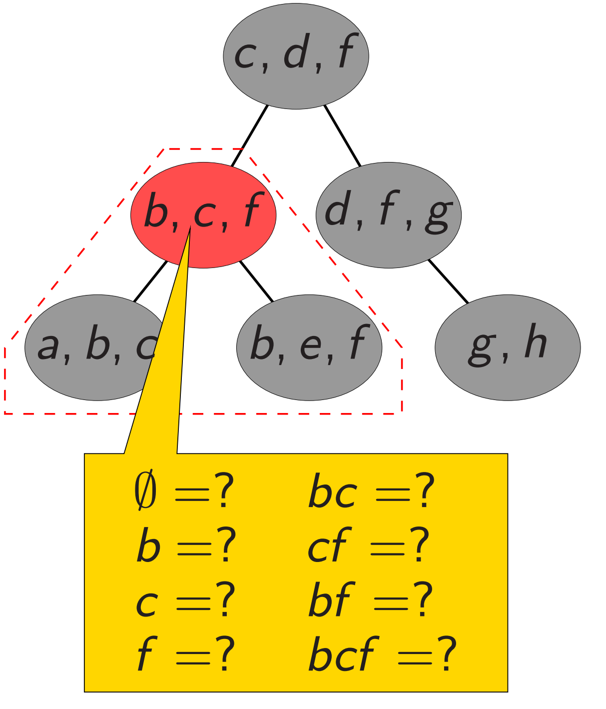
MAX INDEPENDENT SET and tree decompositions

Fact: Given a tree decomposition of width w, MAX INDEPENDENT SET can be solved in time $O(2^w \cdot n)$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

- Define table $M[x, S]$: the maximum weight of an independent set $I \subseteq V_x$ with $I \cap B_x = S$.
- Compute the tables in bottom-up order.
- Size of each table is 2^{w+1}.



Algorithms

Given a tree decomposition of width w, dynamic programming gives:

INDEPENDENT SET	$O(2^w \cdot n)$
DOMINATING SET	$O(3^w \cdot n)$
MAX CUT	$O(2^w \cdot n)$
ODD CYCLE TRANSVERSAL	$O(3^w \cdot n)$
q-COLORING ($q \geq 3$)	$O(q^w \cdot n)$
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Question: Can we improve the base in any of these algorithms?

Supporting evidence: Running time matches the obvious DP table size. But...

Some history

DOMINATING SET

- Obvious approach: 9^w [Telle and Proskurowski ’93]
- More clever algorithm: 4^w [Alber et al. ’02]
- Even more clever algorithm: 3^w [Rooij et al. ’09] using fast subset convolution of [Björklund et al. ’07]

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HAMILTONIAN CYCLE

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DIRECTED FEEDBACK VERTEX SET

- Trivial 2^n algorithm.
- Nontrivial 1.9977^n algorithm [Razgon '07]

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SETH

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> Strong Exponential Time Hypothesis (SETH):
>
> $s_k = \inf\{\delta \mid n\text{-variable } k\text{-SAT } \textit{can be solved in } 2^{\delta n}\}$
>
> **Conjecture:** [Impagliazzo-Paturi '01] $s_k \to 1$

We can use a somewhat weaker assumption:

> No faster SAT:
>
> **Conjecture:** n-variable m-clause SAT (with arbitrary clause length) cannot be solved in time $(2-\epsilon)^n \cdot \text{poly}(m)$ for any $\epsilon > 0$.

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Results

Main result: If the Strong Exponential Time Hypothesis (SETH) is true, then given a tree decomposition of width w,

INDEPENDENT SET		$(2-\epsilon)^w \cdot n^{O(1)}$
DOMINATING SET		$(3-\epsilon)^w \cdot n^{O(1)}$
MAX CUT	cannot be	$(2-\epsilon)^w \cdot n^{O(1)}$
ODD CYCLE TRANSVERSAL	solved in time	$(3-\epsilon)^w \cdot n^{O(1)}$
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The lower bounds match the known algorithms (up to the ϵ in the base).

Note: For some problems, we can obtain stronger results by proving the same lower bound with respect to pathwidth or feedback vertex number.

