Improving local search using parameterized complexity

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Joint work with

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Overview

- Local search algorithms
- Parameterized complexity approach to local search
- Applying this approach for the problem of finding minimum weight solutions for Boolean CSP's.
- Main result: classification theorem.

Local search

Local search: walk in the solution space by iteratively replacing the current solution with a better solution in the local neighborhood.

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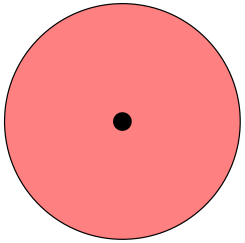

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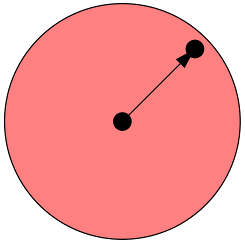

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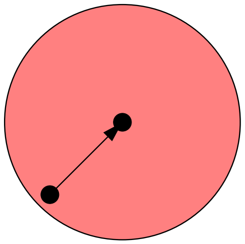

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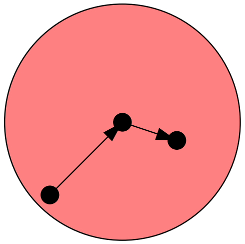

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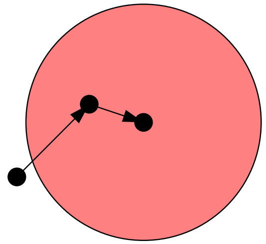

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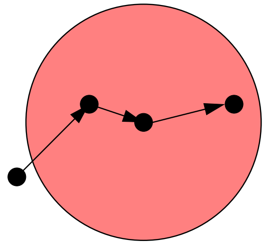

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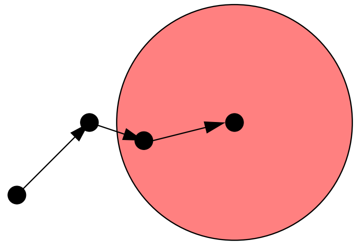

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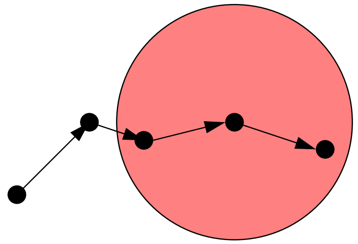

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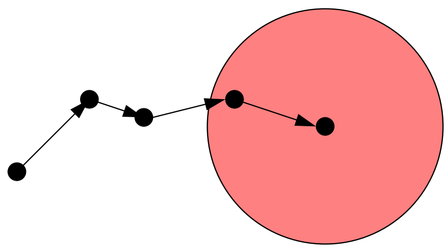

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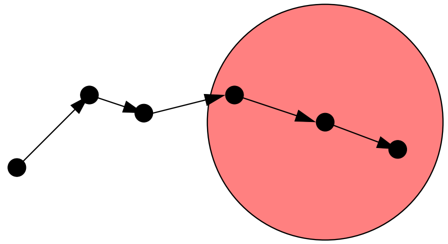

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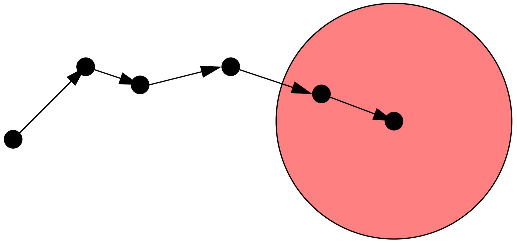

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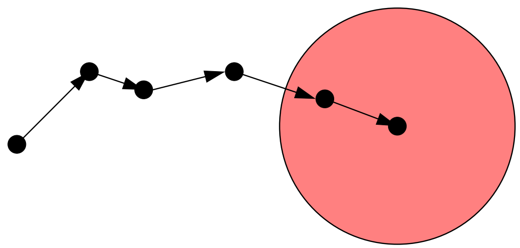

Problem: local search can stop at a local optimum (no better solution in the local neighborhood).

More sophisticated variants: simulated annealing, tabu search, etc.

Local neighborhood

The local neighborhood is defined in a problem-specific way:

- For TSP, the neighbors are obtained by swapping 2 cities or replacing 2 edges.
- For a problem with 0-1 variables, the neighbors are obtained by flipping a single variable.
- For subgraph problems, the neighbors are obtained by adding/removing one edge.

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- For subgraph problems, the neighbors are obtained by adding/removing one edge.

More generally: reordering k cities, flipping k variables, etc.

Larger neighborhood (larger k):

- algorithm is less likely to get stuck in a local optimum,
- it is more difficult to check if there is a better solution in the neighborhood.

Searching the neighborhood

Is there an efficient way of finding a better solution in the k-neighborhood?

We study the complexity of the following problem:

Input:	instance I, solution x, integer k
Decide:	Is there a solution x' with dist$(x, x') \leq k$ that is "better" than x?

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Remark 2: Size of the k-neighborhood is usually $n^{O(k)}$ $\Rightarrow$ local search is polynomial-time solvable for every fixed k, but it is not practical for larger k.

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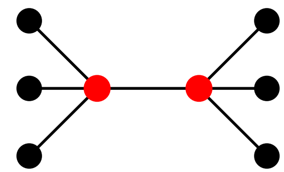
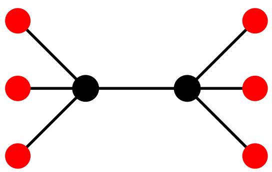
Classical complexity theory does not tell us anything useful about the complexity of local search!

Parameterized complexity

Problem:	MINIMUM VERTEX COVER	MAXIMUM INDEPENDENT SET
Input:	Graph G, integer k	Graph G, integer k
Question:	Is it possible to cover the edges with k vertices? 	Is it possible to find k independent vertices? 
Complexity:	NP-complete	NP-complete

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	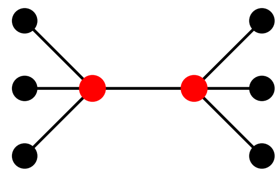	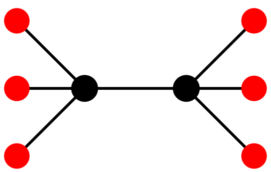
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Complete enumeration:	$O(n^k)$ possibilities	$O(n^k)$ possibilities

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	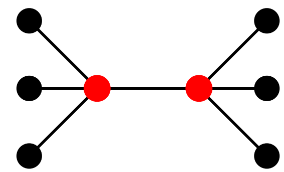	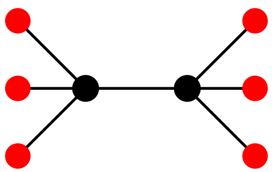
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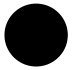
Complete enumeration:	$O(n^k)$ possibilities	$O(n^k)$ possibilities
	$O(2^k n^2)$ algorithm exists	No $n^{o(k)}$ algorithm known
		

