Fine-Grained Complexity and Algorithm Design Boot Camp

Lower Bounds Based on ETH

Dániel Marx

Institute for Computer Science and Control,
Hungarian Academy of Sciences (MTA SZTAKI)
Budapest, Hungary

Simons Institute, Berkeley, CA
September 3, 2015

Exponential Time Hypothesis (ETH)

Hypothesis introduced by Impagliazzo, Paturi, and Zane:

Exponential Time Hypothesis (ETH) [consequence of]

There is no $2^{o(n)}$-time algorithm for n-variable $\mathrm{3SAT}$.

Note: current best algorithm is 1.30704^n [Hertli 2011].

Note: an n-variable $\mathrm{3SAT}$ formula can have $\Omega(n^3)$ clauses.

Exponential Time Hypothesis (ETH)

Hypothesis introduced by Impagliazzo, Paturi, and Zane:

Exponential Time Hypothesis (ETH) [consequence of]

There is no $2^{o(n)}$-time algorithm for n-variable $\mathrm{3SAT}$.

Note: current best algorithm is 1.30704^n [Hertli 2011].

Note: an n-variable $\mathrm{3SAT}$ formula can have $\Omega(n^3)$ clauses.

Sparsification Lemma [Impagliazzo, Paturi, Zane 2001]

There is a $2^{o(n)}$-time algorithm for n-variable $\mathrm{3SAT}$.

$$\Updownarrow$$

There is a $2^{o(m)}$-time algorithm for m-clause $\mathrm{3SAT}$.

$$\begin{tikzcd} \mathrm{ETH} \arrow[r] \arrow[rr, bend right] & 2^{f(n)} \arrow[r] & f(k) \cdot n^{O(1)} \end{tikzcd}$$

Lower bounds for exact and parameterized problems

Lower bounds based on ETH

Exponential Time Hypothesis (ETH)

There is no $2^{o(m)}$-time algorithm for m-clause 3SAT.

The textbook reduction from 3SAT to 3-Coloring:

3SAT formula ϕ n variables m clauses	$\Rightarrow$	Graph G $O(n+m)$ vertices $O(n+m)$ edges

Corollary

Assuming ETH, there is no $2^{o(n)}$ algorithm for 3-Coloring on an n-vertex graph G.

Lower bounds based on ETH

Exponential Time Hypothesis (ETH)

There is no $2^{o(m)}$-time algorithm for m-clause 3SAT.

The textbook reduction from 3SAT to 3-Coloring:

3SAT formula ϕ n variables m clauses	$\Rightarrow$	Graph G $O(m)$ vertices $O(m)$ edges

Corollary

Assuming ETH, there is no $2^{o(n)}$ algorithm for 3-Coloring on an n-vertex graph G.

Transfering bounds

There are polynomial-time reductions from, say, 3-Coloring to many other problems such that the reduction increases the number of vertices by at most a constant factor.

Consequence: Assuming ETH, there is no $2^{o(n)}$ time algorithm on n-vertex graphs for

- Independent Set
- Clique
- Dominating Set
- Vertex Cover
- Hamiltonian Path
- Feedback Vertex Set
- ...

Transfering bounds

There are polynomial-time reductions from, say, 3-Coloring to many other problems such that the reduction increases the number of vertices by at most a constant factor.

Consequence: Assuming ETH, there is no $2^{o(k)} \cdot n^{O(1)}$ time algorithm for

- k-Independent Set
- k-Clique
- k-Dominating Set
- k-Vertex Cover
- k-Path
- k-Feedback Vertex Set
- ...

Transfering bounds

There are polynomial-time reductions from, say, 3-COLORING to many other problems such that the reduction increases the number of vertices by at most a constant factor.

Consequence: Assuming ETH, there is no $2^{o(k)} \cdot n^{O(1)}$ time algorithm for

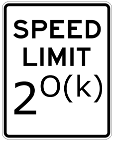
- ~~k-INDEPENDENT SET~~
- ~~k-CLIQUE~~
- ~~k-DOMINATING SET~~
- k-VERTEX COVER
- k-PATH
- k-FEEDBACK VERTEX SET
- ...

Transfering bounds

There are polynomial-time reductions from, say, 3-Coloring to many other problems such that the reduction increases the number of vertices by at most a constant factor.

Consequence: Assuming ETH, there is no $2^{o(k)} \cdot n^{O(1)}$ time algorithm for

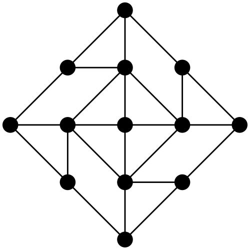
- ~~k-Independent Set~~
- ~~k-Clique~~
- ~~k-Dominating Set~~
- k-Vertex Cover
- k-Path
- k-Feedback Vertex Set
- ...



Lower bounds based on ETH

What about 3-Coloring on planar graphs?

The textbook reduction from 3-Coloring to Planar 3-Coloring uses a "crossover gadget" with 4 external connectors:

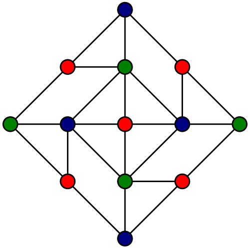


- In every 3-coloring of the gadget, opposite external connectors have the same color.
- Every coloring of the external connectors where the opposite vertices have the same color can be extended to the whole gadget.
- If two edges cross, replace them with a crossover gadget.

Lower bounds based on ETH

What about 3-Coloring on planar graphs?