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Reductions

Suppose we have a reduction:

n-variable SAT instance	$\Rightarrow$	INDEPENDENT SET instance of treewidth $w \leq c \cdot n$.

Then:

$(2-\epsilon)^{c \cdot n}$ algorithm for SAT	$\Leftarrow$	$(2-\epsilon)^{w} \cdot n^{O(1)}$ algorithm for INDEPENDENT SET

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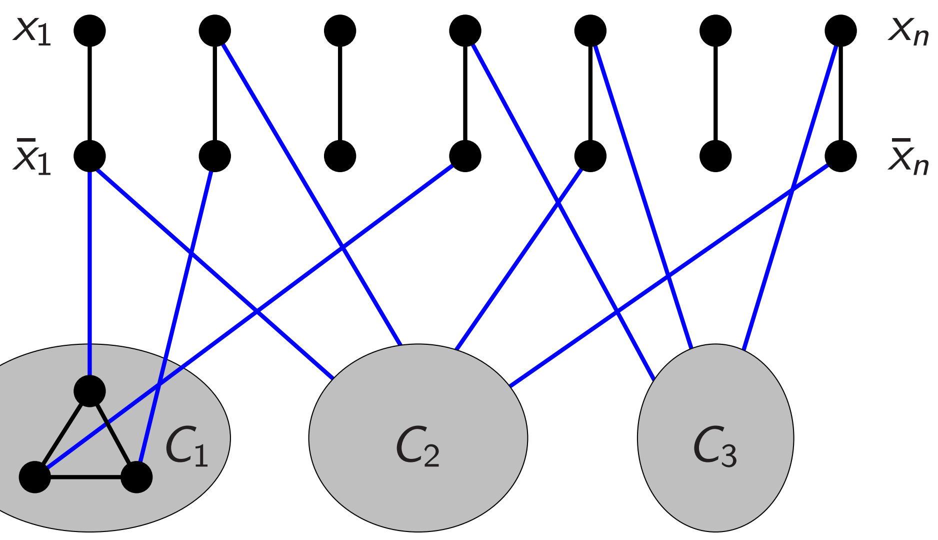
- To get a $(2-\epsilon)^w$ lower bound, we need $c \leq 1$.
- **More generally:** For any c, we get a $(2^{1/c}-\epsilon)^w$ lower bound
  $\Rightarrow$ To get a $(3-\epsilon)^w$ lower bound (e.g., for DOMINATING SET), we need $c \leq \log_3 2 \approx 0.631$.

Textbook reduction

How large is the treewidth in the textbook reduction from SAT to INDEPENDENT SET?



