Graphs, Hypergraphs, and the Complexity of Conjunctive Database Queries

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Conjunctive queries

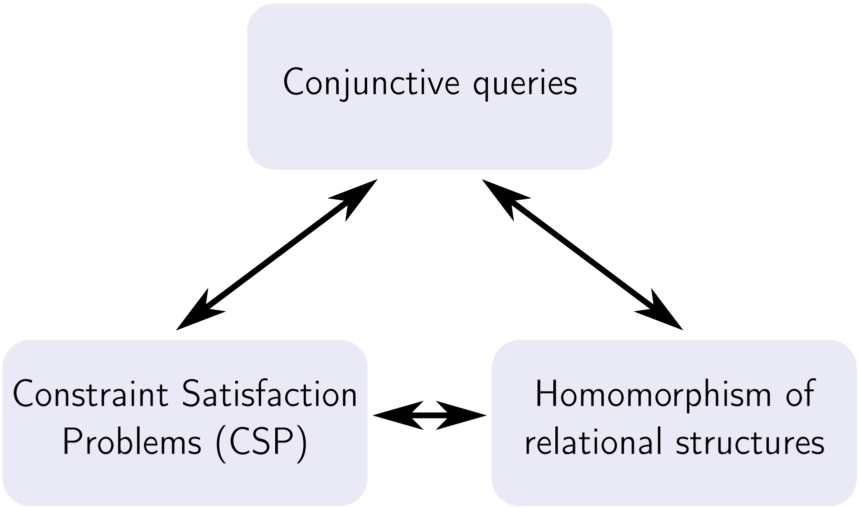
Evaluating conjunctive queries is a fundamental problem.

$$Q = R(A, B, C) \wedge S(C, D) \wedge T(B, C, E)$$

Formally defined as:

$$Q = \{(a, b, c, d, e) \mid (a, b, c) \in R, (c, d) \in S, (b, c, e) \in T\}$$

- Compute the answer relation Q.
- Decide if the relation Q is empty.
- Compute the size of Q.
- ...


Conjunctive queries
Constraint Satisfaction Problems (CSP)
Homomorphism of relational structures

Constraint Satisfaction Problems (CSP)

$$Q = R(A, B, C) \wedge S(C, D) \wedge T(B, C, E)$$

CSP lingo:

- variables A, B, C, D, E
- constraints R, S, T
- find an assignment (a, b, c, d, e) to the variables that satisfies every constraint.

Tasks:

Compute the answer relation.		List the satisfying assignments.
Decide if Q is empty.	$\Leftrightarrow$	Decide if the CSP is satisfiable.
Compute the size of Q.		Count the sat. assignments.

Goal

Goal: understand how efficiently a particular query can be evaluated.

- Worst-case setting: we know the query, but the database relations can be arbitrary.
- Different levels of efficiency: polynomial time, fixed-parameter tractability, linear time.

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Important message:

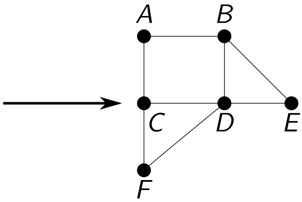
> "Treelikeness" is very helpful!

...because it allows bottom-up dynamic programming.

First: binary relations only

If every relation is binary (i.e., only two variables), then the structure of the query can be described by the **primal graph.**

$$R(A, B) \land R(A, C) \land$$
$$R(B, D) \land R(C, D) \land$$
$$R(B, E) \land R(D, E) \land$$
$$R(C, F) \land R(D, F)$$

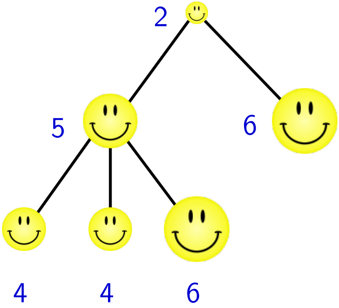


Goal: understand what graph-theoretic properties allow efficient query evaluation.

The Party Problem

Party Problem

Problem:	Invite some colleagues for a party.
Maximize:	The total fun factor of the invited people.
Constraint:	Everyone should be having fun.



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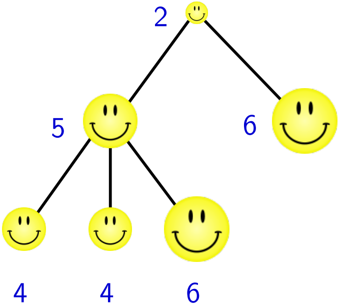
	Do not invite a colleague and his direct boss at the same time!



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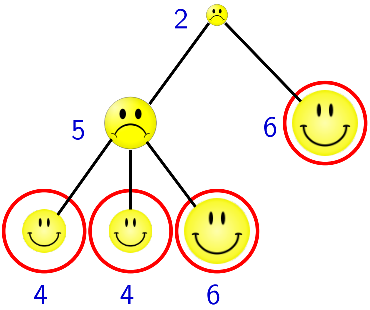

- **Input:** A tree with weights on the vertices.
- **Task:** Find an independent set of maximum weight.

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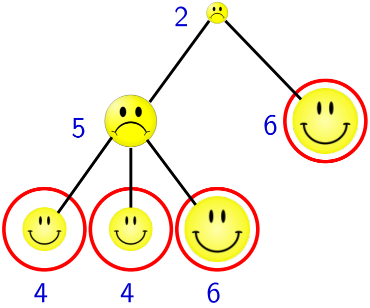

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Solving the Party Problem

Dynamic programming paradigm:
We solve a large number of subproblems that depend on each other. The answer is a single subproblem.