Bounded search tree method

Algorithm for MINIMUM VERTEX COVER:

$$e_1 = x_1 y_1$$



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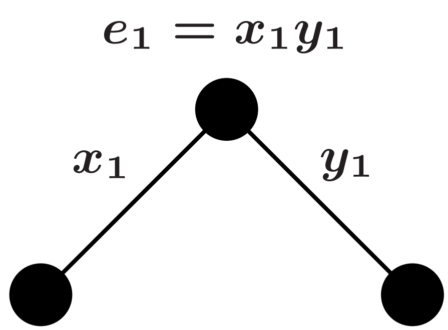

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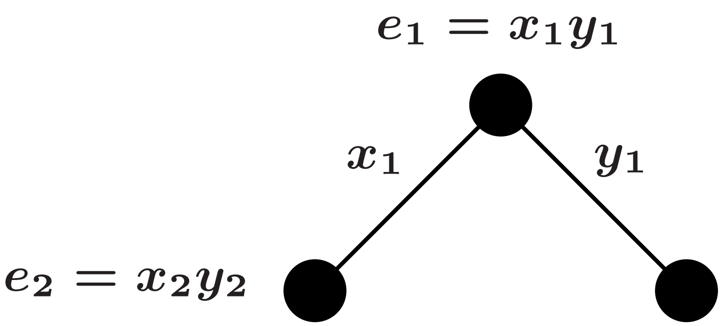

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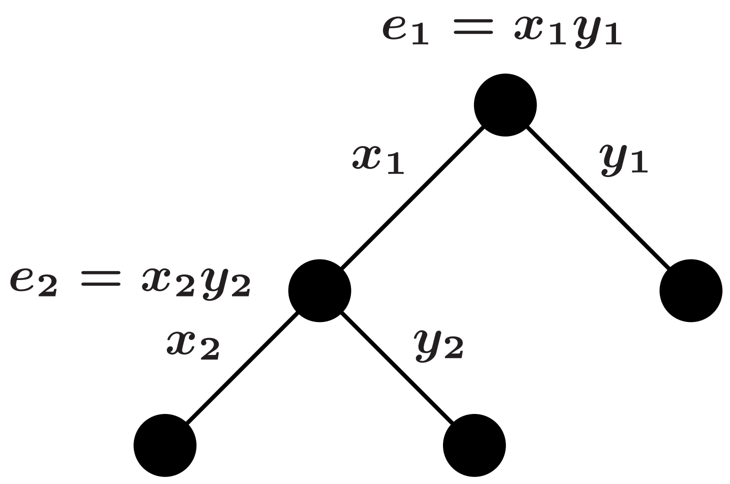

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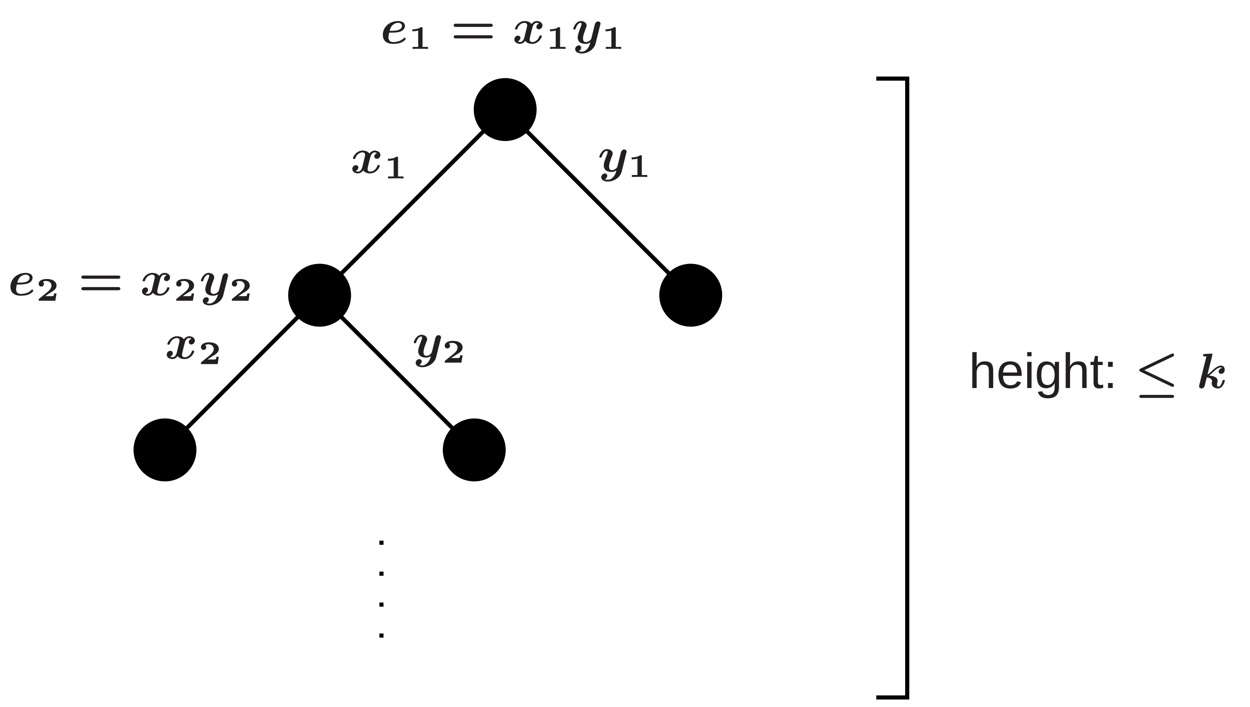

Height of the search tree is $\leq k \Rightarrow$ number of nodes is $O(2^k) \Rightarrow$ complete search requires $2^k \cdot$ poly steps.

Fixed-parameter tractability

Definition: a **parameterized problem** is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant c.

We have seen that MINIMUM VERTEX COVER is in FPT. Best known algorithm: $O(1.2832^k k + k|V|)$ [Niedermeier, Rossmanith, 2003]

Main goal of parameterized complexity: to find FPT problems.