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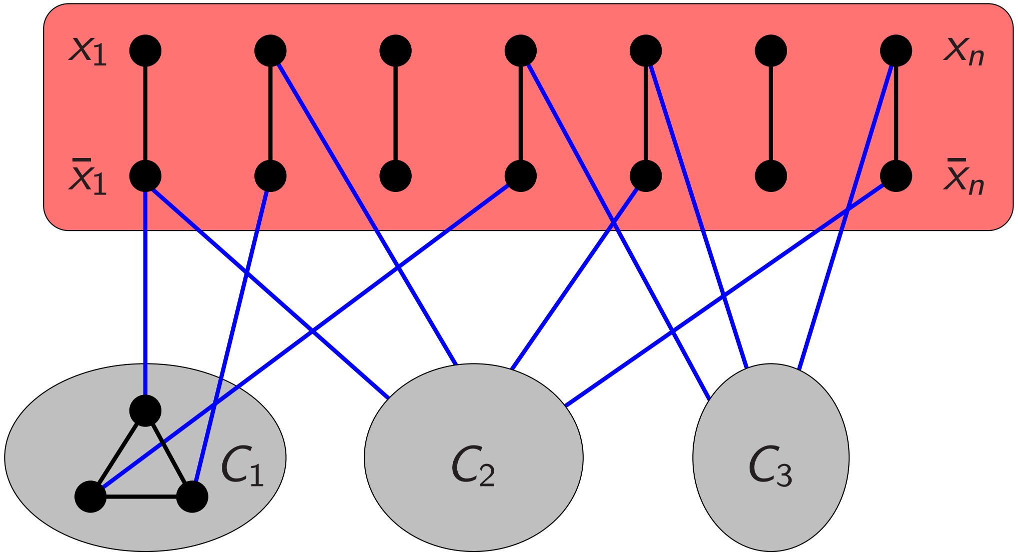
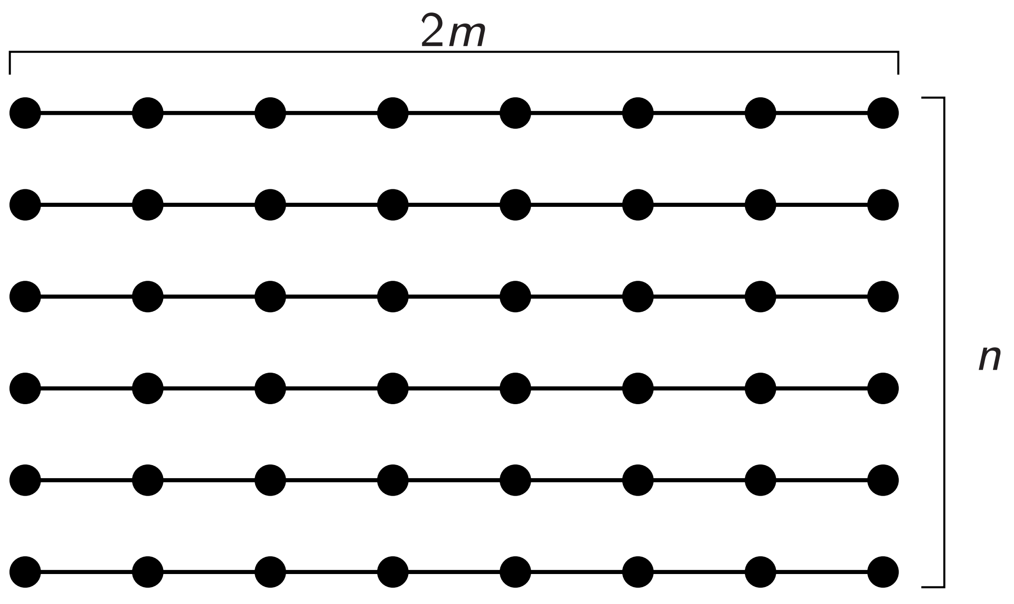


Treewidth is about $2n$, which gives a $(2^{\frac{1}{2}} - \epsilon)^w \approx 1.41^w$ lower bound.

We need treewidth $\leq n$ for the $(2 - \epsilon)^w$ lower bound.

New reduction for INDEPENDENT SET

n variables, m clauses	$\Rightarrow$	n paths of $2m$ vertices each
2 states per each variable	$\Rightarrow$	2 possible states for each path



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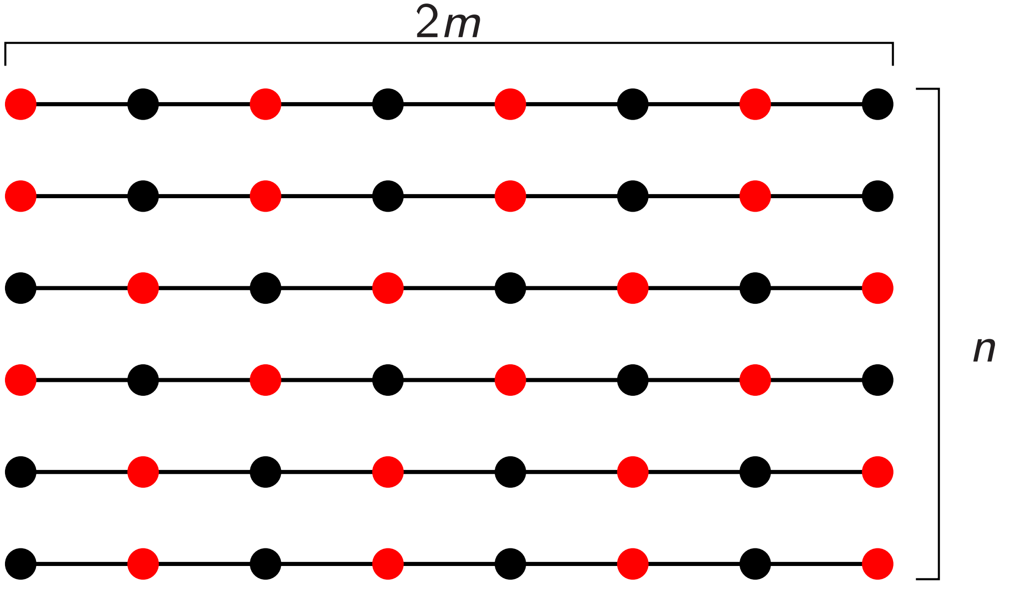