The textbook reduction from 3-Coloring to Planar 3-Coloring uses a "crossover gadget" with 4 external connectors:

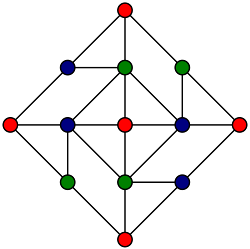


- In every 3-coloring of the gadget, opposite external connectors have the same color.
- Every coloring of the external connectors where the opposite vertices have the same color can be extended to the whole gadget.
- If two edges cross, replace them with a crossover gadget.

Lower bounds based on ETH

What about 3-Coloring on planar graphs?

The textbook reduction from 3-Coloring to Planar 3-Coloring uses a "crossover gadget" with 4 external connectors:



- In every 3-coloring of the gadget, opposite external connectors have the same color.
- Every coloring of the external connectors where the opposite vertices have the same color can be extended to the whole gadget.
- If two edges cross, replace them with a crossover gadget.

Lower bounds based on ETH

- The reduction from 3-COLORING to PLANAR 3-COLORING introduces $O(1)$ new edges/vertices for each crossing.
- A graph with m edges can be drawn with $O(m^2)$ crossings.

3SAT formula ϕ n variables m clauses	$\Rightarrow$	Graph G $O(m)$ vertices $O(m)$ edges	$\Rightarrow$	Planar graph G' $O(m^2)$ vertices $O(m^2)$ edges

Corollary

Assuming ETH, there is no $2^{o(\sqrt{n})}$ algorithm for 3-COLORING on an n-vertex planar graph G.

(Essentially observed by [Cai and Juedes 2001])

Lower bounds for planar problems

Consequence: Assuming ETH, there is no $2^{o(\sqrt{n})}$ time algorithm on n-vertex **planar graphs** for

- Independent Set
- Dominating Set
- Vertex Cover
- Hamiltonian Path
- Feedback Vertex Set
- ...

Lower bounds for planar problems

Consequence: Assuming ETH, there is no $2^{o(\sqrt{k})} \cdot n^{O(1)}$ time algorithm on **planar graphs** for

- k-Independent Set
- k-Dominating Set
- k-Vertex Cover
- k-Path
- k-Feedback Vertex Set
- ...

Lower bounds for planar problems

Consequence: Assuming ETH, there is no $2^{o(\sqrt{k})} \cdot n^{O(1)}$ time algorithm on **planar graphs** for

- k-Independent Set
- k-Dominating Set
- k-Vertex Cover
- k-Path
- k-Feedback Vertex Set
- ...

Note: Reduction to planar graphs does not work for Clique (why?).

Treewidth

Recall from Tuesday:
FPT algorithms parameterized by treewidth.

Treewidth

Given a tree decomposition of width w, FPT algorithms with running time $2^{O(w)} \cdot n^{O(1)}$ for

- Independent Set
- Dominating Set
- 3-Coloring
- Hamiltonian Cycle
- ...

Treewidth

Given a tree decomposition of width w, FPT algorithms with running time $2^{O(w)} \cdot n^{O(1)}$ for

- Independent Set
- Dominating Set
- 3-Coloring
- Hamiltonian Cycle
- ...

Observation: A $2^{o(w)} \cdot n^{O(1)}$ algorithm implies a $2^{o(n)} \cdot n^{O(1)}$ algorithm.

$\Rightarrow$ Assuming ETH, no $2^{o(w)} \cdot n^{O(1)}$ algorithms for these problems!

Treewidth

The following problems have $w^{O(w)} \cdot n^{O(1)} = 2^{O(w \log w)} \cdot n^{O(1)}$ algorithms:

- Vertex Coloring
- Cycle Packing
- Vertex Disjoint Paths

Treewidth

The following problems have $w^{O(w)} \cdot n^{O(1)} = 2^{O(w \log w)} \cdot n^{O(1)}$ algorithms:

- Vertex Coloring
- Cycle Packing
- Vertex Disjoint Paths

... and assuming ETH, they do not have $2^{o(w \log w)} \cdot n^{O(1)}$ algorithms.

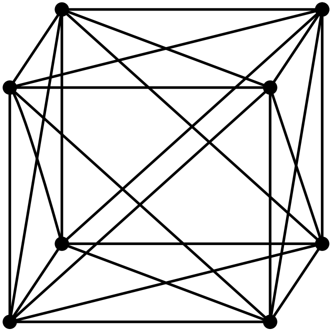
Proof: Reduce an instance of a graph problem on N vertices to an instance with treewidth $O(N/\log N)$.

Edge Clique Cover

Edge Clique Cover: Given a graph G and an integer k, cover the edges of G with at most k cliques.

(the cliques need not be edge disjoint)

Equivalently: can G be represented as an intersection graph over a k element universe?

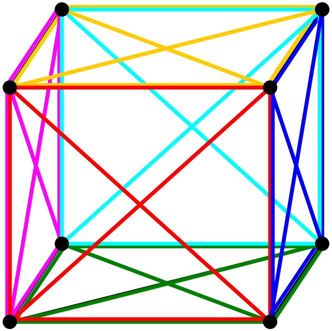


Edge Clique Cover

Edge Clique Cover: Given a graph G and an integer k, cover the edges of G with at most k cliques.

(the cliques need not be edge disjoint)

Equivalently: can G be represented as an intersection graph over a k element universe?

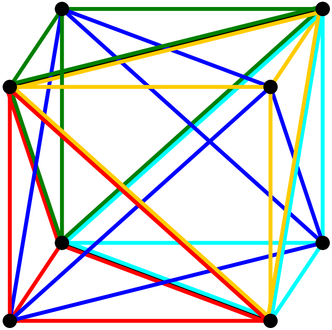


6 cliques

Edge Clique Cover

> **Edge Clique Cover**: Given a graph G and an integer k, cover the edges of G with at most k cliques.
>
> (the cliques need not be edge disjoint)

Equivalently: can G be represented as an intersection graph over a k element universe?