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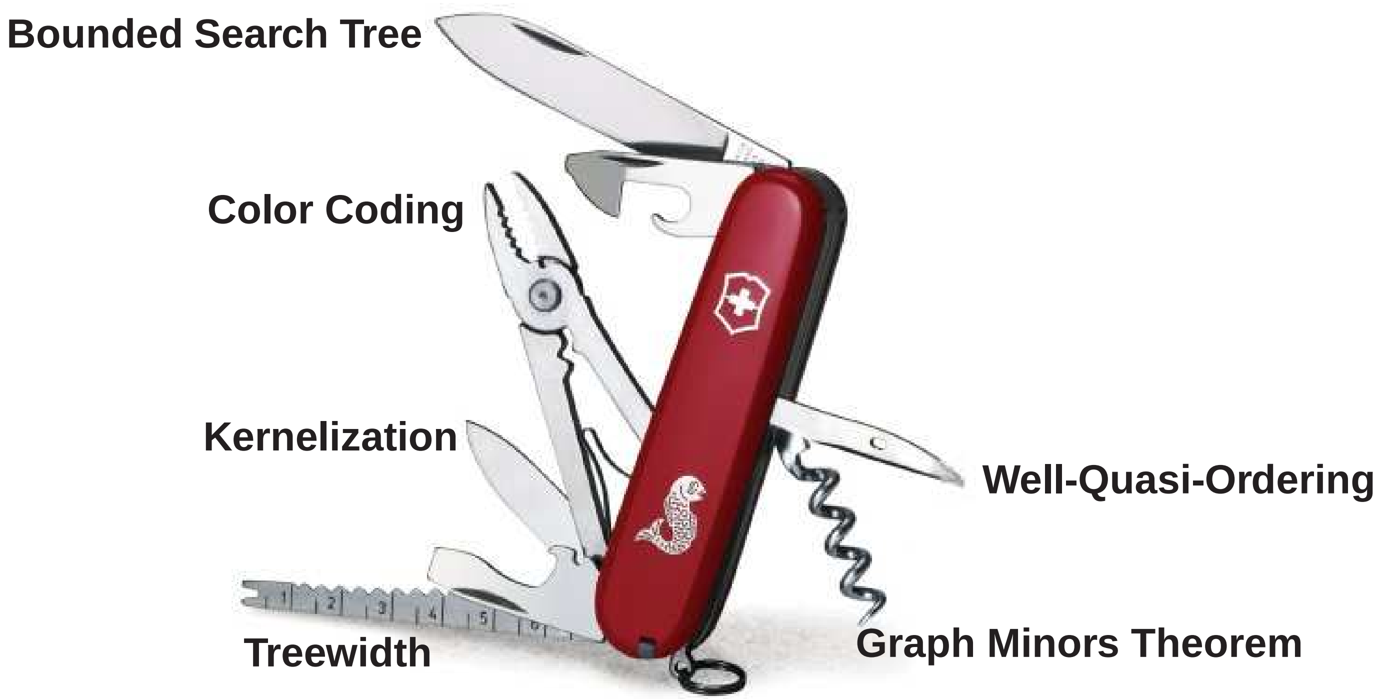
Main goal of parameterized complexity: to find FPT problems.
Examples of NP-hard problems that are FPT:

- Finding a vertex cover of size k.
- Finding a path of length k.
- Finding k disjoint triangles.
- Drawing the graph in the plane with k edge crossing.
- Finding disjoint paths that connect k pairs of points.
- ...

Fixed-parameter tractability (cont.)

- Practical importance: efficient algorithms for small values of k.
- Powerful toolbox for designing FPT algorithms:



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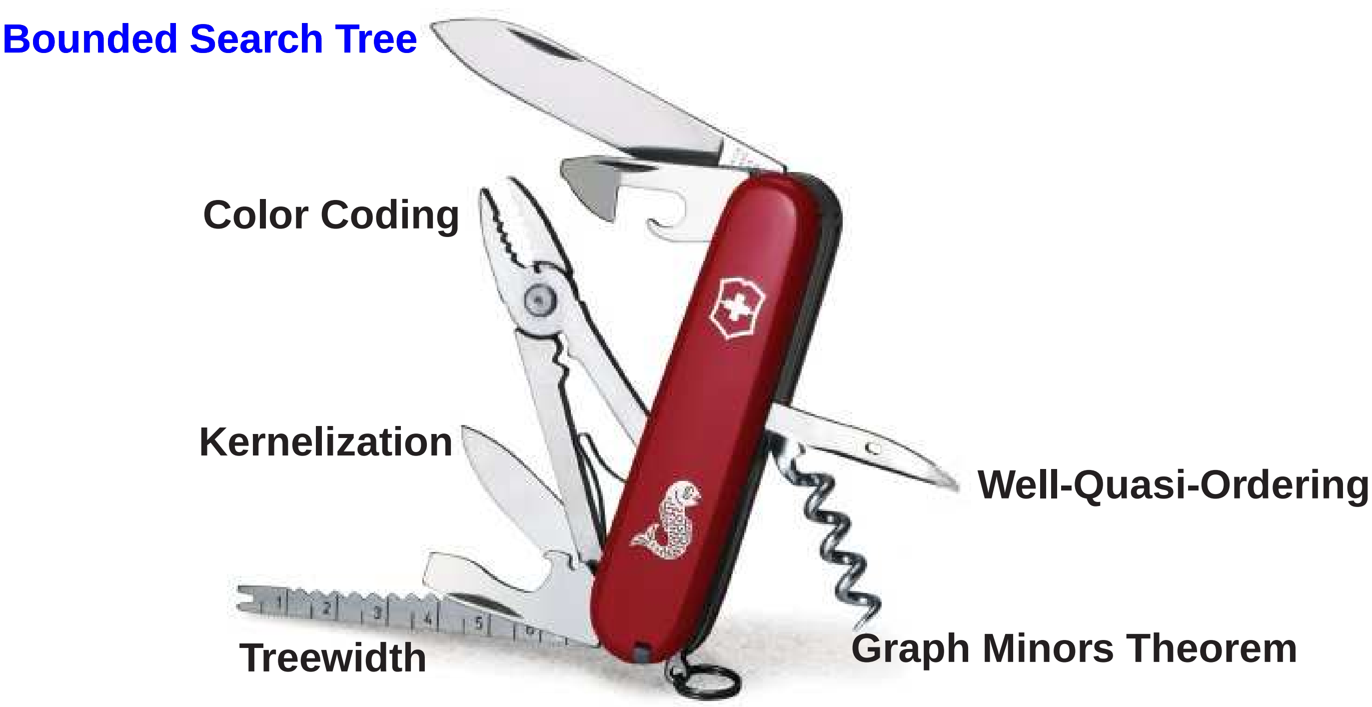

Parameterized intractability

We expect that MAXIMUM INDEPENDENT SET is not fixed-parameter tractable, no $n^{o(k)}$ algorithm is known.

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Parameterized reductions: L_1 is reducible to L_2, if there is a function f that transforms (x, k) to (x', k') such that

- $(x, k) \in L_1$ if and only if $(x', k') \in L_2$,
- f can be computed in $f(k)|x|^c$ time,
- **k' depends only on k**

If L_1 is reducible to L_2, and L_2 is in FPT, then L_1 is in FPT as well.

Most NP-completeness proofs are not good for parameterized reductions.

Parameterized Complexity: Summary

Two key concepts:

- A parameterized problem is **fixed-parameter tractable** if it has an $f(k)n^c$ time algorithm.
- To show that a problem L is hard, we have to give a **parameterized reduction** from a known **W[1]-complete** problem to L.

Parameterized Complexity: Summary

Two key concepts:

- A parameterized problem is **fixed-parameter tractable** if it has an $f(k)n^c$ time algorithm.
- To show that a problem L is hard, we have to give a **parameterized reduction** from a known **W[1]-complete** problem to L.

The question that we want to investigate:

$\boxed{\text{Is } k\text{-local-search fixed-parameter tractable for a particular problem?}}$

If yes, then local search algorithms can consider larger neighborhoods, improving their efficiency.

Important: k is the number of allowed changes and **not** the size of the solution. Relevant even if solution size is large.