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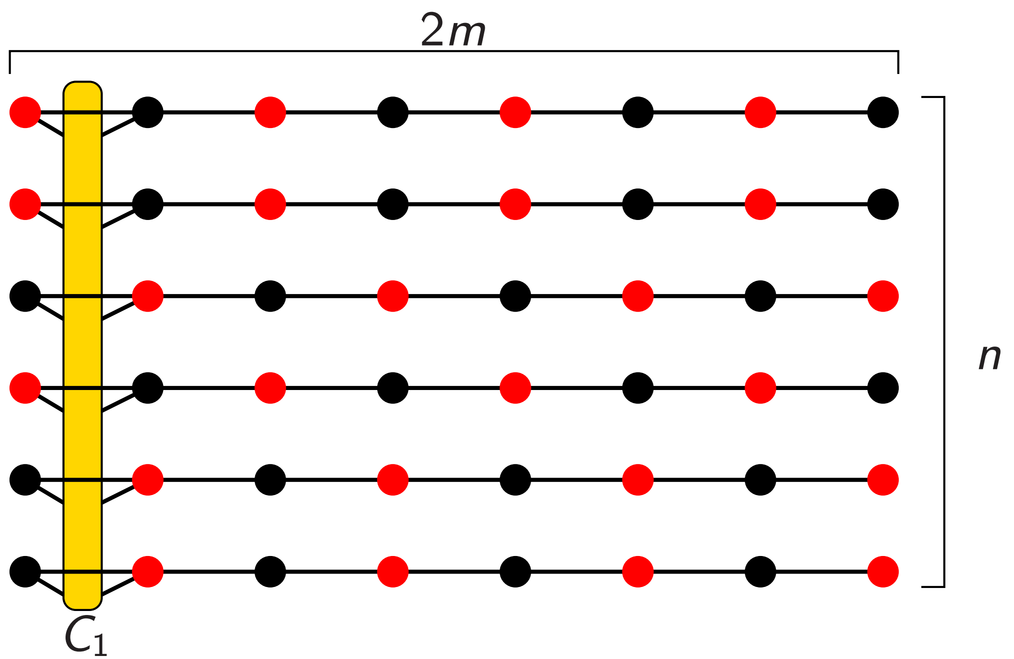

Clause gadgets check that every clause is satisfied.

Treewidth is only $n + O(1)$.

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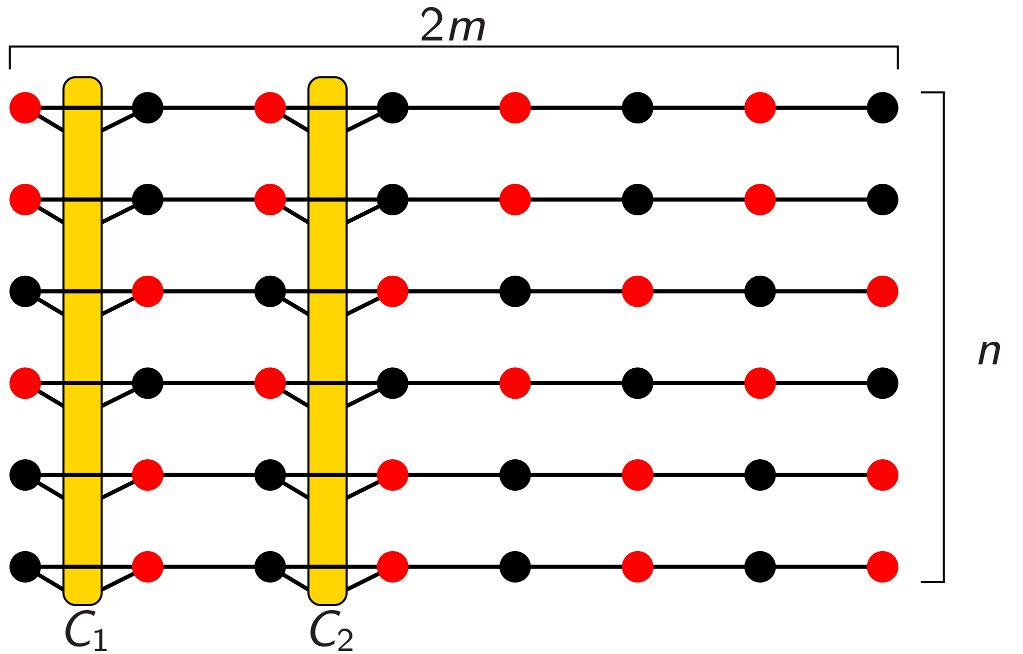