Subproblems:

T_v: the subtree rooted at v.
$A[v]$: max. weight of an independent set in T_v
$B[v]$: max. weight of an independent set in T_v that does not contain v

Goal: determine $A[r]$ for the root r.

Solving the Party Problem

Subproblems:

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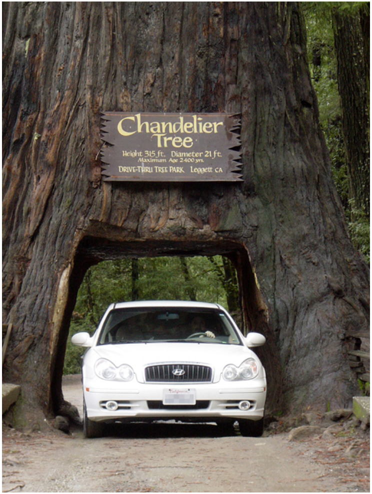
Recurrence:

Assume $v_1, \ldots, v_k$ are the children of v. Use the recurrence relations

$$\boxed{\begin{array}{l} B[v] = \sum_{i=1}^{k} A[v_i] \\ A[v] = \max\{B[v]\ ,\ w(v) + \sum_{i=1}^{k} B[v_i]\} \end{array}}$$

The values $A[v]$ and $B[v]$ can be calculated in a bottom-up order (the leaves are trivial).


Chandelier Tree
Height 315 ft. Diameter 21 ft.
Maximum Age 2400 yrs.
DRIVE-THRU TREE PARK Leggett CA

Treewidth

Generalizing trees

How could we define that a graph is "treelike"?

Generalizing trees

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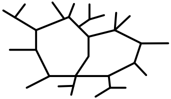
1. Number of cycles is bounded.



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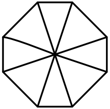

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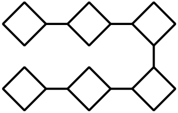

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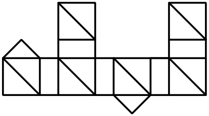

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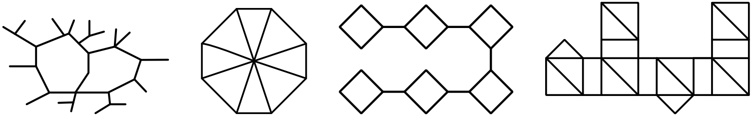

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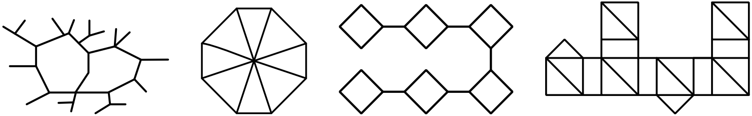
2. Removing a bounded number of vertices makes it acyclic.



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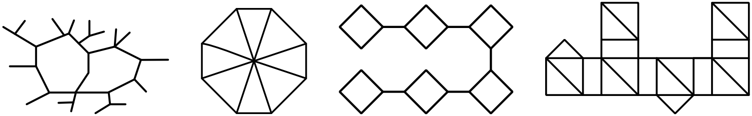

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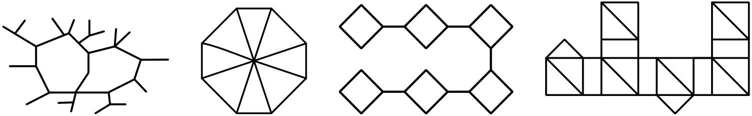

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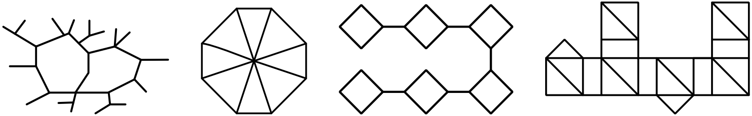
3. Bounded-size parts connected in a tree-like way.



bad bad good good

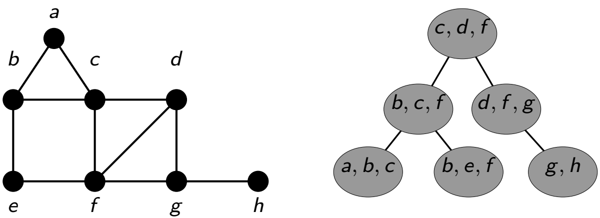
Treewidth — a measure of "tree-likeness"

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. For any edge uv, there is a bag containing both of them.
2. For every v, the bags containing v form a connected subtree.

Width of the decomposition: largest bag size -1.

treewidth: width of the best decomposition.



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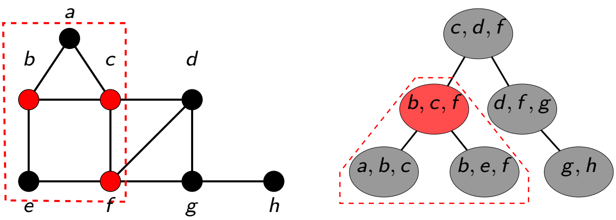


A subtree communicates with the outside world only via the root of the subtree.

