Subexponential parameterized algorithms on planar graphs via low-treewidth pattern covering

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Square root phenomenon

Most NP-hard problems (e.g., 3-Coloring, Independent Set, Hamiltonian Cycle, Steiner Tree, etc.) remain NP-hard on planar graphs,\(^1\) but often get easier on planar graphs in the sense that the running time is still exponential, but significantly smaller:

\[
2^{O(n)} \Rightarrow 2^{O(\sqrt{n})} \\
2^{O(k)} \Rightarrow 2^{O(\sqrt{k})} \\
n^{O(k)} \Rightarrow n^{O(\sqrt{k})} \\
2^{O(k) \cdot n^{O(1)}} \Rightarrow 2^{O(\sqrt{k}) \cdot n^{O(1)}}
\]

This talk: a new technique for such algorithms.

\(^1\)Notable exception: Max Cut is in P for planar graphs.
**Subgraph Isomorphism**

**Input:** Graphs $H$ and $G$

**Decide:** Does $G$ has a subgraph isomorphic to $H$?

Standard dynamic programming:

**Fact**

If connected graph $H$ has $k$ vertices and maximum degree $\Delta$, $G$ has treewidth $w$, then **Subgraph Isomorphism** can be solved

- in time $2^{O(k)} \cdot k^{O(w)} \cdot n$ or
- in time $k^{O(\Delta w)} \cdot n$.

**Remark:** Robust algorithm, can be easily generalized to colored, directed, weighted etc. versions.
**k-outterplanar graphs**

Given a planar embedding, we can define *layers* by iteratively removing the vertices on the infinite face.

**Definition**

A planar graph is *k-outterplanar* if it has a planar embedding having at most *k* layers.

**Fact**

Every *k*-outerplanar graph has treewidth at most $3k + 1$. 

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For a fixed $0 \leq s < k + 1$, delete every layer $L_i$ with $i = s \pmod{k + 1}$
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- For a fixed $0 \leq s < k + 1$, delete every layer $L_i$ with $i = s \pmod{k + 1}$.
- The resulting graph is $k$-outerplanar, hence it has treewidth at most $w := 3k + 1$.
- Using the $2^{O(k)} \cdot k^{O(w)} \cdot n$ time algorithm for \textsc{Subgraph Isomorphism}, the problem can be solved in time $k^{O(k)} \cdot n = 2^{O(k \log k)} \cdot n$. 
Baker’s shifting strategy

We do this for every $0 \leq s < k + 1$: for at least one value of $s$, we do not delete any of the $k$ vertices of the solution

$\downarrow$

We find a copy of $H$ in $G$ if there is one.
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Theorem

**Subgraph Isomorphism** for planar graphs can be solved in time $2^{O(k \log k)} \cdot n$ for $k := |V(H)|$. 

Next: Improved algorithm for the special case $k$-Path via bidimensionality.
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**Next:** Improved algorithm for the special case $k$-Path via bidimensionality.
Planar Excluded Grid Theorem

Theorem [Robertson, Seymour, Thomas 1994]

Every planar graph with treewidth at least $5k$ has a $k \times k$ grid minor.

Note: for general graphs, treewidth at least $k^{19}$ or so guarantees a $k \times k$ grid minor!
Planar Excluded Grid Theorem

**Theorem [Robertson, Seymour, Thomas 1994]**

Every planar graph with treewidth at least $5k$ has a $k \times k$ grid minor.

**Consequence:** every $n$-vertex planar graph has treewidth $O(\sqrt{n})$. 
Subexponential algorithm for $k$-PATH

**Observation:** If the treewidth of a planar graph $G$ is at least $5\sqrt{k}$
⇒ It has a $\sqrt{k} \times \sqrt{k}$ grid minor (Planar Excluded Grid Theorem)
⇒ The grid has a path of length at least $k$.
⇒ $G$ has a path of length at least $k$. 
Subexponential algorithm for \textbf{\(k\)-Path}

\textbf{Observation:} If the treewidth of a planar graph \(G\) is at least \(5\sqrt{k}\)
\(\Rightarrow\) It has a \(\sqrt{k} \times \sqrt{k}\) grid minor (Planar Excluded Grid Theorem)
\(\Rightarrow\) The grid has a path of length at least \(k\).
\(\Rightarrow\) \(G\) has a path of length at least \(k\).