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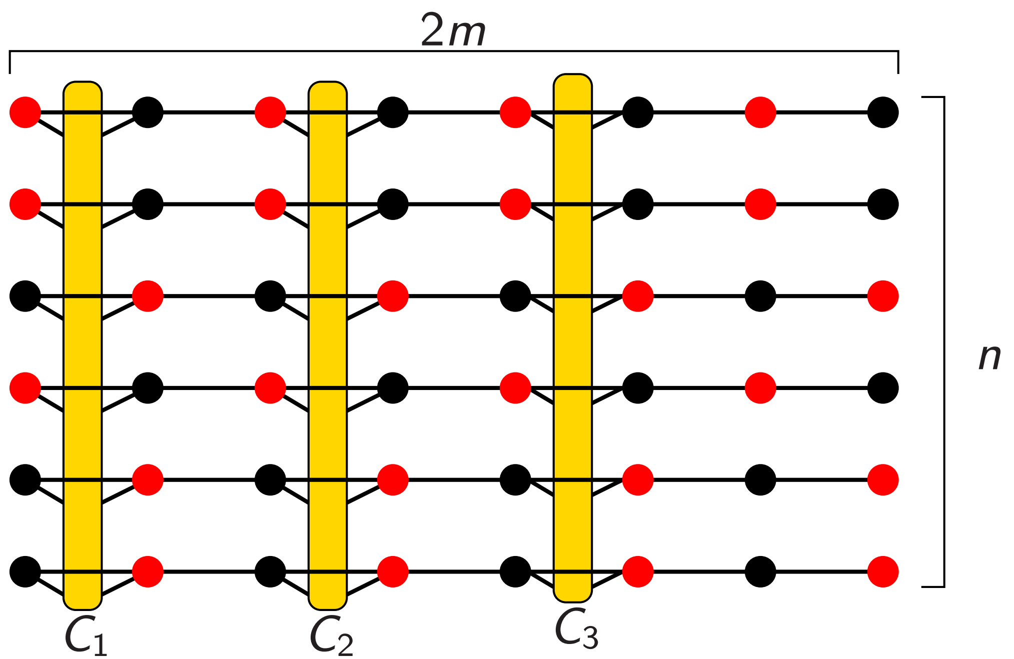

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New reduction for DOMINATING SET