WEIGHTED MAX INDEPENDENT SET and treewidth

Theorem

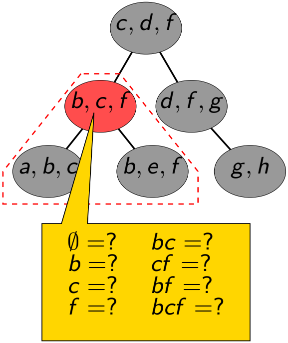
Given a tree decomposition of width w, WEIGHTED MAX INDEPENDENT SET can be solved in time $2^w \cdot w^{O(1)} \cdot n$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

Generalizing our solution for trees:

Instead of computing 2 values $A[v]$, $B[v]$ for each vertex of the tree, we compute $2^{|B_x|} \leq 2^{w+1}$ values for each bag B_x.

$M[x, S]$:
the max. weight of an independent set $I \subseteq V_x$ with $I \cap B_x = S$.



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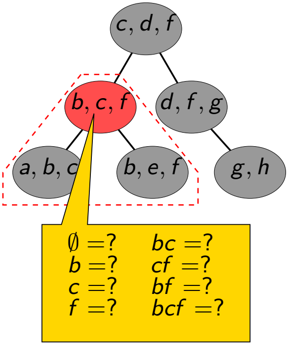
Claim: We can determine $M[x, S]$ if all the values are known for the children of x.



3-Coloring and tree decompositions

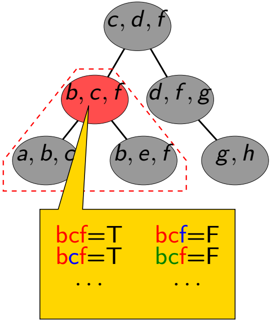
Theorem

Given a tree decomposition of width w, 3-Coloring can be solved in time $3^w \cdot w^{O(1)} \cdot n$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

For every node x and coloring $c : B_x \to \{1, 2, 3\}$, we compute the Boolean value $E[x, c]$, which is true if and only if c can be extended to a proper 3-coloring of V_x.

Claim:
We can determine $E[x, c]$ if all the values are known for the children of x.



Coloring as a CSP

We can interpret 3-coloring as a CSP:

- vertices $\Leftrightarrow$ variables
- domain $D = \{r, g, b\}$
- edges $\Leftrightarrow$ inequality constraints

$$R = \{(x, y) \in D \times D \mid x \neq y\}$$

Straightforward generalization to higher number of colors:

Theorem

Given a tree decomposition of width w, c-Coloring can be solved in time $c^{w+1} \cdot w^{O(1)} \cdot n$.

Coloring as a CSP

We can interpret 3-coloring as a CSP:

- vertices $\Leftrightarrow$ variables
- domain $D = \{r, g, b\}$
- edges $\Leftrightarrow$ inequality constraints

$$R = \{(x, y) \in D \times D \mid x \neq y\}$$

Straightforward generalization to arbitrary binary CSPs:

Theorem

Given a tree decomposition of width w, binary CSP over domain D can be solved in time $|D|^{w+1} \cdot w^{O(1)} \cdot n$.

Coloring as a database query

- vertices $\Leftrightarrow$ variables
- edges $\Leftrightarrow$ relation $R = \{rg, rb, gr, gb, br, bg\}$

$$R(A,B) \wedge R(A,C) \wedge$$
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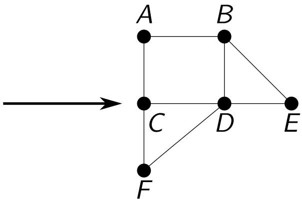

Straightforward generalization to arbitrary binary queries:

Theorem

Given a tree decomposition of width w, a Boolean Conjunctive Query where every variable allows at most N different values can can be solved in time $N^{w+1} \cdot |Q|^{O(1)}$.

Projections

Projecting the relation $R(A, B, C, D)$ to $\{A, B\}$:

$$R_{|AB} = \{(a, b) \mid \exists c, d : (a, b, c, d) \in R\}$$

Projection of the query to a set S: projecting every relation.

$$Q = R(A, B, C) \wedge S(C, D) \wedge T(B, C, E)$$

$$\begin{aligned} Q_{|AB} &= R_{|AB}(A, B, C) \wedge S_{|AB}(C, D) \wedge T_{|AB}(B, C, E) \\ &= R_{|AB}(A, B, C) \wedge T_{|B}(B, C, E) \end{aligned}$$

Easy: If $(a, b, c) \in Q$, then $(a, b) \in Q_{|AB}$, but not necessarily the other way around!

Boolean Conjunctive Queries and tree decompositions

Theorem

Given a tree decomposition of width w, a Boolean Conjunctive Query where every variable allows at most N different values can can be solved in time $N^{w+1} \cdot |Q|^{O(1)}$.

B_x: vertices appearing in node x.
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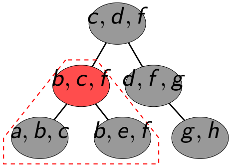
For every node x and tuple $t \in Q_{|B_x}$, we compute the Boolean value $E[x, t]$, which is true if and only if t can be extended to a tuple of $Q_{|V_x}$.



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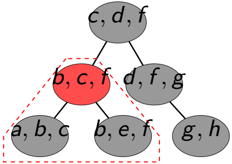

Running time:
Dominating factor is the size of $Q_{|B_x}$, which can be bounded by $N^{|B_x|} \leq N^{w+1}$.

Tractable classes

We have seen that for every fixed bound on the treewidth, BCQ is polynomial-time solvable in the size of the database.

Are there other properties that make the problem polynomial-time solvable?

Tractable classes

Formally:

If $\mathcal{G}$ is a class of graphs with bounded treewidth, then BCQ restricted $\mathcal{G}$ (we call it $\mathrm{BCQ}(\mathcal{G})$) is polynomial-time solvable.
Are there other such classes?