5 cliques

Edge Clique Cover

Edge Clique Cover: Given a graph G and an integer k, cover the edges of G with at most k cliques.

(the cliques need not be edge disjoint)

Simple algorithm (sketch)

- If two adjacent vertices have the same neighborhood ("twins"), then remove one of them.
- If there are no twins and $|V(G)| > 2^k$, then there is no solution.
- Use brute force.

Running time: $2^{2^{O(k)}} \cdot n^{O(1)}$ — double exponential dependence on k!

Edge Clique Cover

Edge Clique Cover: Given a graph G and an integer k, cover the edges of G with at most k cliques.

(the cliques need not be edge disjoint)

Double-exponential dependence on k cannot be avoided!

Theorem [Cygan, Pilipczuk, Pilipczuk 2013]

Assuming ETH, there is no $2^{2^{o(k)}} \cdot n^{O(1)}$ time algorithm for Edge Clique Cover.

Proof: Reduce an n-variable 3SAT instance into and instance of Edge Clique Cover with $k = O(\log n)$.

ETH $\longrightarrow$ $n^{f(k)}$

Lower bounds for W[1]-hard problems

Exponential Time Hypothesis

Engineers' Hypothesis

k-Clique cannot be solved in time $f(k) \cdot n^{O(1)}$.

$\Updownarrow$

Theorists' Hypothesis

k-Step Halting Problem (is there a path of the given NTM that stops in k steps?) cannot be solved in time $f(k) \cdot n^{O(1)}$.

$\Uparrow$

Exponential Time Hypothesis (ETH)

n-variable 3SAT cannot be solved in time $2^{o(n)}$.

What do we have to show to prove that ETH implies Engineers' Hypothesis?

Exponential Time Hypothesis

Engineers' Hypothesis

k-CLIQUE cannot be solved in time $f(k) \cdot n^{O(1)}$.

$\Updownarrow$

Theorists' Hypothesis

k-STEP HALTING PROBLEM (is there a path of the given NTM that stops in k steps?) cannot be solved in time $f(k) \cdot n^{O(1)}$.

$\Uparrow$

Exponential Time Hypothesis (ETH)

n-variable 3SAT cannot be solved in time $2^{o(n)}$.

What do we have to show to prove that ETH implies Engineers' Hypothesis?

We have to show that an $f(k) \cdot n^{O(1)}$ algorithm implies that there is a $2^{o(n)}$ time algorithm for n-variable 3SAT.

Exponential Time Hypothesis

Engineers' Hypothesis

k-Clique cannot be solved in time $f(k) \cdot n^{O(1)}$.

$\Updownarrow$

Theorists' Hypothesis

k-Step Halting Problem (is there a path of the given NTM that stops in k steps?) cannot be solved in time $f(k) \cdot n^{O(1)}$.

$\Uparrow$

Exponential Time Hypothesis (ETH)

n-variable 3SAT cannot be solved in time $2^{o(n)}$.

We actually show something much stronger and more interesting:

Theorem [Chen et al. 2004]

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for k-Clique for any computable function f.

Lower bound on the exponent

Theorem [Chen et al. 2004]

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for k-Clique for any computable function f.

Suppose that k-Clique can be solved in time $f(k) \cdot n^{k/s(k)}$, where $s(k)$ is a monotone increasing unbounded function. We use this algorithm to solve 3-Coloring on an n-vertex graph G in time $2^{o(n)}$.

Lower bound on the exponent

Theorem [Chen et al. 2004]

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for k-Clique for any computable function f.

Suppose that k-Clique can be solved in time $f(k) \cdot n^{k/s(k)}$, where $s(k)$ is a monotone increasing unbounded function. We use this algorithm to solve 3-Coloring on an n-vertex graph G in time $2^{o(n)}$.

Let k be the largest integer such that $f(k) \leq n$ and $k^{k/s(k)} \leq n$. Function $k := k(n)$ is monotone increasing and unbounded.

Split the vertices of G into k groups. Let us build a graph H where each vertex corresponds to a proper 3-coloring of one of the groups. Connect two vertices if they are not conflicting.

Lower bound on the exponent

Theorem [Chen et al. 2004]

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for k-Clique for any computable function f.

Suppose that k-Clique can be solved in time $f(k) \cdot n^{k/s(k)}$, where $s(k)$ is a monotone increasing unbounded function. We use this algorithm to solve 3-Coloring on an n-vertex graph G in time $2^{o(n)}$.

Let k be the largest integer such that $f(k) \le n$ and $k^{k/s(k)} \le n$. Function $k := k(n)$ is monotone increasing and unbounded.

Split the vertices of G into k groups. Let us build a graph H where each vertex corresponds to a proper 3-coloring of one of the groups. Connect two vertices if they are not conflicting.

Every k-clique of H corresponds to a proper 3-coloring of G.

$\Rightarrow$ A 3-coloring of G can be found in time $f(k) \cdot |V(H)|^{k/s(k)} \le n \cdot (k3^{n/k})^{k/s(k)} = n \cdot k^{k/s(k)} \cdot 3^{n/s(k)} = 2^{o(n)}$.

Tight bounds

Theorem [Chen et al. 2004]

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for k-Clique for any computable function f.

Transfering to other problems:

k-Clique (x, k)	$\Rightarrow$	Problem A $(x', O(k))$
$f(k) \cdot n^{o(k)}$ algorithm	$\Leftarrow$	$f(k) \cdot n^{o(k)}$ algorithm

Tight bounds

Theorem [Chen et al. 2004]

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for k-Clique for any computable function f.