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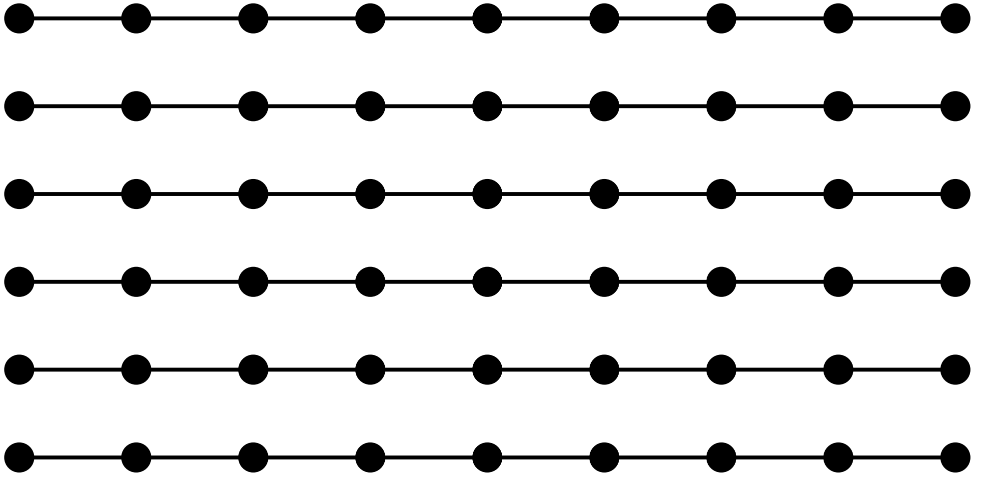

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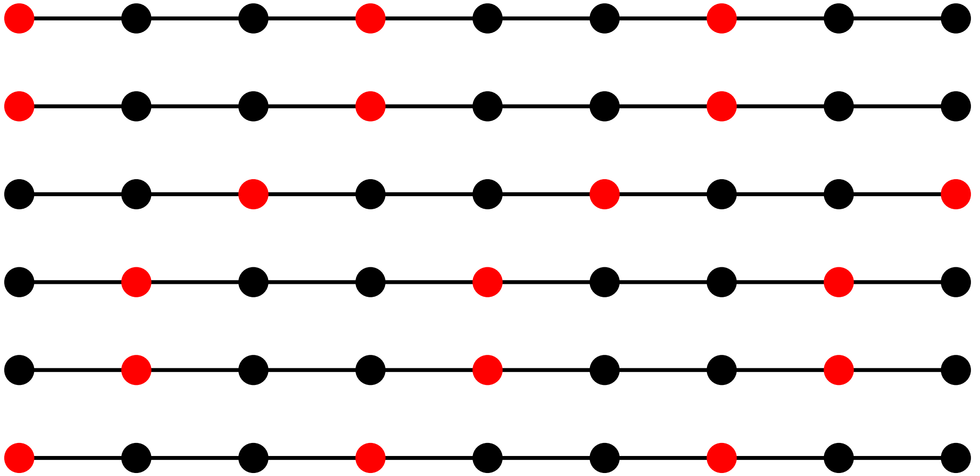

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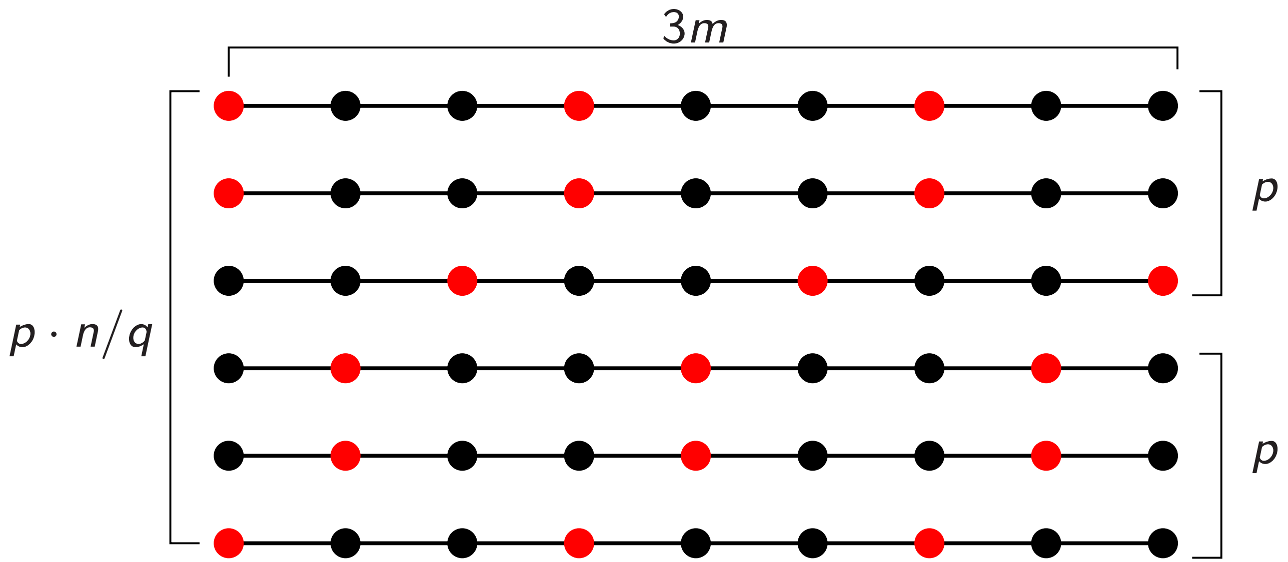