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Are there other such classes?

An equally interesting question: we can relax polynomial time and allow arbitrary dependence on the length of the query.

$\Rightarrow$ Fixed-parameter tractability

Fixed-parameter tractability

Main definition

A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant c.

Main goal of parameterized complexity: to find FPT problems.

Fixed-parameter tractability

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A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant c.

Main goal of parameterized complexity: to find FPT problems.

Examples of NP-hard problems that are FPT:

- Finding a vertex cover of size k.
- Finding a path of length k.
- Finding k disjoint triangles.
- Drawing the graph in the plane with k edge crossings.
- Finding disjoint paths that connect k pairs of points.
- ...

W[1]-hardness

Negative evidence similar to NP-completeness. If a problem is **W[1]-hard,** then the problem is not FPT unless FPT=W[1].

Some W[1]-hard problems:

- Finding a clique/independent set of size k.
- Finding a dominating set of size k.
- Finding k pairwise disjoint sets.
- ...

Tractable classes

Theorem [Grohe, Schwentick, Segoufin 2001]

Let $\mathcal{G}$ be a computable class of graphs. Then assuming FPT $\neq W[1]$, the following are equivalent:

- BCQ($\mathcal{G}$) is polynomial-time solvable.
- BCQ($\mathcal{G}$) is FPT.
- $\mathcal{G}$ has bounded treewidth.

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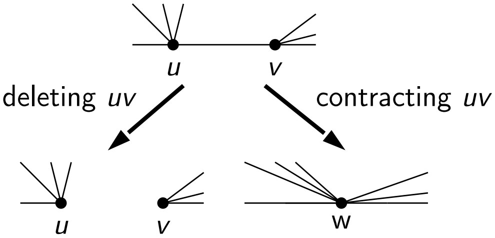
Two surprises:

- Treewidth-based algorithms already solve every polynomial-time solvable case.
- FPT does not give us extra power over polynomial time.

Minors

Definition

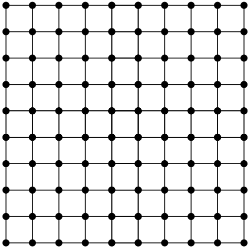
Graph H is a **minor** of G ($H \le G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.



Excluded Grid Theorem

Theorem [Chuzhoy 2016] [Chekuri and Chuzhoy 2014]

Every graph with treewidth at least $k^{19}\text{polylog}(k)$ has a $k \times k$ grid minor.



The $k \times k$ grid has treewidth exactly k.

Tractable classes

Theorem [Grohe, Schwentick, Segoufin 2001]

Let $\mathcal{G}$ be a computable class of graphs. Then assuming $\mathsf{FPT} \neq W[1]$, the following are equivalent:

- $\mathsf{BCQ}(\mathcal{G})$ is polynomial-time solvable.
- $\mathsf{BCQ}(\mathcal{G})$ is FPT.
- $\mathcal{G}$ has bounded treewidth.

Tractable classes

Theorem [Grohe, Schwentick, Segoufin 2001]

Let $\mathcal{G}$ be a computable class of graphs with unbounded treewidth. Then assuming FPT $\neq W[1]$, BCQ($\mathcal{G}$) is not FPT.

- Assuming FPT $\neq W[1]$, k-Clique is not FPT.
- k-Clique can be simulated by a BCQ whose primal graph is a $k \times k$ grid.
- $\mathcal{G}$ has unbounded treewidth
  $\Rightarrow$ Excluded Grid Theorem
  $\Rightarrow$ $\mathcal{G}$ contains graphs with a $k \times k$ grid minor
  $\Rightarrow$ BCQ($\mathcal{G}$) can simulate BCQ's with $k \times k$ grid structure.

Can you beat treewidth?

We have seen that treewidth-based algorithms discover every polynomial time solvable class.

- Is there a class $\mathcal{G}$ where we can be significantly faster than the treewidth-based algorithm? E.g., running time $N^{\sqrt{\mathrm{tw}(Q)}}$ or $N^{(\mathrm{tw}(\mathrm{Q}))^{1/100}}$ or $N^{(\log \log \mathrm{tw}(\mathrm{Q}))}$.

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Theorem [M. 2007]

Let $\mathcal{G}$ be a computable class of graphs. Assuming the Exponential-time Hypothesis, there is no algorithm for $\mathrm{BCQ}(\mathcal{G})$ with running time $f(Q)N^{o(\mathrm{tw}(Q)/\log \mathrm{tw(Q)})}$.

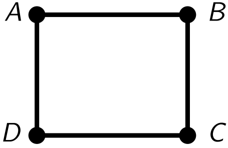
Exponential-time Hypothesis:
There is no $2^{o(n)}$ time algorithm for n-variable 3SAT.

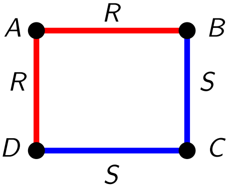
Proof requires a tighter combinatorial understanding of what large treewidth means.

Homomorphisms

The primal graph loses information if some relation appears more than once in the query.

$$Q = R(A, B) \wedge S(B, C) \wedge R(A, D) \wedge S(D, C)$$





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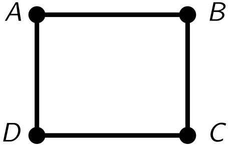
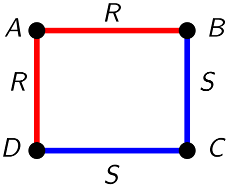




This is empty if and only if

$$Q' = R(A, B) \wedge S(B, C)$$

is empty!