Transfering to other problems:

k-Clique (x, k)	$\Rightarrow$	Problem A $(x', g(k))$
$f(k) \cdot n^{o(k)}$ algorithm	$\Leftarrow$	$f(k) \cdot n^{o(g^{-1}(k))}$ algorithm

Tight bounds

Theorem [Chen et al. 2004]

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for k-Clique for any computable function f.

Transfering to other problems:

k-Clique (x, k)	$\Rightarrow$	Problem A (x', k^2)
$f(k) \cdot n^{o(k)}$ algorithm	$\Leftarrow$	$f(k) \cdot n^{o(\sqrt{k})}$ algorithm

Tight bounds

Theorem [Chen et al. 2004]

Assuming ETH, there is no $f(k) \cdot n^{o(k)}$ algorithm for k-Clique for any computable function f.

Transfering to other problems:

k-Clique (x, k)	$\Rightarrow$	Problem A (x', k^2)
$f(k) \cdot n^{o(k)}$ algorithm	$\Leftarrow$	$f(k) \cdot n^{o(\sqrt{k})}$ algorithm

Bottom line:

- To rule out $f(k) \cdot n^{o(k)}$ algorithms, we need a parameterized reduction that blows up the parameter at most *linearly.*
- To rule out $f(k) \cdot n^{o(\sqrt{k})}$ algorithms, we need a parameterized reduction that blows up the parameter at most *quadratically.*

Tight bounds

Assuming ETH, there is no $f(k)n^{o(k)}$ time algorithms for

- Set Cover
- Hitting Set
- Connected Dominating Set
- Independent Dominating Set
- Partial Vertex Cover
- Dominating Set in bipartite graphs
- ...

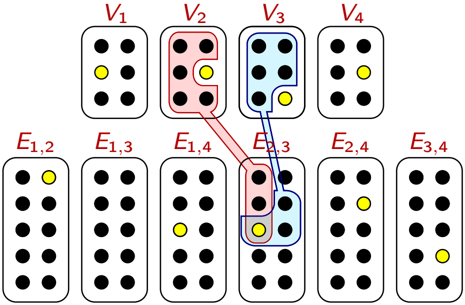
The odd case of ODD SET

ODD SET: Given a set system $\mathcal{F}$ over a universe U and an integer k, find a set S of at most k elements such that $|S \cap F|$ is odd for every $F \in \mathcal{F}$.

We have seen:

Theorem

ODD SET is W[1]-hard parameterized by k.



New parameter: $k' := k + \binom{k}{2} = O(k^2)$.

The odd case of Odd Set

Odd Set: Given a set system $\mathcal{F}$ over a universe U and an integer k, find a set S of at most k elements such that $|S \cap F|$ is odd for every $F \in \mathcal{F}$.

We have seen:

Theorem

Odd Set is W[1]-hard parameterized by k.

We immediately get:

Corollary

Assuming ETH, there is no $f(k)n^{o(\sqrt{k})}$ time algorithm for Odd Set.

But this does not seem to be tight...

Problem: k-Clique is a very densely constrained problem, which makes the reduction very expensive.

SUBGRAPH ISOMORPHISM

SUBGRAPH ISOMORPHISM: Given two graphs H and G, decide if H is isomorphic to a subgraph of G.

Trivial reduction from k-CLIQUE:

Corollary (parameterized by no. of **vertices** of H)

Assuming ETH, SUBGRAPH ISOMORPHISM parameterized by $k := |V(H)|$ has no $f(k)n^{o(k)}$ time algorithm.

Subgraph Isomorphism

Subgraph Isomorphism: Given two graphs H and G, decide if H is isomorphic to a subgraph of G.

Trivial reduction from k-Clique:

Corollary (parameterized by no. of **edges** of H)

Assuming ETH, Subgraph Isomorphism parameterized by $k := |E(H)|$ has no $f(k)n^{o(\sqrt{k})}$ time algorithm.

Is this tight?

Subgraph Isomorphism

Subgraph Isomorphism: Given two graphs H and G, decide if H is isomorphic to a subgraph of G.

Trivial reduction from k-Clique:

Corollary (parameterized by no. of **edges** of H)

Assuming ETH, Subgraph Isomorphism parameterized by $k := |E(H)|$ has no $f(k)n^{o(\sqrt{k})}$ time algorithm.

Is this tight?

An almost tight result:

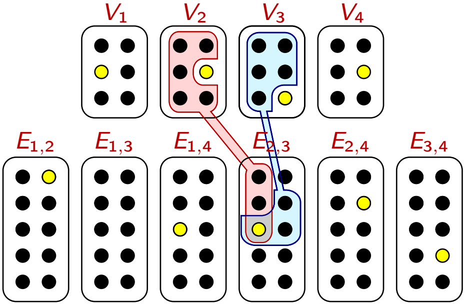
Theorem [M. 2010]

Assuming ETH, Subgraph Isomorphism parameterized by $k := |E(H)|$ has no $f(k)n^{o(k/\log k)}$ time algorithm.

Open question: can we remove the $\log k$ from this lower bound?

ODD SET

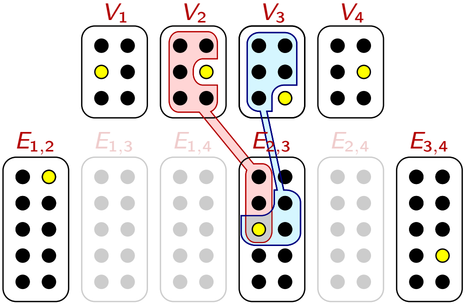
Reduction from k-CLIQUE to ODD SET:



New parameter: $k' := k + \binom{k}{2} = O(k^2)$.

