Solving Planar k-Terminal Cut in $O(n^{c\sqrt{k}})$ time

Philip N. Klein[1] Dániel Marx[2]

[1]Computer Science Department,
Brown University
Providence, RI

[2]Computer and Automation Research Institute,
Hungarian Academy of Sciences (MTA SZTAKI)
Budapest, Hungary

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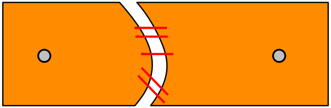
A classical problem

$s - t$ Cut

Input: A graph G, an integer p, vertices s and t

Output: A set S of at most p edges such that removing S separates s and t.



Fact

A minimum $s - t$ cut can be found in polynomial time.

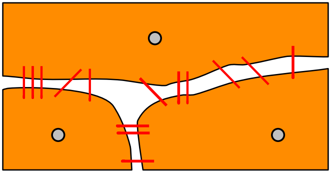
What about separating more than two terminals?

More than two terminals

Multiway Cut

Input: A graph G, an integer p, and a set T of terminals
Output: A set S of at most p edges such that removing S separates any two vertices of T

Note: Also called Multiterminal Cut or k-Terminal Cut.



Theorem [Dalhaus et al. 1994]

NP-hard already for $|T| = 3$.

Planar graphs

Theorem [Dalhaus et al. 1994] [Hartvigsen 1998] [Bentz 2012]

k-Terminal Cut can be solved in time $n^{O(k)}$ on **planar graphs.**

Main result

k-Terminal Cut can be solved in time $c^k \cdot n^{O(\sqrt{k})}$ on **planar graphs.**

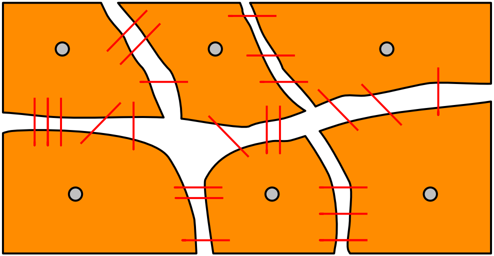
The improvement in the exponent is best possible:

Previous talk

Assuming ETH, k-Terminal Cut on planar graphs cannot be solved in time $f(k) \cdot n^{o(\sqrt{k})}$ for any computable function $f(k)$.

Dual graph

The previous algorithms (as well as ours) look at the solution in the dual graph



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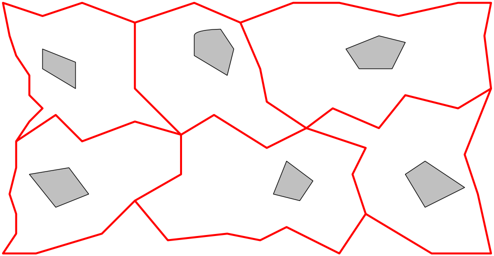

Recall:

Primal graph		Dual graph
vertices	$\Leftrightarrow$	faces
faces	$\Leftrightarrow$	vertices
edges	$\Leftrightarrow$	edges

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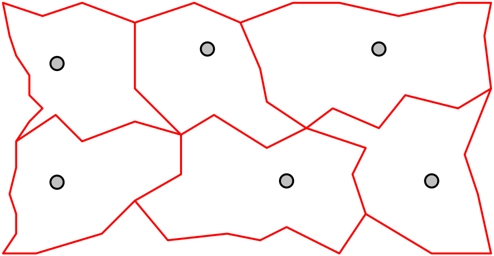

Recall:

Primal graph		Dual graph
vertices	$\Leftrightarrow$	faces
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edges	$\Leftrightarrow$	edges

We slightly transform the problem in such a way that the terminals are represented by **vertices** in the dual graph (instead of faces).

Previous approaches

[Dalhaus et al. 1994] [Hartvigsen 1998] [Bentz 2012]

1. The dual solution has $O(k)$ branch vertices.
2. Guess the location of branch vertices ($n^{O(k)}$ guesses).
3. Deep magic to find the paths connecting the branch vertices (shortest paths are not necessarily good!)

New idea:

Fact

A planar graph with k vertices has treewidth $O(\sqrt{k})$.

The dual solution has treewidth $O(\sqrt{k})$, so instead of guessing, let's find the vertices in a dynamic programming on the tree decomposition.

Problem: How to implement the deep magic in a DP?