Homomorphisms

A **homomorphism** from Q to Q' is a mapping ϕ of the variables of Q to the variables of Q' such that if $R(A, B)$ appears in Q, then $R(\phi(A), \phi(B))$ appears in Q'.

Observation:

- If there is a homomorphism $Q \to Q'$ and Q' is nonempty, then Q is nonempty as well.
- If there is a homomorphism from Q to a subquery Q', then Q is empty $\Leftrightarrow$ Q' is empty.

Homomorphisms

A **homomorphism** from Q to Q' is a mapping ϕ of the variables of Q to the variables of Q' such that if $R(A, B)$ appears in Q, then $R(\phi(A), \phi(B))$ appears in Q'.

Observation:

- If there is a homomorphism $Q \to Q'$ and Q' is nonempty, then Q is nonempty as well.
- If there is a homomorphism from Q to a subquery Q', then Q is empty $\Leftrightarrow$ Q' is empty.

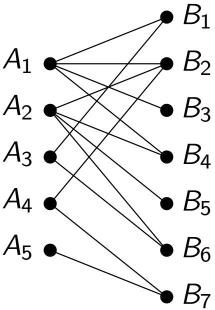
Fact: Every query Q has a unique (up to isomorphism) smallest subquery Q' with a homomorphism $Q \to Q'$. This is the **core** of Q.

For Boolean Conjunctive Queries, it is only the core of the query that matters!

Homomorphisms

What is the core of

$$Q = R(A_1, B_1) \wedge R(A_1, B_2) \wedge R(A_2, B_2) \wedge R(A_1, B_3) \wedge R(A_1, B_4) \wedge R(A_2, B_4) \wedge R(A_2, B_5) \wedge R(A_2, B_6) \wedge R(A_3, B_1) \wedge R(A_3, B_6) \wedge R(A_4, B_2) \wedge R(A_4, B_7) \wedge R(A_5, B_7)?$$



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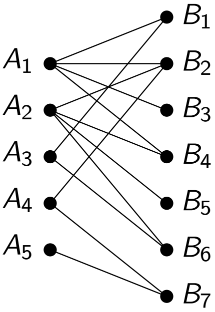

It is just $R(A_1, B_1)$! (As the graph is bipartite.)

Homomorphisms

Theorem [Grohe 2003]

Let $\mathcal{Q}$ be a computable class of queries with binary relations. Then assuming FPT $\neq W[1]$, the following are equivalent:

- BCQ restricted to queries $\mathcal{Q}$ is is polynomial-time solvable.
- BCQ restricted to queries $\mathcal{Q}$ is FPT.
- The primal graph of the core of every query in $\mathcal{Q}$ has bounded treewidth.

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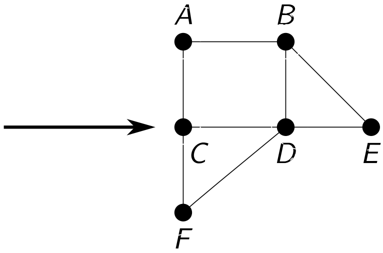
Theorem [M. 2007]

Let $\mathcal{Q}$ be a computable class of queries with binary relations. Assuming the Exponential-time Hypothesis, there is no algorithm for BCQ restricted to $\mathcal{Q}$ with running time $f(Q)N^{o(\text{ctw}(Q)/\log \text{ctw}(Q))}$, where $\text{ctw}(Q)$ is the treewidth of the core of the primal graph of Q.

Next: relations of arbitrary arity

Primal graph: vertices are the variables, two vertices are adjacent if they appear in a common relation of the query.

$R(A, B) \wedge R(A, C) \wedge$
$R(B, D, E) \wedge R(C, D, F)$



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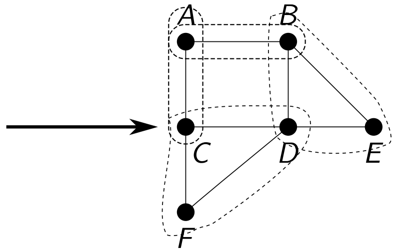


Most of the theoretical results go through for fixed constant arity.

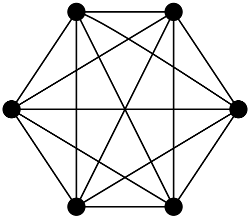
But for undbounded arities we need to look at the **hypergraph** of the query!

Primal graph vs. hypergraphs

The primal graph loses a lot of information if arity is unbounded.

$$Q_1 = \bigwedge_{i \neq j} R(A_i, A_j)$$

$$Q_2 = R(A_1, \ldots, A_k)$$



- Queries of the form Q_1 are hard: binary relations with large treewidth.
- Queries of the form Q_2 are trivial: N tuples to consider.

Primal graph vs. hypergraphs

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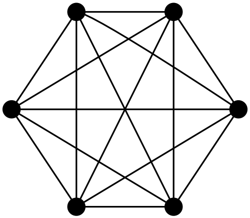
$$Q_1 = \bigwedge_{i \neq j} R(A_i, A_j)$$

$$Q_2 = R(A_1, \dots, A_k) \wedge S(A_2, A_3, A_5) \wedge T(A_3, A_8) \dots$$



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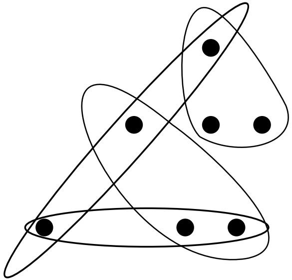
- Queries of the form Q_2 are trivial: N tuples to consider.