Odd Set

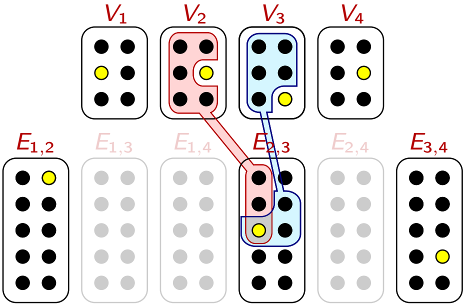
Reduction from Subgraph Isomorphism to Odd Set:



New parameter: $k' := |V(H)| + |E(H)| = O(k)$.
(where $k := |E(H)|$)

Odd Set

Reduction from Subgraph Isomorphism to Odd Set:



New parameter: $k' := |V(H)| + |E(H)| = O(k)$.
(where $k := |E(H)|$)

Theorem

Assuming ETH, there is no $f(k)n^{o(k/\log k)}$ time algorithm for Odd Set.

Tight bounds

Assuming ETH, there is no $f(k)n^{o(k)}$ time algorithms for

- SET COVER
- HITTING SET
- CONNECTED DOMINATING SET
- INDEPENDENT DOMINATING SET
- PARTIAL VERTEX COVER
- DOMINATING SET in bipartite graphs
- ...

Tight bounds

Assuming ETH, there is no $f(k)n^{o(k)}$ time algorithms for

- Set Cover
- Hitting Set
- Connected Dominating Set
- Independent Dominating Set
- Partial Vertex Cover
- Dominating Set in bipartite graphs
- ...

What about planar problems?

- More problems are FPT, more difficult to prove W[1]-hardness.
- The problem Grid Tiling is the key to many of these results.

Grid Tiling

Grid Tiling

Input: A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

Find: A pair $s_{i,j} \in S_{i,j}$ for each cell such that

- Vertical neighbors agree in the 1st coordinate.
- Horizontal neighbors agree in the 2nd coordinate.

(1,1) (3,1) (2,4)	(5,1) (1,4) (5,3)	(1,1) (2,4) (3,3)
(2,2) (1,4)	(3,1) (1,2)	(2,2) (2,3)
(1,3) (2,3) (3,3)	(1,1) (1,3)	(2,3) (5,3)

$k = 3$, $D = 5$

Grid Tiling

Grid Tiling

Input: A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

Find: A pair $s_{i,j} \in S_{i,j}$ for each cell such that

- Vertical neighbors agree in the 1st coordinate.
- Horizontal neighbors agree in the 2nd coordinate.

(1,1) (3,1) (2,4)	(5,1) (1,4) (5,3)	(1,1) (2,4) (3,3)
(2,2) (1,4)	(3,1) (1,2)	(2,2) (2,3)
(1,3) (2,3) (3,3)	(1,1) (1,3)	(2,3) (5,3)

$k = 3,\ D = 5$

Grid Tiling

Grid Tiling

Input: A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

Find: A pair $s_{i,j} \in S_{i,j}$ for each cell such that

- Vertical neighbors agree in the 1st coordinate.
- Horizontal neighbors agree in the 2nd coordinate.

Simple proof:

Fact

There is a parameterized reduction from k-Clique to $k \times k$ Grid Tiling.

Grid Tiling is W[1]-hard

Reduction from k-Clique

Definition of the sets:

- For $i = j$: $(x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $(x, y) \in S_{i,j} \iff x$ and y are adjacent.

	(v_i, v_i)			

Each diagonal cell defines a value v_i . . .

Grid Tiling is W[1]-hard

Reduction from k-Clique

Definition of the sets:

- For $i = j$: $(x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $(x, y) \in S_{i,j} \iff x$ and y are adjacent.

	$(v_i, .)$			
$(., v_i)$	(v_i, v_i)	$(., v_i)$	$(., v_i)$	$(., v_i)$
	$(v_i, .)$			
	$(v_i .,)$			
	$(v_i, .)$			

. . . which appears on a "cross"

Grid Tiling is W[1]-hard

Reduction from k-Clique

Definition of the sets:

- For $i = j$: $(x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $(x, y) \in S_{i,j} \iff x$ and y are adjacent.

	$(v_i, .)$			
$(., v_i)$	(v_i, v_i)	$(., v_i)$	$(., v_i)$	$(., v_i)$
	$(v_i, .)$			
	$(v_i, .)$		(v_j, v_j)	
	$(v_i, .)$			

v_i and v_j are adjacent for every $1 \leq i < j \leq k$.

Grid Tiling is W[1]-hard

Reduction from k-Clique

Definition of the sets:

- For $i = j$: $(x, y) \in S_{i,j} \iff x = y$
- For $i \neq j$: $(x, y) \in S_{i,j} \iff x$ and y are adjacent.

	$(v_i, .)$		$(v_j, .)$	
$(., v_i)$	(v_i, v_i)	$(., v_i)$	(v_j, v_i)	$(., v_i)$
	$(v_i, .)$		$(v_j, .)$	
$(., v_j)$	(v_i, v_j)	$(., v_j)$	(v_j, v_j)	$(., v_j)$
	$(v_i, .)$		$(v_j, .)$	

v_i and v_j are adjacent for every $1 \leq i < j \leq k$.

Grid Tiling and planar problems

Theorem

$k \times k$ Grid Tiling is W[1]-hard and, assuming ETH, cannot be solved in time $f(k)n^{o(k)}$ for any function f.

This lower bound is the key for proving hardness results for planar graphs.