Results on parameterized local search

- **Task:** find a spanning tree maximizing the number of vertices having full degree.

  Local search is FPT: given a solution, it can be checked in time $O(n^2 + nf(k))$ if it is possible to obtain a better solution by replacing at most k edges [Khuller, Bhatia, and Pless 2003].

- **Task:** TSP with distances satisfying the triangle inequality.

  Local search is hard: it is W[1]-hard to check if it is possible to obtain a shorter tour by replacing at most k arcs [M. 2008].

Results on parameterized local search (cont.)

- **Task:** find a minimum dominating set/minimum r-center/minimum vertex cover in a planar graph.

  Local search is FPT. [Fellows et al., 2008].

Results on parameterized local search (cont.)

- **Task:** find a minimum dominating set/minimum r-center/minimum vertex cover in a planar graph.

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- **Task:** find a maximum stable assignment in the “Hospitals/Residents with Couples” problem (a variant of Stable Marriage).

  - Local search is W[1]-hard:
    There is no $f(k) \cdot n^{O(1)}$ algorithm for deciding whether an assignment can be improved by at most k changes.
  - Local search is FPT if the number ℓ of couples is also a parameter:
    There is an $f(k, \ell) \cdot n^{O(1)}$ for deciding whether an assignment can be improved by at most k changes. [M. and Schlotter 2008].

Boolean CSP

Topic of this talk: investigating the parameterized complexity of local search for the problem of finding a minimum weight solution for a Boolean constraint satisfaction problem (CSP).

Boolean CSP: generalization of SAT. Input is a conjunction of constraints over a set of Boolean variables.

$$R_1(x_1, x_4, x_5) \wedge R_2(x_2, x_1) \wedge R_1(x_3, x_3, x_3) \wedge R_3(x_5, x_1, x_4, x_1)$$

Constraints can be arbitrary Boolean relations.

Problem is too general!

Boolean CSP

If Γ is a set of Boolean relations, then a **Γ-formula** is a conjunction of relations in Γ:

$$R_1(x_1, x_4, x_5) \wedge R_2(x_2, x_1) \wedge R_1(x_3, x_3, x_3) \wedge R_3(x_5, x_1, x_4, x_1)$$

Γ-SAT

- Given: an Γ-formula φ
- Find: a variable assignment satisfying φ

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Γ-SAT

- Given: an Γ-formula φ
- Find: a variable assignment satisfying φ

$\Gamma = \{a \neq b\} \Rightarrow$ Γ-SAT = 2-coloring of a graph
$\Gamma = \{a \vee b,\ a \vee \bar{b},\ \bar{a} \vee \bar{b}\} \Rightarrow$ Γ-SAT = 2SAT
$\Gamma = \{a \vee b \vee c, a \vee b \vee \bar{c}, a \vee \bar{b} \vee \bar{c}, \bar{a} \vee \bar{b} \vee \bar{c}\} \Rightarrow$ Γ-SAT = 3SAT

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Question: Γ-SAT is polynomial time solvable for which Γ?
It is NP-complete for which Γ?

Schaefer's Dichotomy Theorem (1978)

For every finite Γ, the Γ-SAT problem is polynomial time solvable if one of the following holds, and NP-complete otherwise:

- Every relation is satisfied by the all 0 assignment
- Every relation is satisfied by the all 1 assignment
- Every relation can be expressed by a 2SAT formula
- Every relation can be expressed by a Horn formula
- Every relation can be expressed by an anti-Horn formula
- Every relation is an affine subspace over $GF(2)$

Other dichotomy results

- Approximability of MAX-SAT, MIN-UNSAT [Khanna et al., 2001]
- Approximability of MAX-ONES, MIN-ONES [Khanna et al., 2001]
- Generalization to 3 valued variables [Bulatov, 2002]
- Inverse satisfiability [Kavvadias and Sideri, 1999]
- Parameterized complexity of weight k solutions [M., 2005]
- Counting solutions [Bulatov, 2008]
- etc.

Minimizing weight

Γ-MIN-ONES: find a solution of a Γ-SAT formula that minimizes the weight (= the number of 1's).

Theorem: [Khanna et al., 2001] For every finite Γ, the Γ-MIN-ONES problem is polynomial time solvable if one of the following holds, and NP-complete otherwise:

- Every relation is satisfied by the all 0 assignment
- Every relation can be expressed by a Horn formula
- Every relation is width-2 affine (= can be expressed by constants, $=$, $\neq$).

Our goal: characterize those sets Γ where local search for Γ-MIN-ONES is fixed-parameter tractable.

Losing weight

$\mathbf{\Gamma}$-LOSE-WEIGHT	
Input:	A $\mathbf{\Gamma}$-formula φ, a solution x for φ, and an integer k.
Decide:	Is there a solution x' of φ with dist$(x, x') \leq k$ and weight$(x') <$ weight(x)?

dist(x, x'): Hamming distance of x and x'.

weight(x): number of 1's in x.

Losing weight

Γ-LOSE-WEIGHT

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$\mathrm{dist}(x, x')$: Hamming distance of x and x'.
$\mathrm{weight}(x)$: number of 1's in x.

Main result:

Theorem: For every finite set Γ, Γ-LOSE-WEIGHT is either fixed-parameter tractable or W[1]-hard.

+ a simple characterization of the FPT cases.

Horn constraints

Definition: A relation is **Horn** (or weakly negative) if it can be expressed as the conjunction of clauses with at most one positive literal in each clause.

$$(x_1 \vee \bar{x}_2) \wedge (x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4) \wedge (\bar{x}_2)$$

A relation is Horn if and only if it is closed under componentwise AND.

Flip sets

Definition: Let R be an r-ary relation and $(a_1, \ldots, a_r) \in R$. A set $S \subseteq \{1, \ldots, r\}$ is a **flip set** of $(a_1, \ldots, a_r)$ (with respect to R) if flipping the coordinates corresponding to S gives another tuple in R.

Example:

$$R(x_1, x_2, x_3, x_4)$$
$$(0, 0, 1, 0)$$
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$$(0, 1, 1, 1)$$
$$(1, 0, 0, 0)$$
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Example:

$R(x_1, x_2, x_3, x_4)$	Flip sets of $(1, 0, 1, 0)$
$(0, 0, 1, 0)$	$\{1\}$
$(1, 0, 1, 0)$	
$(0, 1, 1, 1)$	$\{1, 2, 4\}$
$(1, 0, 0, 0)$	$\{3\}$
$(0, 1, 1, 0)$	$\{1, 2\}$
$(1, 0, 1, 1)$	$\{4\}$

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$(0,0,1,0)$	$\{1\}$	$\{2,3\}$
$(1,0,1,0)$		$\{1,2,4\}$
$(0,1,1,1)$	$\{1,2,4\}$	
$(1,0,0,0)$	$\{3\}$	$\{1,2,3,4\}$
$(0,1,1,0)$	$\{1,2\}$	$\{4\}$
$(1,0,1,1)$	$\{4\}$	$\{1,2\}$

Flip separable

Definition: An r-ary relation R is **flip separable** if whenever $S_1 \subset S_2 \subseteq \{1, \ldots, r\}$ are flip sets of a tuple $(x_1, \ldots, x_r)$, then $S_2 \setminus S_1$ is also a flip set.