What do we know about bounding the size of the answer?

(…and enumerating all solutions)

Upper bound

Observation: If the hypergraph has edge cover number ρ and every relation has size at most N, then there are at most N^{ρ} tuples in the answer.



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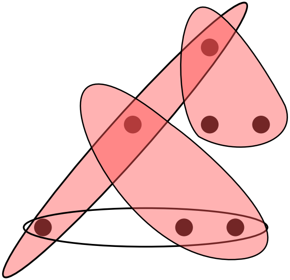
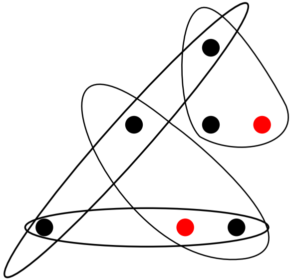


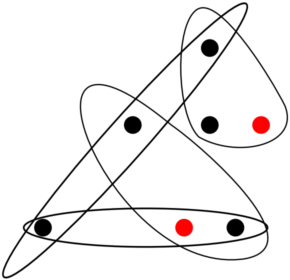
Lower bound

Observation: If the hypergraph has independence number α, then one can construct an instance where every relation has size N at the answer has size N^{α}.

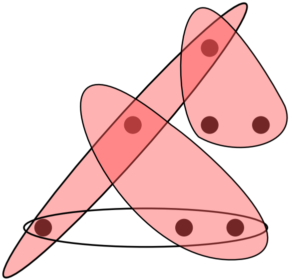


Definition of the relations:

- If variable A is in the independent set, then it can take any value in $[N]$.
- Otherwise it is forced to 1.



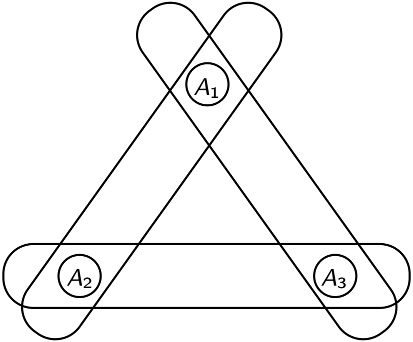
N^2



N^3

Which is tight: the upper bound or the lower bound?

Example: triangles



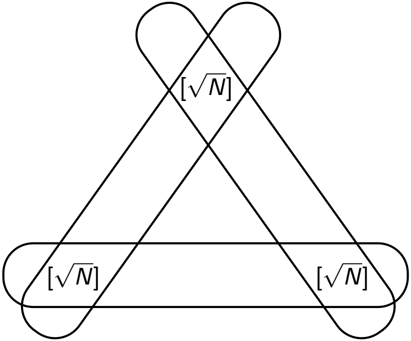
Upper bound

Two kind of values for A_1:

- Light: can be extended to at most $\sqrt{N}$ ways to A_2.
  $\Rightarrow \leq N \cdot \sqrt{N}$ answers with light A_1
- Heavy: can be extended to at least $\sqrt{N}$ ways to A_2.
  $\Rightarrow \leq \sqrt{N}$ heavy values $\Rightarrow \leq \sqrt{N} \cdot N$ answers with heavy A_1

$\Rightarrow$ At most $2 \cdot N^{3/2}$ answers.

Example: triangles



Lower bound

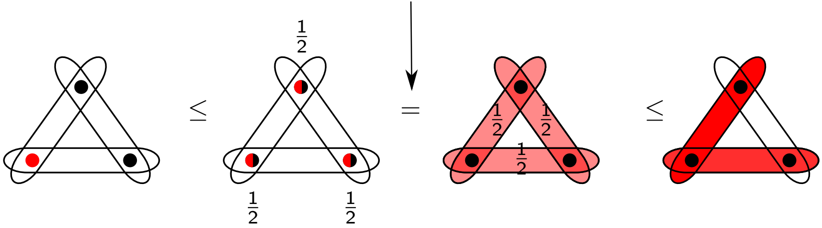
Allow every variable to be any value from $[\sqrt{N}] \Rightarrow N^{3/2}$ answers.

The correct bound $N^{3/2}$ is between
$N^{\alpha} = N^1$ and $N^{\rho} = N^2$.

Fractional values

- α: independence number
- α^*: fractional independence number
  (max. weight of vertices s.t. each edge contains weight ≤ 1)
- ρ^*: fractional edge cover number
  (min. weight of edges s.t. each vertex receives weight ≥ 1)
- ρ: edge cover number

LP duality!



$\alpha = 1$ $\qquad$ $\alpha^* = 3/2$ $\qquad$ $\rho^* = 3/2$ $\qquad$ $\rho = 2$

Tight bound

Theorem [Atserias, Grohe, M. 2008]

Consider a query with fractional edge cover number ρ^*.

- If every relation has size at most N, there are at most N^{ρ^*} answers.
- For every N, one can construct relations of size $\leq N$ such that there are $\approx N^{\rho^*}$ answers.

Upper bound

Follows from classic combinatorial/probabilistic/geometric results (Shearer's Lemma, Submodularity of Entropy, Loomis-Whitney, ...)

Tight bound

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Lower bound

Let f be a max. fractional independent set. Allow variable A to have any value from $[N^{f(A)}]$.

Size of relation R:

$$\prod_{A \text{ in } R} N^{f(A)} = N^{\sum_{A \in a(R)} f(A)} \leq N^1$$

Answer size:

$$\prod_{A} N^{f(A)} = N^{\sum_A f(A)} = N^{\alpha^*} = N^{\rho^*}$$

Enumerating all solutions

Can we find all solutions in time roughly N^{ρ^*}?