Examples:

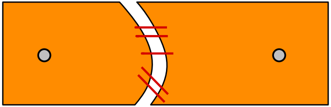
- Multiway Cut on planar graphs with k terminals
- Independent Set for unit disks

A classical problem

$s - t$ CUT

Input: A graph G, an integer p, vertices s and t

Output: A set S of at most p edges such that removing S separates s and t.



Theorem [Ford and Fulkerson 1956]

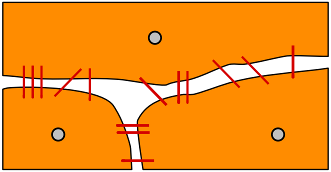
A minimum $s - t$ cut can be found in polynomial time.

What about separating more than two terminals?

More than two terminals

k-Terminal Cut (aka Multiway Cut)

Input:	A graph G, an integer p, and a set T of k terminals
Output:	A set S of at most p edges such that removing S separates any two vertices of T



Theorem [Dalhaus et al. 1994]

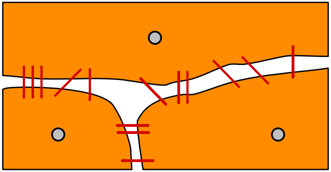
NP-hard already for $k = 3$.

More than two terminals

k-Terminal Cut (aka Multiway Cut)

Input: A graph G, an integer p, and a set T of k terminals

Output: A set S of at most p edges such that removing S separates any two vertices of T



Theorem [Dalhaus et al. 1994] [Hartvigsen 1998] [Bentz 2012]

Planar k-Terminal Cut can be solved in time $n^{O(k)}$.

Theorem [Klein and M. 2012]

Planar k-Terminal Cut can be solved in time $2^{O(k)} \cdot n^{O(\sqrt{k})}$.

Lower bounds

Theorem [Klein and M. 2012]

Planar k-Terminal Cut can be solved in time $2^{O(k)} \cdot n^{O(\sqrt{k})}$.

Natural questions:

- Is there an $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm?
- Is there an $f(k) \cdot n^{O(1)}$ time algorithm (i.e., is it fixed-parameter tractable)?

Lower bounds

Theorem [Klein and M. 2012]

PLANAR k-TERMINAL CUT can be solved in time $2^{O(k)} \cdot n^{O(\sqrt{k})}$.

Natural questions:

- Is there an $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm?
- Is there an $f(k) \cdot n^{O(1)}$ time algorithm (i.e., is it fixed-parameter tractable)?

Lower bounds:

Theorem [M. 2012]

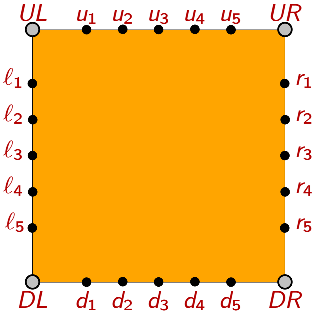
PLANAR k-TERMINAL CUT is W[1]-hard and has no $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm (assuming ETH).

Reduction from $k \times k$ Grid Tiling to Planar k^2-Terminal Cut

For every set $S_{i,j}$, we construct a gadget with 4 terminals such that

- for every $(x, y) \in S_{i,j}$, there is a minimum multiway cut that represents (x, y).
- every minimum multiway cut represents some $(x, y) \in S_{i,j}$.

Main part of the proof: constructing these gadgets.



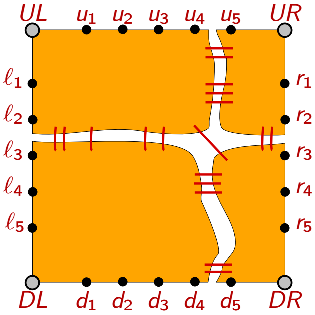
The gadget.

Reduction from $k \times k$ Grid Tiling to Planar k^2-Terminal Cut

For every set $S_{i,j}$, we construct a gadget with 4 terminals such that

- for every $(x, y) \in S_{i,j}$, there is a minimum multiway cut that represents (x, y).
- every minimum multiway cut represents some $(x, y) \in S_{i,j}$.

Main part of the proof: constructing these gadgets.



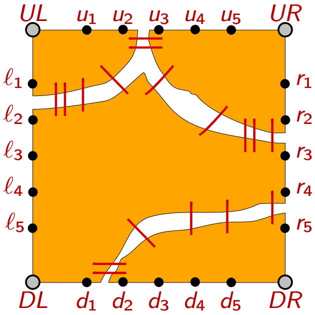
A cut representing $(4, 2)$.

Reduction from $k \times k$ Grid Tiling to Planar k^2-Terminal Cut

For every set $S_{i,j}$, we construct a gadget with 4 terminals such that

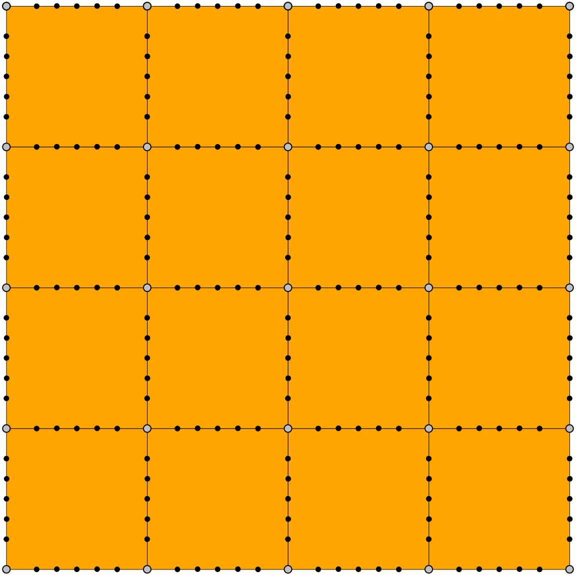
- for every $(x, y) \in S_{i,j}$, there is a minimum multiway cut that represents (x, y).
- every minimum multiway cut represents some $(x, y) \in S_{i,j}$.