Partition variables into n/q groups of size $q = O(1)$. The 2^q possibilities for a group of variables are represented by a group of p paths, where $2^q \leq 3^p$, i.e., $p = \lceil \log_3 2^q \rceil \approx 0.631q$.

$\Rightarrow$ Treewidth is $n \cdot \log_3 2$ and the $(3-\epsilon)^w$ bound follows.

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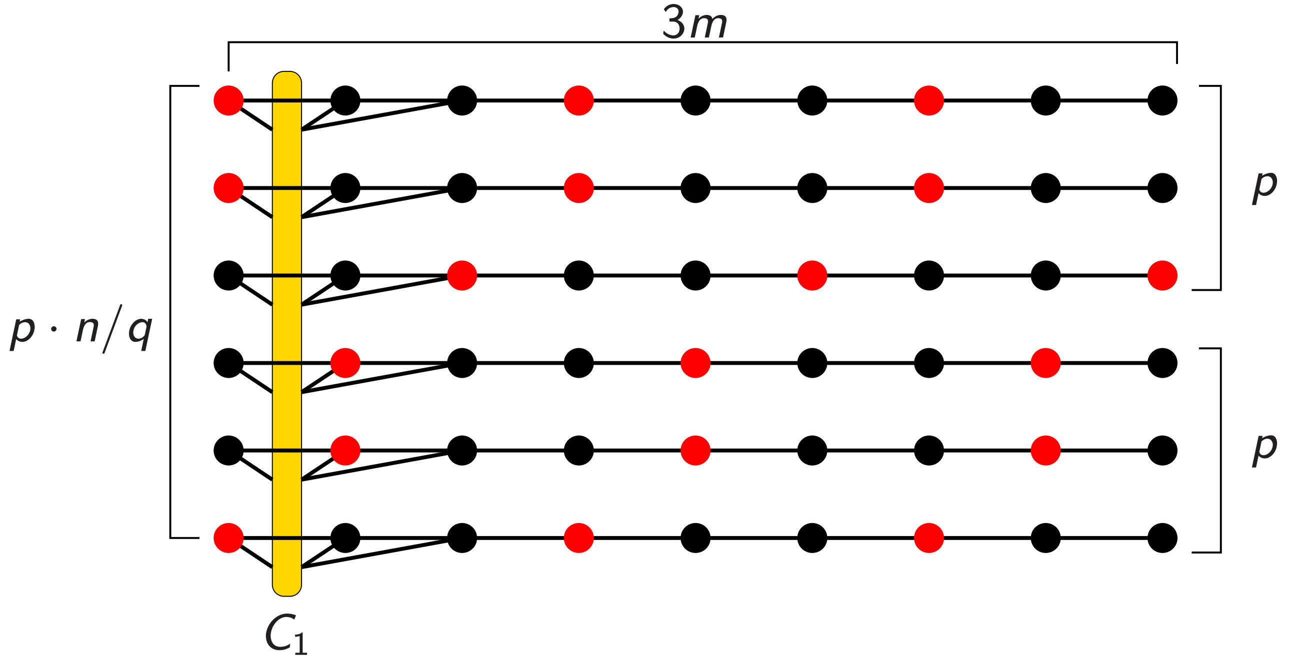