Example: Flip sets of

$R(x_1, x_2, x_3, x_4)$	$(1, 0, 1, 0)$
$(0, 0, 1, 0)$	$\{1\}$
$(1, 0, 1, 0)$	
$(0, 1, 1, 1)$	$\{1, 2, 4\}$
$(1, 0, 0, 0)$	$\{3\}$
$(0, 1, 1, 0)$	$\{1, 2\}$
$(1, 0, 1, 1)$	$\{4\}$

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Example:

$R(x_1, x_2, x_3, x_4)$	Flip sets of $(1, 0, 1, 0)$	
$(0, 0, 1, 0)$	$\{1\}$	
$(1, 0, 1, 0)$		R is not
$(0, 1, 1, 1)$	$\{1, 2, 4\}$	flip separable!
$(1, 0, 0, 0)$	$\{3\}$	
$(0, 1, 1, 0)$	$\{1, 2\}$	
$(1, 0, 1, 1)$	$\{4\}$	

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Example:

EVEN(x_1, x_2, x_3, x_4)

$(0,0,0,0)$

$(1,1,0,0)$

$(1,0,1,0)$

$(1,0,0,1)$

$(0,1,1,0)$

$(0,1,0,1)$

$(0,0,1,1)$

$(1,1,1,1)$

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Example:

EVEN(x_1, x_2, x_3, x_4)	Flip sets of $(1,1,0,0)$
$(0,0,0,0)$	$\{1,2\}$
$(1,1,0,0)$	
$(1,0,1,0)$	$\{2,3\}$
$(1,0,0,1)$	$\{2,4\}$
$(0,1,1,0)$	$\{1,3\}$
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$(0,0,0,0)$	$\{1,2\}$	$\{1,2,3,4\}$
$(1,1,0,0)$		$\{3,4\}$
$(1,0,1,0)$	$\{2,3\}$	$\{2,4\}$
$(1,0,0,1)$	$\{2,4\}$	$\{2,3\}$
$(0,1,1,0)$	$\{1,3\}$	$\{1,4\}$
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	Flip sets of	Flip sets of
EVEN(x_1, x_2, x_3, x_4)	$(1,1,0,0)$	$(1,1,1,1)$
$(0,0,0,0)$	$\{1,2\}$	$\{1,2,3,4\}$
$(1,1,0,0)$		$\{3,4\}$
$(1,0,1,0)$	$\{2,3\}$	$\{2,4\}$
$(1,0,0,1)$	$\{2,4\}$	$\{2,3\}$
$(0,1,1,0)$	$\{1,3\}$	$\{1,4\}$
$(0,1,0,1)$	$\{1,4\}$	$\{1,3\}$
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EVEN is flip separable!

Flip separable

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Example:

1-IN-4(x_1, x_2, x_3, x_4)

$$(1, 0, 0, 0)$$
$$(0, 1, 0, 0)$$
$$(0, 0, 1, 0)$$
$$(0, 0, 0, 1)$$

Flip separable

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Example:

	Flip sets of
1-IN-4(x_1, x_2, x_3, x_4)	$(1, 0, 0, 0)$
$(1, 0, 0, 0)$	
$(0, 1, 0, 0)$	$\{1, 2\}$
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Example:

1-IN-4(x_1, x_2, x_3, x_4)	Flip sets of $(1,0,0,0)$	Flip sets of $(0,1,0,0)$
$(1,0,0,0)$		$\{1,2\}$
$(0,1,0,0)$	$\{1,2\}$	
$(0,0,1,0)$	$\{1,3\}$	$\{2,3\}$
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Example:

1-IN-4(x_1, x_2, x_3, x_4)	Flip sets of $(1,0,0,0)$	Flip sets of $(0,1,0,0)$	
$(1,0,0,0)$		$\{1,2\}$	1-IN-4 is
$(0,1,0,0)$	$\{1,2\}$		flip separable!
$(0,0,1,0)$	$\{1,3\}$	$\{2,3\}$	
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Example:

$x_1 \vee x_2$

$(1, 0)$

$(0, 1)$

$(1, 1)$

Flip separable

Definition: An r-ary relation R is **flip separable** if whenever $S_1 \subset S_2 \subseteq \{1, \ldots, r\}$ are flip sets of a tuple $(x_1, \ldots, x_r)$, then $S_2 \setminus S_1$ is also a flip set.

Example: Flip sets of

$x_1 \vee x_2$	$(1,0)$
$(1,0)$	
$(0,1)$	$\{1,2\}$
$(1,1)$	$\{2\}$

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Example: Flip sets of

$x_1 \vee x_2$	$(1,0)$
$(1,0)$	
$(0,1)$	$\{1,2\}$
$(1,1)$	$\{2\}$

$x_1 \vee x_2$ is not

flip separable!

Main result

Theorem: For every finite set Γ, Γ-LOSE-WEIGHT is fixed-parameter tractable if one of the following holds, and W[1]-hard otherwise:

- Every relation can be expressed by a Horn formula.
- Every relation is flip separable.

Some FPT cases:

- EVEN and ODD constraints.
- affine constraints.
- p-IN-q constraints.

Some hard cases:

- $x_1 \vee x_2$ (= MINIMUM VERTEX COVER)
- 3SAT

Algorithm

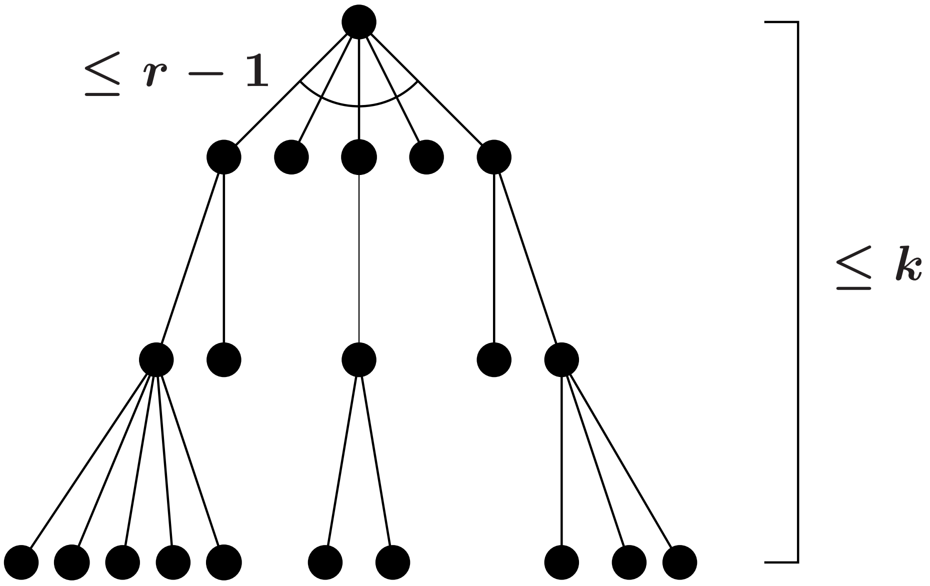
Task: given a formula with flip separable constraints and a satisfying assignment, decrease the weight by flipping at most k variables.