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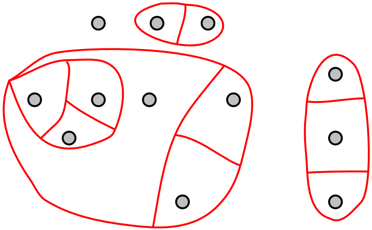
2-connectivity

In general, the dual solution is not 2-connected.



2-connected problem

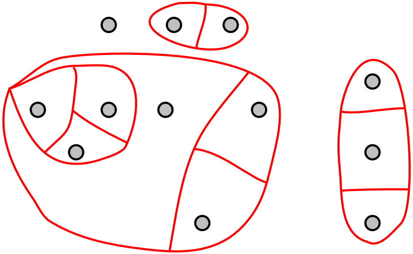
Find a 2-connected dual solution that separates a subset X of terminals from each other and from every other terminal.

A simple DP reduces the original problem to the 2-connected problem.

2-connectivity

$a(X)$: cost of separating the terminals in X from each other.
$b(X)$: cost of separating X from each other and from every other terminal with a solution that is 2-connected in the dual.

$$a(T) = \min_{\emptyset \neq X \subseteq T} (b(X) + a(T \setminus X))$$



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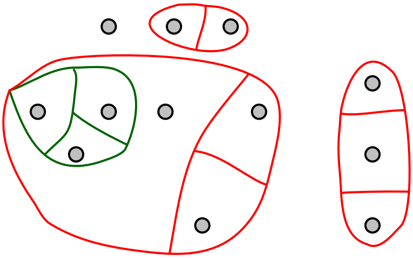
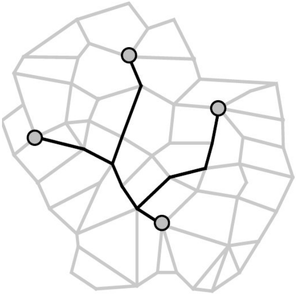


The Steiner tree

We find a minimum cost Steiner tree T of the terminals in the **dual** and cut open the graph along the tree.
(Steiner tree: $3^k \cdot n^{O(1)}$ time by [Dreyfus-Wagner 1972] or $2^k \cdot n^{O(1)}$ time by [Björklund 2007])



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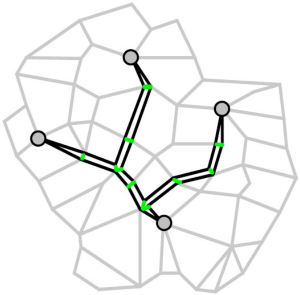

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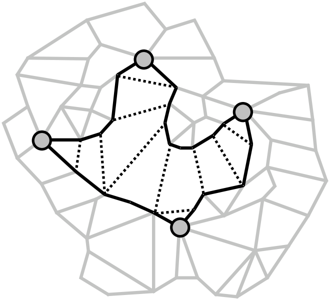

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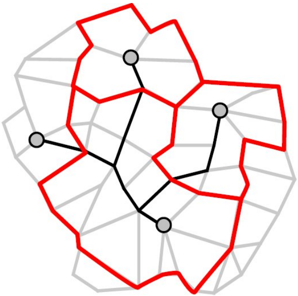

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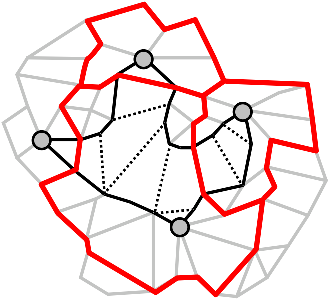

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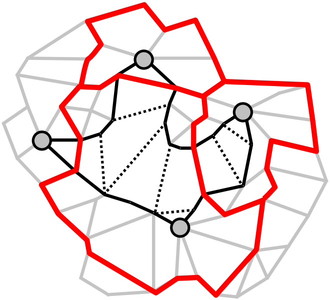

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Topology

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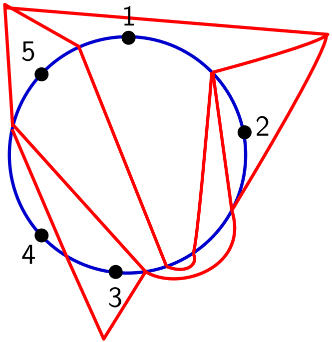
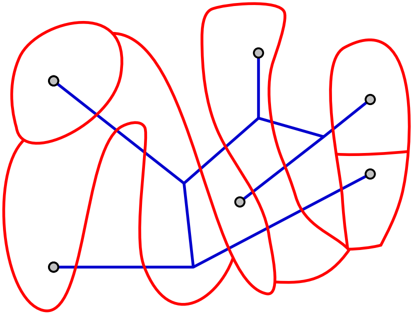


Thus a solution can be completely described by the location of these points and which of them are connected.
A "topology" just describes the connections without the locations.