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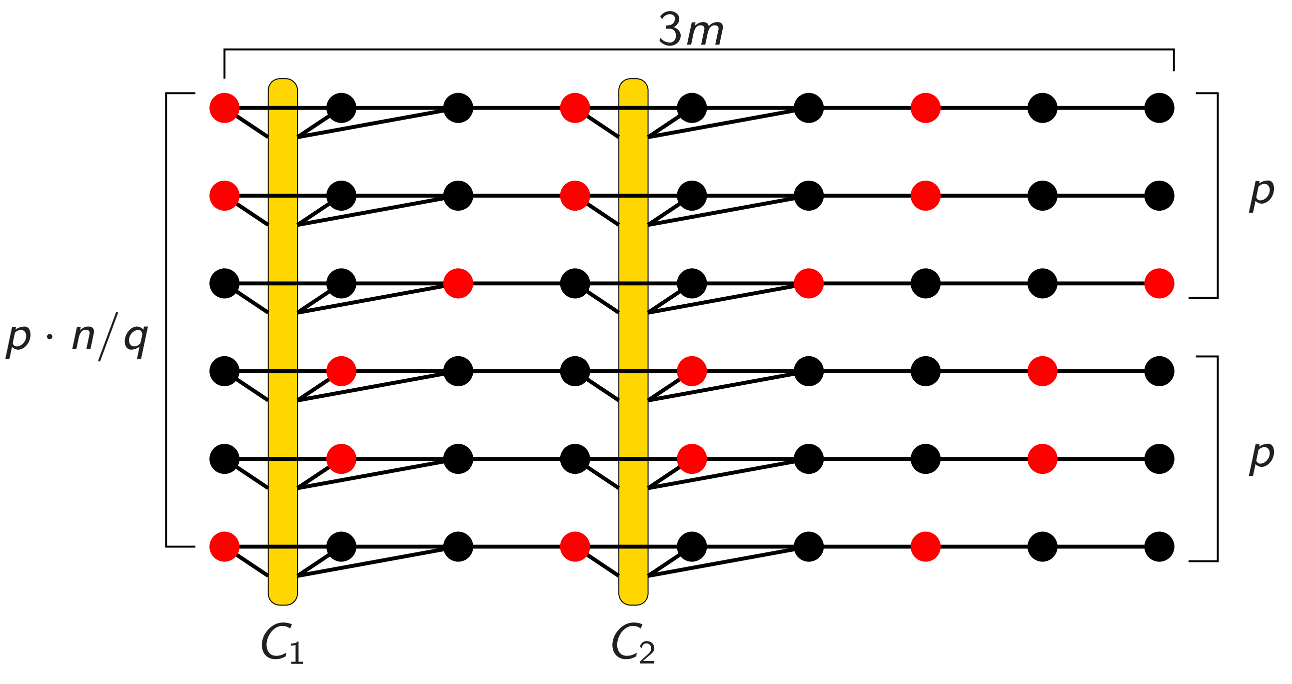

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Decompositions?

We know that INDEPENDENT SET

- Can be solved in time $2^w \cdot n$ if a tree decomposition of width w is given in the input.
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What if the graph has treewidth w, but no tree decomposition is given in the input?

Theorem: [Bodlaender '96] Width w decomposition in time $2^{O(w^3)} \cdot n$.
Theorem: [Robertson and Seymour '95] 4-approximation in time $3^{3w} \cdot \mathrm{poly} n$.
Theorem: [Feige et al. '05] $\sqrt{\log w}$ approximation in polynomial time.

To have a $2^{(1+o(1))w}$ algorithm, we would need a $(1+o(1))$ approximation in time $2^{(1+o(1))w}$.

Conclusions

- Tight lower bounds for several basic problems on tree decompositions.
- Are there other problems where we can show that there is no $(c-\epsilon)^k \cdot n^{O(1)}$ time algorithm (where k is something else than treewidth)?
  Example: Can we solve STEINER TREE with k terminals in time $(2-\epsilon)^k \cdot n^{O(1)}$?

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  Example: Can we solve STEINER TREE with k terminals in time $(2-\epsilon)^k \cdot n^{O(1)}$?
- Results are conditional on SETH.
  - If you believe SETH: our results are strong lower bounds.
  - If you don't believe SETH: our results show that improving the algorithms requires an improved general SAT algorithm, and hence not a graph theory/treewidth related question.