Bounded search tree algorithm:

- Flip a variable with value 1 to 0 (at most n possible choices).
- If a clause is not satisfied, flip one of its variables that was not yet flipped (at most $r - 1$ possible choices if maximum arity is r).
- Repeat until
  - more than k variables are flipped $\Rightarrow$ terminate this branch.
  - every clause is satisfied $\Rightarrow$ check if the satisfying assignment has strictly smaller weight than the original assignment.

Algorithm

Running time: After the initial flip, the search tree has size at most $(r-1)^k$:



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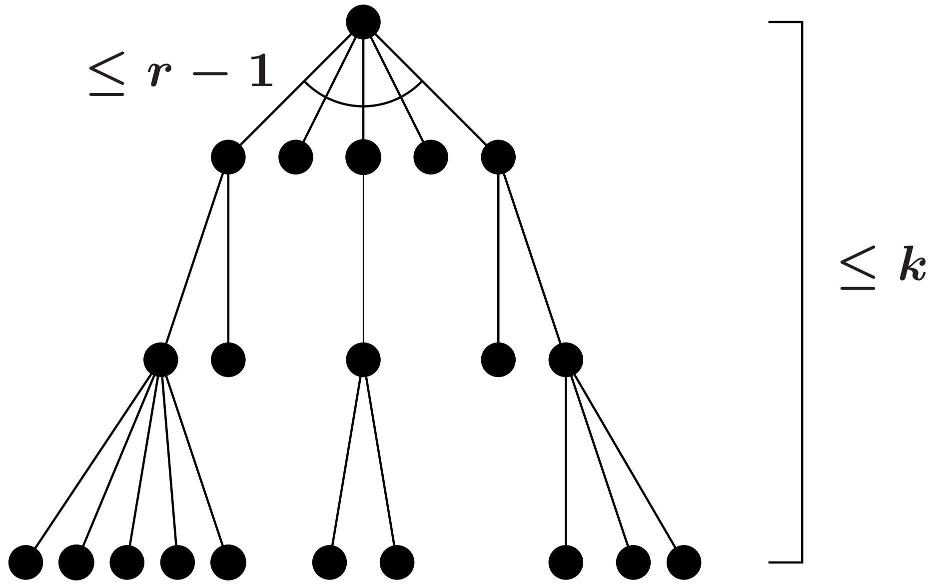

Running time is $f(k,r)\cdot n^c \Rightarrow f'(k)\cdot n^c$ for a fixed Γ.

Algorithm

Correctness: is it true that we always find a solution if it exits?

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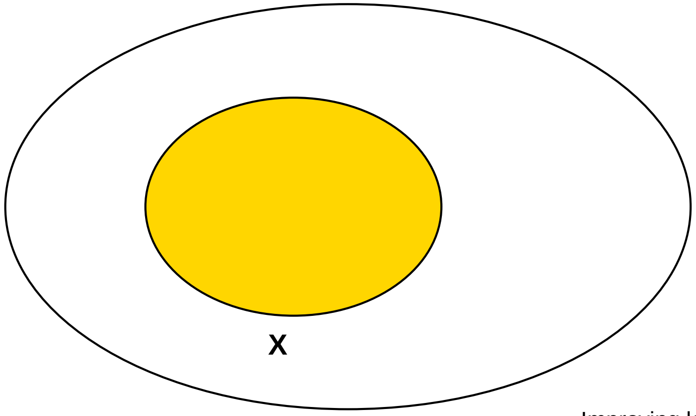
- Let X be a solution that decreases the weight most ($|X| \leq k$, flipping X gives a satisfying assignment).



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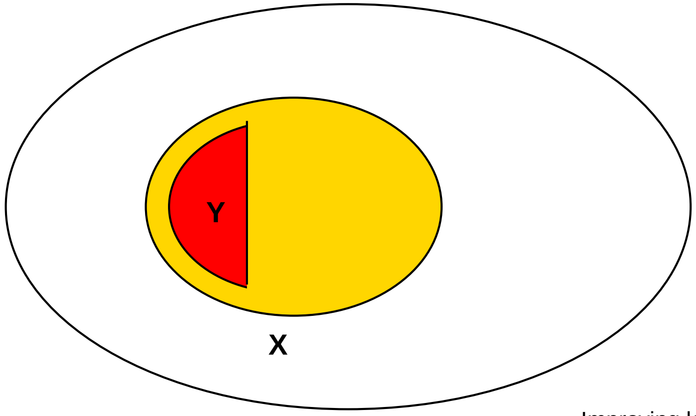
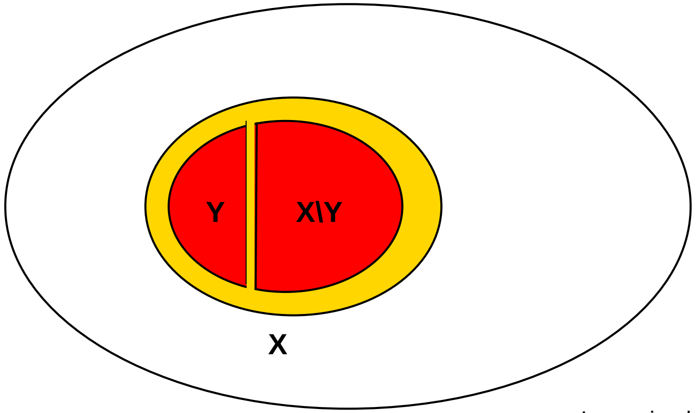
- Let X be a solution that decreases the weight most ($|X| \leq k$, flipping X gives a satisfying assignment).
- There is a branch of the algorithm that flips only a subset $Y \subseteq X$.



Algorithm

Correctness: is it true that we always find a solution if it exits?

- Let X be a solution that decreases the weight most ($|X| \leq k$, flipping X gives a satisfying assignment).
- There is a branch of the algorithm that flips only a subset $Y \subseteq X$.
- Flipping $X \setminus Y$ is also a solution (constraints are flip separable).
- If flipping Y does not decrease the weight, then flipping $X \setminus Y$ decreases the weight more than Y.



Hardness proof

Hardness proof: if Γ contains a relation that is not Horn and a relation that is not flip separable, then local search is W[1]-hard.

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Step 1: Direct proof for $x \vee y$.

$\Rightarrow$ Given a vertex cover S and an integer k, it is W[1]-hard to decide if it is possible to decrease the vertex cover by adding/removing at most k vertices.

$\Rightarrow$ Given an independent set S and an integer k, it is W[1]-hard to decide if it is possible to increase the independent set cover by adding/removing at most k vertices.

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Note: These results hold even for bipartite graphs.

