Known Algorithms on Graphs of Bounded Treewidth are Probably Optimal

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Joint work with

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**Treewidth**: A measure of how “tree-like” the graph is. (Introduced by Robertson and Seymour in the Graph Minors project.)

Significance:

- Appears naturally in graph structure theory.
- Polynomial or even linear algorithms for NP-hard problems on bounded treewidth graphs.
- Crucial tool for planar approximation schemes.
- Useful for fixed-parameter tractability results.
**Tree decomposition:** Vertices are arranged in a tree structure satisfying the following properties:

1. If $u$ and $v$ are neighbors, then there is a bag containing both of them.
2. For every vertex $v$, the bags containing $v$ form a connected subtree.
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**Fact:** Given a tree decomposition of width $w$, MAX INDEPENDENT SET can be solved in time $O(2^w \cdot n)$.

$B_x$: vertices appearing in node $x$.
$V_x$: vertices appearing in the subtree rooted at $x$.

1. Define table $M[x, S]$: the maximum weight of an independent set $I \subseteq V_x$ with $I \cap B_x = S$.
2. Compute the tables in bottom-up order.
3. Size of each table is $2^{w+1}$.
Given a tree decomposition of width $w$, dynamic programming gives:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Time Complexity</th>
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<tbody>
<tr>
<td>INDEPENDENT SET</td>
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<tr>
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</tr>
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</tr>
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[Various authors]
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**Question:** Can we improve the base in any of these algorithms?

**Supporting evidence:** Running time matches the obvious DP table size. But...
DOMINATING SET

- Obvious approach: $9^w$ [Telle and Proskurowski ’93]
- More clever algorithm: $4^w$ [Alber et al. ’02]
- Even more clever algorithm: $3^w$ [Rooij et al. ’09] using fast subset convolution of [Björklund et al. ’07]
Some history

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DIRECTED FEEDBACK VERTEX SET

- Trivial $2^n$ algorithm.
- Nontrivial $1.9977^n$ algorithm [Razgon '07]
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**Strong Exponential Time Hypothesis (SETH):**

$$s_k = \inf \{ \delta \mid \text{n-variable } k\text{-SAT can be solved in } 2^{\delta n} \}$$

**Conjecture:** [Impagliazzo-Paturi ’01] $s_k \to 1$

We can use a somewhat weaker assumption:

**No faster SAT:**

**Conjecture:** $n$-variable $m$-clause SAT (with arbitrary clause length) cannot be solved in time $(2 - \epsilon)^n \cdot \text{poly}(m)$ for any $\epsilon > 0$. 
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Results

Main result: If the Strong Exponential Time Hypothesis (SETH) is true, then given a tree decomposition of width $w$,

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The lower bounds match the known algorithms (up to the $\epsilon$ in the base).

Note: For some problems, we can obtain stronger results by proving the same lower bound with respect to pathwidth or feedback vertex number.
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Suppose we have a reduction:

\[
\begin{align*}
&\text{n-variable SAT instance} \\
\Rightarrow \\
&\text{INDEPENDENT SET instance of treewidth } w \leq c \cdot n.
\end{align*}
\]

Then:

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\begin{align*}
&(2 - \epsilon)^{c \cdot n} \text{ algorithm for SAT} \\
\iff \\
&(2 - \epsilon)^w \cdot n^{O(1)} \text{ algorithm for INDEPENDENT SET}
\end{align*}
\]

To get a \((2 - \epsilon)^w\) lower bound, we need \(c \leq 1\).
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\[ n \]-variable \text{SAT} \text{ instance} \Rightarrow \text{INDEPENDENT SET instance of treewidth } w \leq c \cdot n. \]

Then:

\[ (2 - \epsilon)^c \cdot n \text{ algorithm for SAT} \iff (2 - \epsilon)^w \cdot n^{O(1)} \text{ algorithm for INDEPENDENT SET} \]

- To get a \((2 - \epsilon)^w\) lower bound, we need \(c \leq 1\).

- **More generally:** For any \(c\), we get a \((2^{1/c} - \epsilon)^w\) lower bound.
  \[ \Rightarrow \] To get a \((3 - \epsilon)^w\) lower bound (e.g., for DOMINATING SET), we need \(c \leq \log_3 2 \approx 0.631\).
How large is the treewidth in the textbook reduction from SAT to INDEPENDENT SET?
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Treewidth is about $2n$, which gives a $(2^{\frac{3}{2}} - \epsilon)^w \approx 1.41^w$ lower bound. We need treewidth $\leq n$ for the $(2 - \epsilon)^w$ lower bound.
New reduction for INDEPENDENT SET

\[ n \text{ variables, } m \text{ clauses} \Rightarrow n \text{ paths of } 2m \text{ vertices each} \]

\[ 2 \text{ states per each variable} \Rightarrow 2 \text{ possible states for each path} \]
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Clauses gadgets check that every clause is satisfied. 
Treewidth is only $n + O(1)$. 

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Now there are 3 possible optimal states for each path:

- edge connections diagram
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Partition variables into $n/q$ groups of size $q = O(1)$. The $2^q$ possibilities for a group of variables are represented by a group of $p$ paths, where $2^q \leq 3^p$, i.e.,

\[ p = \lceil \log_3 2^q \rceil \approx 0.631 q. \]

⇒ Treewidth is $n \cdot \log_3 2$ and the $(3 - \epsilon)^w$ bound follows.
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\[ \Rightarrow \] Treewidth is \( n \cdot \log_3 2 \) and the \((3 - \epsilon)^w\) bound follows.
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Decompositions?

We know that INDEPENDENT SET

- Can be solved in time $2^w \cdot n$ if a tree decomposition of width $w$ is given in the input.
- Cannot be solved in time $(2 - \epsilon)^w \cdot n^{O(1)}$ for any $\epsilon > 0$ even if a tree decomposition of width $w$ is given input.
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What if the graph has treewidth $w$, but no tree decomposition is given in the input?

**Theorem:** [Bodlaender ’96] Width $w$ decomposition in time $2^{O(w^3)} \cdot n$.

**Theorem:** [Robertson and Seymour ’95] 4-approximation in time $3^{3w} \cdot \text{poly} n$.

**Theorem:** [Feige et al. ’05] $\sqrt{\log w}$ approximation in polynomial time.

To have a $2^{(1+o(1))w}$ algorithm, we would need a $(1+o(1))$ approximation in time $2^{(1+o(1))w}$.
Conclusions

- Tight lower bounds for several basic problems on tree decompositions.

- Are there other problems where we can show that there is no 
  
  \[(c - \epsilon)^k \cdot n^{O(1)}\] 
  
  time algorithm (where \(k\) is something else than treewidth)?

**Example:** Can we solve STEINER TREE with \(k\) terminals in time 

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  **Example:** Can we solve STEINER TREE with \(k\) terminals in time 
  \((2 - \epsilon)^k \cdot n^{O(1)}\)?
- Results are conditional on SETH.
  - If you believe SETH: our results are strong lower bounds.
  - If you don’t believe SETH: our results show that improving the 
    algorithms requires an improved general SAT algorithm, and hence not 
    a graph theory/treewidth related question.