Possible approaches:

- Join plan
- Join-Project plan
- Something else

Join-Project plans

$Q_i = Q_{|A_1,\dots,A_i}$ — projection to the first i variables.

Observation 1:
$\rho^*(Q_i) \leq \rho^*(Q)$, so the N^{ρ^*} upper bound holds for every Q_i.

Join-Project plans

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Observation 2:
Q_i can be computed from Q_{i-1} in time N^{ρ^*+1}:

$$Q_i = ((\ldots(Q_{i-1} \bowtie R_{1|A_1,\ldots,A_i})) \bowtie R_{2|A_1,\ldots,A_i})\ldots \bowtie R_{m|A_1,\ldots,A_i}$$

$\Rightarrow$ Simple Join-Project plan in N^{ρ^*+1} time.

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$\Rightarrow$ Simple Join-Project plan in N^{ρ^*+1} time.

- Do we need projections?
- Can we get rid of the $+1$?

Example

Our "favorite hypergraph": $2m$ relations, $\binom{2m}{m}$ variables, each contained in exactly m relations.

$m = 2$:

$$R_1(A_{12}, A_{13}, A_{14}) \wedge R_2(A_{12}, A_{23}, A_{24}) \wedge$$
$$R_3(A_{13}, A_{23}, A_{34}) \wedge R_4(A_{14}, A_{24}, A_{34})$$

Example

Our "favorite hypergraph": $2m$ relations, $\binom{2m}{m}$ variables, each contained in exactly m relations.

$$
\begin{aligned}
m = 3\text{:}\quad & R_1(A_{123}, A_{124}, A_{125}, A_{126}, A_{134}, A_{135}, A_{136}, A_{145}, A_{146}, A_{156}) \wedge \\
& R_2(A_{123}, A_{124}, A_{125}, A_{126}, A_{234}, A_{235}, A_{236}, A_{245}, A_{246}, A_{256}) \wedge \\
& R_3(A_{123}, A_{134}, A_{135}, A_{136}, A_{234}, A_{235}, A_{236}, A_{245}, A_{246}, A_{256}) \wedge \\
& R_4(A_{124}, A_{134}, A_{145}, A_{146}, A_{234}, A_{245}, A_{246}, A_{345}, A_{346}, A_{456}) \wedge \\
& R_5(A_{125}, A_{135}, A_{145}, A_{156}, A_{235}, A_{245}, A_{256}, A_{345}, A_{356}, A_{456}) \wedge \\
& R_6(A_{126}, A_{136}, A_{146}, A_{156}, A_{236}, A_{246}, A_{256}, A_{346}, A_{356}, A_{456})
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Edge cover number

$\rho = m + 1$: if you pick e.g., $R_1, \ldots, R_m$, then $A_{m+1,\ldots,2m}$ is not covered.

Fractional edge cover number

$\rho^* = 2$: weight $1/m$ for every relation, every variable is in m relations.

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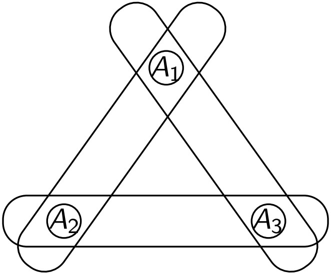
\end{aligned}
$$

Join plans

- There is a point where we have joined roughly $m/2$ relations, say, $R_1 \wedge \ldots \wedge R_{m/2}$.
- This hypergraph has an independent set of size $m/2$: variables $A_{i,m+1,\ldots,2m}$ are independent for $1 \le i \le m/2$.
- One can use this to make sure that there are $N^{m/2}$ solutions.

Join-Project plans are suboptimal



$$R = ([N/2] \times [1]) \cup ([1] \times [N/2])$$

Join-Project plan first joins two relations:

$$R(A_1, A_2) \bowtie R(A_2, A_3) = ([N/2] \times 1 \times [N/2]) \cup (1 \cup \times [N/2] \cup 1)$$

Has size $\Omega(N^2)$ — but the upper bound is $N^{3/2}$.

Optimal join algorithms

We can get rid of the $+1$ in the exponent, but these are not Join-Project algorithms.

- Ngo, Porat, Ré, and Rutra [PODS 2012]
- Veldhuizen [ICDT 2014]
- Ngo and Rudra [Sigmod Record 13]

Back to Boolean Conjunctive Queries

We have seen that treewidth of the primal graph is not a good measure of the complexity of BCQ with unbounded arities.

Tree decomposition $+$ Size bounds $=$?

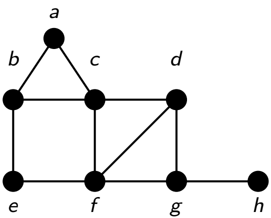
Treewidth — a measure of “tree-likeness”

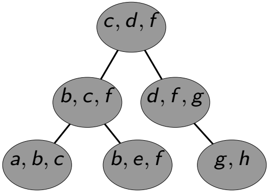
Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. For any hyperedge e, there is a bag containing e.
2. For every v, the bags containing v form a connected subtree.

Width of the decomposition: largest bag size -1.

treewidth: width of the best decomposition.