A combinatorial lemma



Lemma

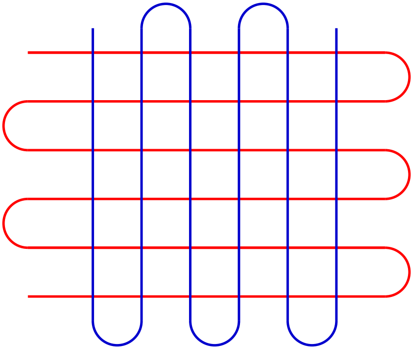
There is an optimum dual solution S that has $O(k)$ branch vertices and “crosses the tree” $O(k)$ times.

Proof uses

- the minimality of T,
- the minimality of S,
- the 2-connectivity of S,
- Euler’s formula.

A proof idea

Why this cannot happen?



There are no red-blue-red-blue faces:

- If $x < y$, then we can get a better solution S.
- If $x > y$, then we can get a better Steiner tree T.

A proof idea

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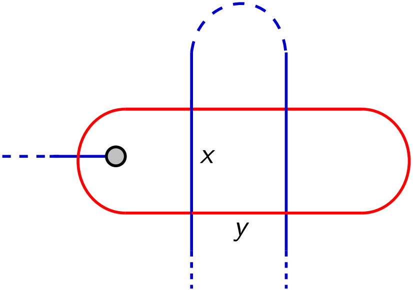

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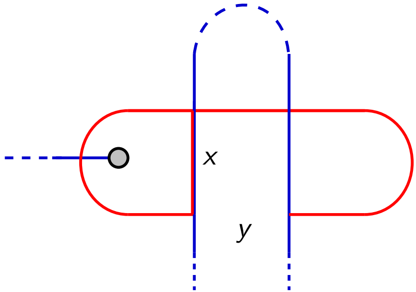

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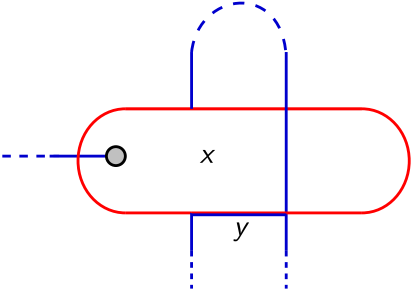

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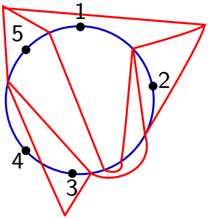
Realizing a topology

Lemma

Given a topology of size p, we can find a minimum cost realization in time $n^{O(\sqrt{p})}$.

- p branch points/crossing points $\Rightarrow$ treewidth is $O(\sqrt{p})$.
- Fairly standard DP on the tree decomposition.
- In each bag of the tree decomposition, we have to keep track of the location of $O(\sqrt{p})$ points $\Rightarrow n^{O(\sqrt{p})}$ possibilities.
- We need that the crossing points and the terminals are in the right order, but that is easy.



Algorithm

For the 2-connected problem:

1. Find the Steiner tree T ($2^k \cdot n^{O(1)}$ time).
2. Cut along T.
3. Guess a "topology" of size $O(k)$ (c^k guesses).
4. Find a minimum cost realization of the topology using DP on the tree decomposition ($n^{O(\sqrt{k})}$ time).

For the general problem:

1. Solve 2^k instances of the 2-connected problem.
2. Solved the general problem for every subset using DP.

Conclusions

- A $c^k \cdot n^{O(\sqrt{k})}$ time algorithm for k-terminal planar Multiway Cut.
- Is there an $n^{O(\sqrt{k})}$ time algorithm?
- Eventually boils down to the $O(\sqrt{n})$ treewidth bound on planar graphs, but not just a trivial application of bidimensionality.
- It seems hard to prove lower bounds better than $\Omega(\sqrt{k})$ for planar problems. There should be $O(\sqrt{k})$ algorithms for **all** these problems!