Main part of the proof: constructing these gadgets.

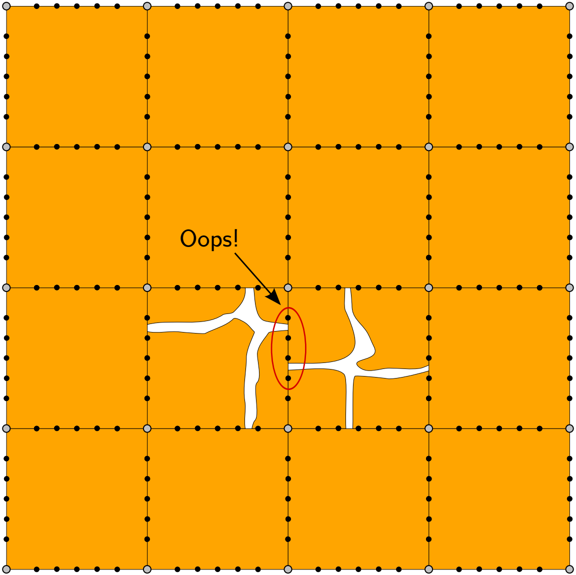


A cut not representing any pair.

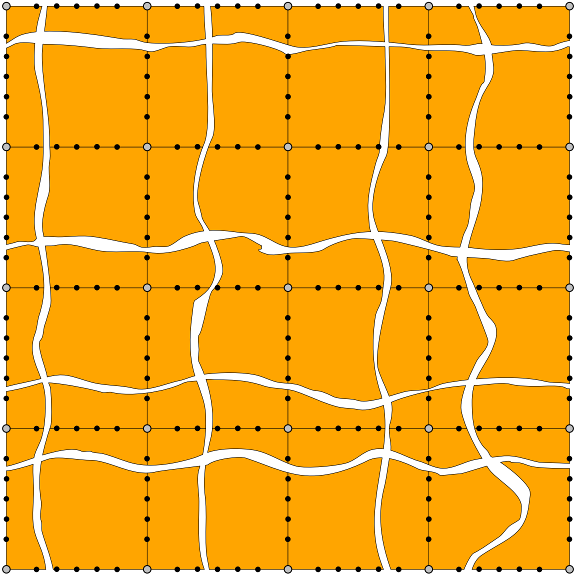
Putting together the gadgets



Putting together the gadgets



Putting together the gadgets



Grid Tiling with $\leq$

Grid Tiling with $\leq$

Input: A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

Find: A pair $s_{i,j} \in S_{i,j}$ for each cell such that

- 1st coordinate of $s_{i,j} \leq$ 1st coordinate of $s_{i+1,j}$.
- 2nd coordinate of $s_{i,j} \leq$ 2nd coordinate of $s_{i,j+1}$.

(5,1) (1,2) (3,3)	(4,3) (3,2)	(2,3) (2,5)
(2,1) (5,5) (3,5)	(4,2) (5,3)	(5,1) (3,2)
(5,1) (2,2) (5,3)	(2,1) (4,2)	(3,1) (3,2) (3,3)

$k = 3$, $D = 5$

Grid Tiling with $\leq$

Grid Tiling with $\leq$

Input: A $k \times k$ matrix and a set of pairs $S_{i,j} \subseteq [D] \times [D]$ for each cell.

Find: A pair $s_{i,j} \in S_{i,j}$ for each cell such that

- 1st coordinate of $s_{i,j} \leq$ 1st coordinate of $s_{i+1,j}$.
- 2nd coordinate of $s_{i,j} \leq$ 2nd coordinate of $s_{i,j+1}$.

Variant of the previous proof:

Theorem

There is a parameterized reduction from $k \times k$-Grid Tiling to $O(k) \times O(k)$ Grid Tiling with $\leq$.

Very useful starting point for geometric problems!

k-Independent Set for unit disks

Theorem

Given a set of n unit disks in the plane, we can find k independent disks in time $n^{O(\sqrt{k})}$.

k-Independent Set for unit disks

Theorem

Given a set of n unit disks in the plane, we can find k independent disks in time $n^{O(\sqrt{k})}$.

Matching lower bound:

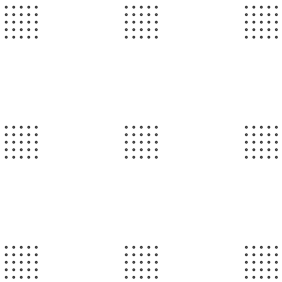
Theorem

There is a reduction from $k \times k$ Grid Tiling with $\leq$ to k^2-Independent Set for unit disks. Consequently, Independent Set for unit disks is

- is W[1]-hard, and
- cannot be solved in time $f(k)n^{o(\sqrt{k})}$ for any function f.

Reduction to unit disks

(5,1) (1,2) (3,3)	(4,3) (3,2)	(2,3) (2,5)
(2,1) (5,5) (3,5)	(4,2) (5,3)	(5,1) (3,2)
(5,1) (2,2) (5,3)	(2,1) (4,2)	(3,1) (3,2) (3,3)

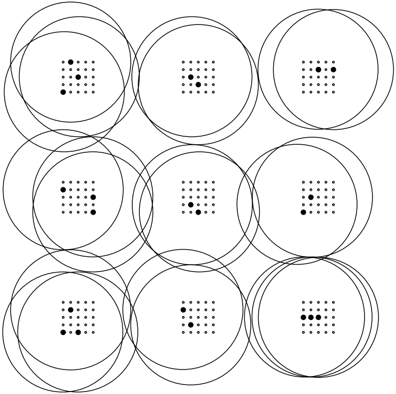


Every pair is represented by a unit disk in the plane.