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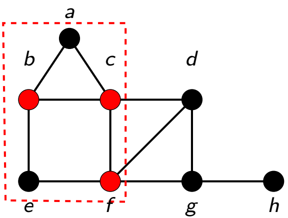
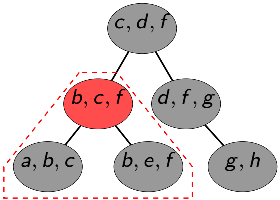




A subtree communicates with the outside world only via the root of the subtree.

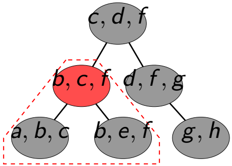
Boolean Conjunctive Queries and tree decompositions

Theorem

Given a tree decomposition of width w, a Boolean Conjunctive Query where every variable allows at most N different values can can be solved in time $N^{w+1} \cdot |Q|^{O(1)}$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

For every node x and tuple $t \in Q_{|B_x}$, we compute the Boolean value $E[x,t]$, which is true if and only if t can be extended to a tuple of $Q_{|V_x}$.



Claim:
We can determine $E[x,t]$ if all the values are known for the children of x.

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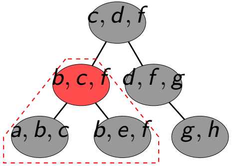

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Running time:
Dominating factor is the size of $Q_{|B_x}$, which can be bounded by $N^{|B_x|} \leq N^{w+1}$.

Fractional hypertree width

Fractional hypertree width: every bag has fractional edge cover number at most k.

Theorem [Grohe and M. 2006]

Given a fractional hypertree decomposition of width k, a Boolean Conjunctive Query where every variable allows at most N different values can can be solved in time $N^k \cdot |Q|^{O(1)}$.

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Generalized hypertree width: every bag has edge cover number at most k.

Hypertree width: same as generalized hypertree width, with an additional "special condition."

Acyclic hypergraphs: hypetree width $=$ generalized hypertree width $= 1$.

Finding decompositions

- If we want fixed-parameter tractability, then we can find an optimal decomposition in time $f(H)$.
- For polynomial-time algorithms, we need to find good decompositions in polynomial time.

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Treewidth

- optimal decomposition in time n^k [Robertson and Seymour].
- optimal decomposition in time $2^{O(k^3)} \cdot n$ [Bodlaender 1996].
- 5-approximate decomposition in time $2^{O(k)} \cdot n$ [Bodlaender et al. 2013].
- $O(\sqrt{\log k})$-approximation in polynomial time [Feige, Hajiaghayi, Lee 2008].

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Hypertree width

- optimal decomposition in time n^k [Gottlob, Leone, and Scarcello 2002]
- W[1]-hard $\Rightarrow$ no FPT algorithm.

Finding decompositions

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Generalized hypertree width

- NP-hard even for $k \geq 3$ [Gottlob, Miklós, Schwentick PODS 2007] and for $w = 2$ [Fischl, Gottlob, and Pichler 2016]
- But $ghw \leq hw \leq 3 \cdot ghw \Rightarrow$ Hypertree width gives a 3-approximation!

Finding decompositions

- If we want fixed-parameter tractability, then we can find an optimal decomposition in time $f(H)$.
- For polynomial-time algorithms, we need to find good decompositions in polynomial time.

Fractional hypertree width

- For every $k \geq 1$, there is a polynomial-time algorithm computing a decomposition of width $O(k^3)$ [M. 2009].

Theorem

If class $\mathcal{H}$ has bounded fractional hypertree width, then $\mathrm{BCQ}(\mathcal{H})$ can be solved in polynomial time.

- NP-hard for every $k \geq 2$ [Fischl, Gottlob, and Pichler 2016]

Better decompositions?

Fractional hypertree decomposition is the **best possible** tree decomposition in a formal sense.

Observation: If a tree decomposition guarantees that the projection to every bag has at most N^w solutions, then the decomposition has fractional hypertree width at most w.

(If a bag has fractional edge cover number ρ^*, we can construct an instance where it has N^{ρ^*} solutions.)

Better decompositions?

Fractional hypertree decomposition is the **best possible** tree decomposition in a formal sense.

How can we move beyond fractional hypertree decompositions?

- **Idea 1:** Look at the database, and choose a decomposition based on that (not only on the query).
- **Idea 2:** Branch and partition the solution space (e.g., light-heavy) and choose different decompositions.

Submodular width

Theorem [M. 2010]

Let $\mathcal{H}$ be a computable class of hypergraphs. Assuming the Exponential-Time Hypothesis, the following are equivalent:

- BCQ($\mathcal{H}$) is fixed-parameter tractable (solvable in time $f(Q) \cdot N^{O(1)}$).
- $\mathcal{H}$ has bounded submodular width.

Definition: H has submodular width $\leq w$ if for any function $f : 2^{V(H)} \to \mathbb{R}^+$ that is

- **monotone** ($f(X) \geq f(Y)$ for any $X \supset Y$),
- **submodular** ($f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$), and
- **edge dominated** ($f(e) \leq 1$ for any edge $e \in E(H)$)

there is a tree decomposition of H with $f(B) \leq w$ for every bag B.

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Intuitive algorithmic idea: we imagine

$$f(X) \approx \frac{\log \#\text{ solutions in } Q_{|X}}{\log N}$$

Then there is a decomposition where $f(B) \leq w$ for every bag, so $|Q_{|B}| \leq N^w$.

Conclusions

Messages

- Treelike decompositions can make the problem easy.
- You may want to look at the data and choose a decomposition based on that.
- You may want to branch and choose different decompositions in the different branches.

Topics not covered: counting, enumeration, quantification, functional dependencies, parallel algorithms ...