Hardness proof

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Step 2: Suppose that there is a relation $R \in \Gamma$ that is not Horn, i.e., it is not closed under componentwise AND.

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Step 2: Suppose that there is a relation $R \in \Gamma$ that is not Horn, i.e., it is not closed under componentwise AND.

$(1, 0, 0, 1) \in R$
$(0, 1, 0, 1) \in R$
$(0, 0, 0, 1) \notin R$

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$(1, 0, 0, 1) \in R$
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$(0, 0, 0, 1) \notin R$

either

$(1, 1, 0, 1) \in R$
$\Rightarrow R(x, y, 0, 1) \equiv x \vee y$, we can "almost express" relation $x \vee y$ (DONE).

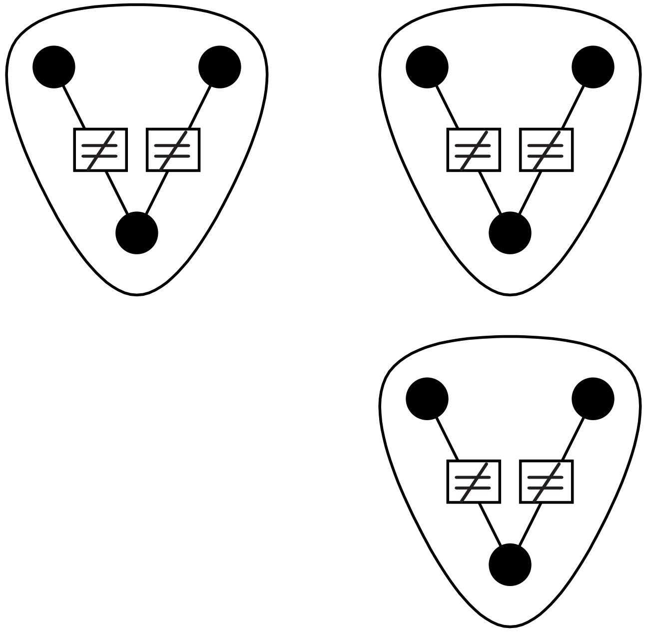
$(1, 1, 0, 1) \notin R$
$\Rightarrow R(x, y, 0, 1) \equiv x \neq y$, we can "almost express" relation $\neq$.

Hardness proof

Hardness proof: if Γ contains a relation that is not Horn and a relation that is not flip separable, then local search is W[1]-hard.

Step 3: Suppose that there is a relation $R \in \Gamma$ that is not flip separable and we can use $\neq$.

- Reduction from $x \vee y$.
- Replace each variable with 3 variables
- Two states for each triple.
- Changing a triple changes the weight by 1.



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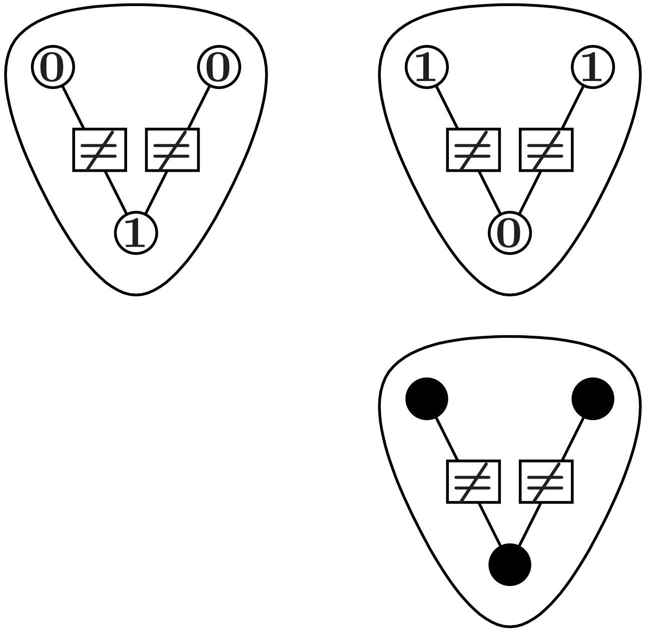
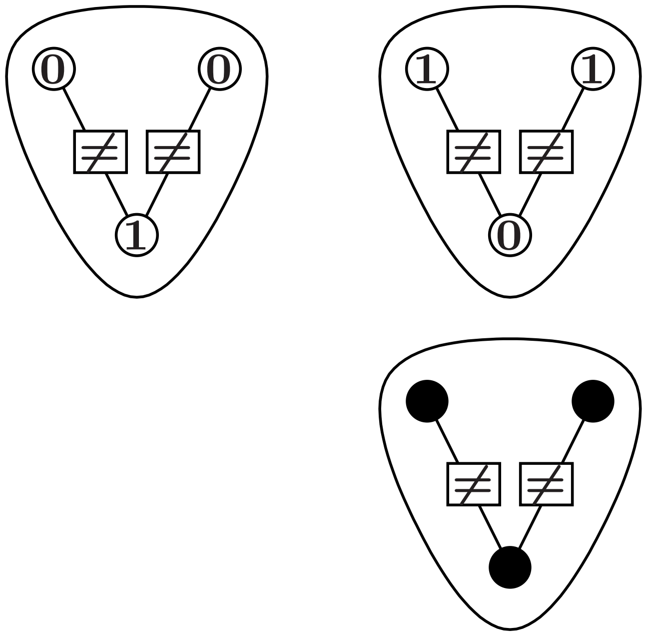


Hardness proof

Hardness proof: if Γ contains a relation that is not Horn and a relation that is not flip separable, then local search is W[1]-hard.

Suppose there is a counterexample to the fact that $R \in \Gamma$ is flip separable:

$(0,1,0,1) \in R$
$(1,0,0,1) \in R$
$(1,0,1,0) \in R$
$(0,1,1,0) \notin R$



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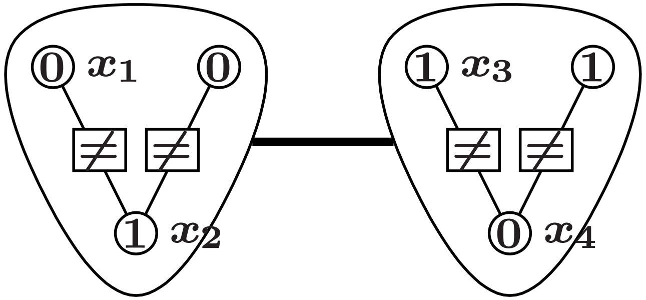
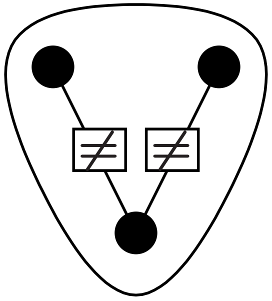


We represent the edge by constraint $R(x_1, x_2, x_4, x_3)$.