$\leq$ relation between coordinates $\iff$ disks do not intersect.

Reduction to unit disks

(5,1) (1,2) (3,3)	(4,3) (3,2)	(2,3) (2,5)
(2,1) (5,5) (3,5)	(4,2) (5,3)	(5,1) (3,2)
(5,1) (2,2) (5,3)	(2,1) (4,2)	(3,1) (3,2) (3,3)

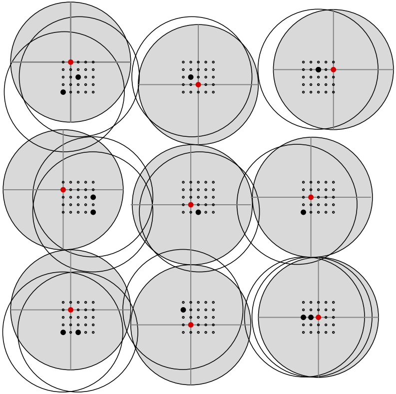


Every pair is represented by a unit disk in the plane.

$\leq$ relation between coordinates $\iff$ disks do not intersect.

Reduction to unit disks

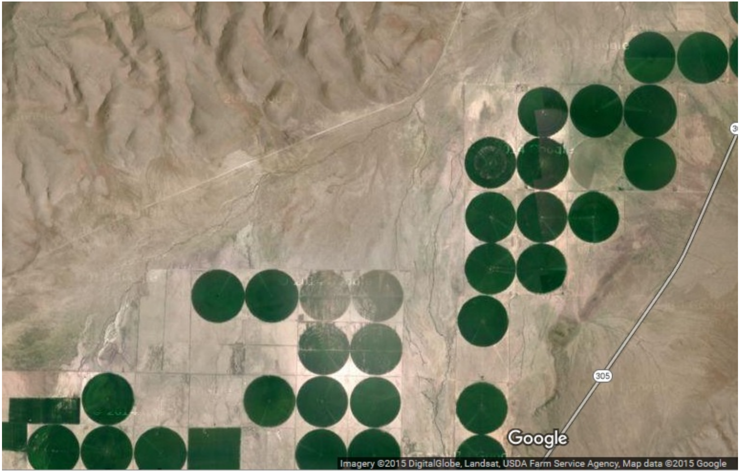
(5,1) (1,2) (3,3)	(4,3) (3,2)	(2,3) (2,5)
(2,1) (5,5) (3,5)	(4,2) (5,3)	(5,1) (3,2)
(5,1) (2,2) (5,3)	(2,1) (4,2)	(3,1) (3,2) (3,3)



Every pair is represented by a unit disk in the plane.

$\leq$ relation between coordinates $\iff$ disks do not intersect.

Center-pivot irrigation



Higher dimensions

Bidimensionalty for planar graphs:

- $2^{O(\sqrt{n})}$, $2^{O(\sqrt{k})} \cdot n^{O(1)}$, $n^{O(\sqrt{k})}$ time algorithms.
- There is no tridimensionalty!

Higher dimensions

Bidimensionality for 2-dimensional geometric problems:

- $2^{O(\sqrt{n})}$, $2^{O(\sqrt{k})} \cdot n^{O(1)}$, $n^{O(\sqrt{k})}$ time algorithms.
- What about higher dimensions?

Higher dimensions

Bidimensionality for 2-dimensional geometric problems:

- $2^{O(\sqrt{n})}$, $2^{O(\sqrt{k})} \cdot n^{O(1)}$, $n^{O(\sqrt{k})}$ time algorithms.
- What about higher dimensions?

"Limited blessing of low dimensionality:"

Theorem

Independent Set for unit spheres in d dimensions can be solved in time $n^{O(k^{1-1/d})}$.

Matching lower bound:

Theorem [M. and Sidiropoulos 2014]

Assuming ETH, Independent Set for unit spheres in d dimensions cannot be solved in time $n^{o(k^{1-1/d})}$.

Higher dimensions

Bidimensionality for 2-dimensional geometric problems:

- $2^{O(\sqrt{n})}$, $2^{O(\sqrt{k})} \cdot n^{O(1)}$, $n^{O(\sqrt{k})}$ time algorithms.
- What about higher dimensions?

"Limited blessing of low dimensionality:"

Theorem [Smith and Wormald 1998]

Euclidean TSP in d dimensions can be solved in time $2^{O(n^{1-1/d+\epsilon})}$.

Matching lower bound:

Theorem [M. and Sidiropoulos 2014]

Assuming ETH, Euclidean TSP in d dimension cannot be solved in time $2^{O(n^{1-1/d-\epsilon})}$ for any $\epsilon > 0$.

Summary

We used ETH to rule out

1. $2^{o(n)}$ time algorithms for, say, Independent Set.
2. $2^{o(\sqrt{n})}$ time algorithms for, say, Independent Set on planar graphs.
3. $2^{o(k)} \cdot n^{O(1)}$ time algorithms for, say, Vertex Cover.
4. $2^{o(\sqrt{k})} \cdot n^{O(1)}$ time algorithms for, say, Vertex Cover on planar graphs.
5. $f(k)n^{o(k)}$ time algorithms for Clique.
6. $f(k)n^{o(\sqrt{k})}$ time algorithms for planar problems such as k-Terminal Cut and Independent Set for unit disks.

Other tight lower bounds on $f(k)$ having the form $2^{o(k \log k)}$, $2^{o(k^2)}$, or $2^{2^{o(k)}}$ exist.