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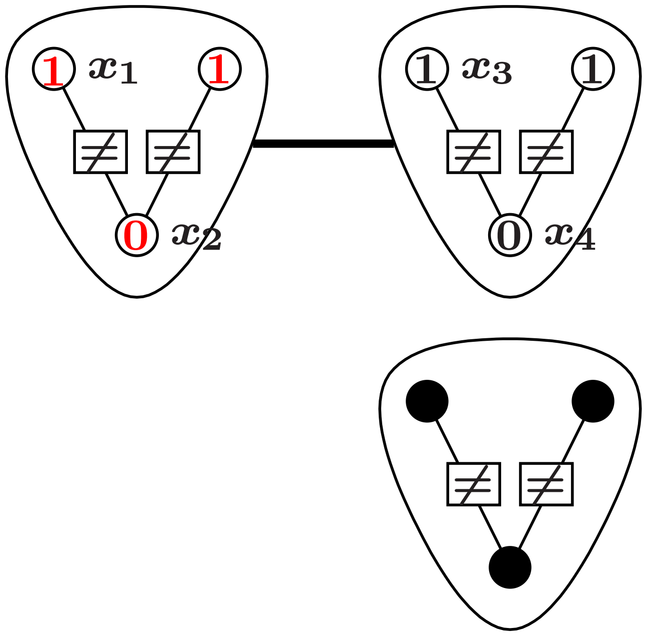
$$
\begin{aligned}
&(0,1,0,1) \in R \\
&(1,0,0,1) \in R \Leftarrow \\
&(1,0,1,0) \in R \\
&(0,1,1,0) \notin R
\end{aligned}
$$



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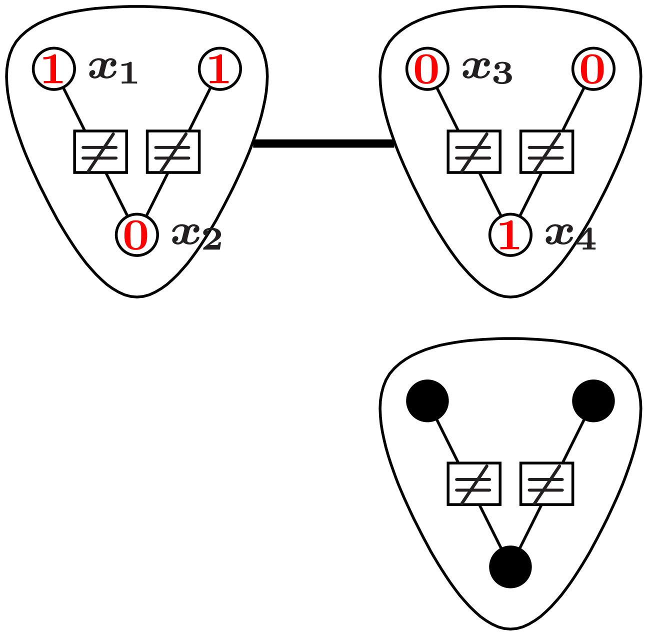

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Flipping both gadgets is allowed...

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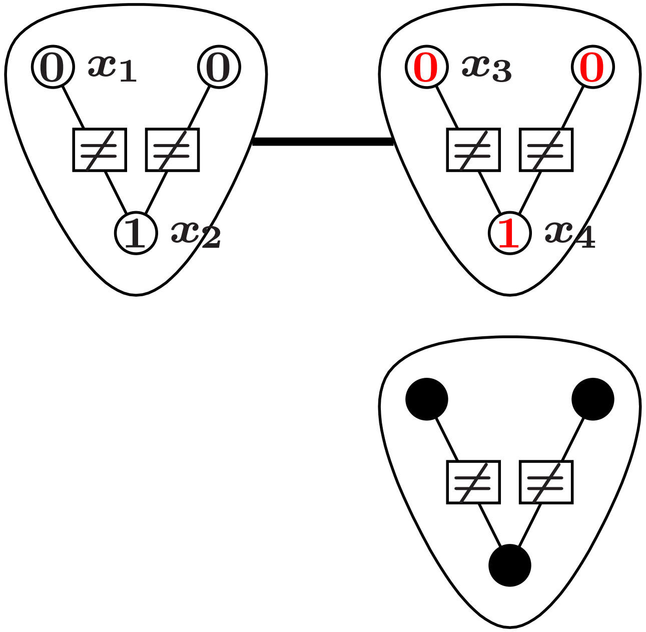
$(0,1,1,0) \notin R \Leftarrow$



We represent the edge by constraint $R(x_1, x_2, x_4, x_3)$.

Flipping the first gadget is allowed...
Flipping both gadgets is allowed...
But second gadget cannot be flipped!

Main result

We have completed the complexity characterization of Γ-LOSE-WEIGHT:

Theorem: For every finite set Γ, Γ-LOSE-WEIGHT is fixed-parameter tractable if one of the following holds, and W[1]-hard otherwise:

- Every relation can be expressed by a Horn formula.
- Every relation is flip separable.

But something is strange...

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We have seen that local search is W[1]-hard for MINIMUM VERTEX COVER, even if the graph is bipartite.

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The relation $x \vee y \vee \bar{z}$ is not Horn and not flip separable (for the tuple $(1, 0, 1)$, $\{2\}$ and $\{1, 2\}$ are flip sets but $\{1\}$ is not), thus local search is hard.

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Counterintuitive results: finding a local improvement is hard, but finding the global optimum is easy.

We are answering the wrong question!

Strict vs. permissive

So far, we investigated **strict** local search algorithms:

Input:	A Γ-formula φ, a solution x for φ, and an integer k.
Task:	If there is a solution x' of φ with dist$(x, x') \leq k$ and weight$(x') <$ weight(x), then find such an x'.

Strict vs. permissive

So far, we investigated **strict** local search algorithms:

Input:	A Γ-formula φ, a solution x for φ, and an integer k.
Task:	If there is a solution x' of φ with $\text{dist}(x, x') \leq k$ and $\text{weight}(x') < \text{weight}(x)$, then find such an x'.

But a **permissive** local search algorithm would be equally useful:

Input:	A Γ-formula φ, a solution x for φ, and an integer k.
Task:	If there is a solution x' of φ with $\text{dist}(x, x') \leq k$ and $\text{weight}(x') < \text{weight}(x)$, then find any x'' with $\text{weight}(x'') < \text{weight}(x)$.

Our hardness result for strict local search does not rule out the possibility of a permissive algorithm.

Revised result

Theorem: For every finite set Γ, **strict** Γ-LOSE-WEIGHT is fixed-parameter tractable if one of the following holds, and W[1]-hard otherwise:

- Every relation can be expressed by a Horn formula.
- Every relation is flip separable.

Theorem: For every finite set Γ, **permissive** Γ-LOSE-WEIGHT is fixed-parameter tractable if one of the following holds, and W[1]-hard otherwise:

- Every relation can be expressed by a Horn formula.
- Every relation is flip separable.
- Every relation is 0-valid.

Conclusions

- Is it possible to efficiently search the local neighborhood?
- Parameterized complexity is the natural way to study.
- Might apply to YOUR problem as well!
- Schaefer-style classification for decreasing the weight of a solution in Boolean CSP.
- Main new definition: flip separable relations.
- Distinction between strict and permissive local search.