We use this observation to find a path of length at least \(k\) on planar graphs:

- Set \(w := 5\sqrt{k}\).
- Find an \(O(1)\)-approximate tree decomposition.
  - If treewidth is at least \(w\): we answer “there is a path of length at least \(k\).”
  - If we get a tree decomposition of width \(O(w)\), then we can solve the problem in time
    \[k^{O(\Delta w)} \cdot n^{O(1)} = 2^{O(\sqrt{k \log k})} \cdot n^{O(1)}.\]
Lower bounds based on ETH

Lower bound technology introduced by Impagliazzo, Paturi, and Zane:

Exponential-Time Hypothesis

There is no $2^{o(n)}$-time algorithm for $n$-variable 3SAT.
Lower bounds based on ETH

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**Exponential-Time Hypothesis + Sparsification Lemma**

There is no $2^{o(n+m)}$-time algorithm for $n$-variable $m$-clause 3SAT.
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Textbook reduction from 3SAT to Planar Hamiltonian Path:

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- $n$ variables
- $m$ clauses

$\Rightarrow$

Planar graph $G'$
- $O((n + m)^2)$ vertices
- $O((n + m)^2)$ edges

Corollary

Assuming ETH, there is no $2^{o(\sqrt{n})}$ algorithm for Planar Hamiltonian Path on an $n$-vertex planar graph $G$. 
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**Corollary**

Assuming ETH, there is no $2^{o(\sqrt{k})} \cdot n^{O(1)}$ algorithm for $k$-Path on an $n$-vertex planar graph $G$.

Our $2^{O(\sqrt{k \log k})} \cdot n^{O(1)}$ algorithm is essentially best possible.
Other problems:

Good news:
- Same algorithm works for finding a cycle of length at least $k$.

Bad news:
- Does not work for finding a cycle of length exactly $k$. 
- Does not work for finding a $s-t$ path of length at least/exactly $k$. 
- Does not work for finding a minimum weight $k$-path. 
- Does not work for finding a directed $k$-path.
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- ...
Main combinatorial result

Theorem

There is a randomized polynomial-time algorithm that, given a planar graph $G$ and an integer $k$, computes an induced subgraph $G'$ such that

1. $G'$ has treewidth $O(\sqrt{k} \cdot \text{polylog}(k))$ and
2. for any connected subgraph $H \subseteq G$ with at most $k$ vertices, we have $H \subseteq G'$ with probability at least $(2^{O(\sqrt{k} \cdot \text{polylog}(k))} \cdot n^{O(1)})^{-1}$. 
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Thus the **Subgraph Isomorphism** problem for connected $H$ can be solved by restriction to $G'$.

Theorem

**Subgraph Isomorphism** for planar graphs can be solved in time $2^{O(\Delta \sqrt{k} \cdot \text{polylog}(k))} \cdot n^{O(1)}$ if $H$ is connected with maximum degree $\Delta$.

Remark: Robust algorithm, can be easily generalized to colored, directed, weighted etc. versions.
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Theorem [Bodlaender, Nederlof, van der Zanden 2016]

Assuming ETH, there is no $2^{o(k/\log k)} \cdot n^{O(1)}$ time algorithm for planar Subgraph Isomorphism, even when

- $H$ is a forest of maximum degree 3, or
- $H$ is a tree with only one vertex having degree larger than 3.
Example: grids

1. Guess an index $0 \leq i < \sqrt{k}$ such that rows $i \mod \sqrt{k}$ contain a total of $\sqrt{k}$ vertices of $H$.

2. Guess the at most $\sqrt{k}$ columns where $H$ appears in these rows.

3. Graph falls apart into $\sqrt{k}$-tall grids connected by $\sqrt{k}$ vertices.
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Example: complete binary tree

Ball-growing argument: there is an index $\sqrt{k} \leq i \leq O(\sqrt{k} \log k)$ such that the first $i$ rows in total contain $\sqrt{k}$ times more vertices of the solution than row $i$. 
Example: complete binary tree

**Ball-growing argument:** there is an index $\sqrt{k} \leq i \leq O(\sqrt{k \log k})$ such that the first $i$ rows in total contain $\sqrt{k}$ times more vertices of the solution than row $i$. 
Duality result

Theorem

Given a graph $G$ and two sets of vertices $S$ and $T$ there is either

- a family $P_1, \ldots, P_C$ of “almost-disjoint” $S - T$ paths such that $\exists A_i \subseteq P_i$ with $\sum |A_i| \leq \ell C$ and $P_i \setminus A_i$’s are pairwise disjoint or
- a family $S_1, \ldots, S_\ell$ of disjoint “small” $S - T$ separators with $|S_i| \leq C$.
Theorem

Given a graph $G$ and two sets of vertices $S$ and $T$ there is either

- a family $P_1, \ldots, P_{C/2}$ of “almost-disjoint” $S - T$ paths such that $\exists A_i \subseteq P_i$ with $|A_i| \leq \ell$ and $P_i \setminus A_i$’s are pairwise disjoint or

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![Graph diagram]

We will use the duality with $C = 2(k + 1)^2$ and $\ell = \sqrt{k}$. 

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**Theorem**

Given a graph $G$ and two sets of vertices $S$ and $T$ there is either

- a family $P_1, \ldots, P_{k+1}$ of “almost-disjoint” $S - T$ paths such that $\exists A_i \subseteq P_i$ with $|A_i| \leq \sqrt{k}$ and $P_i \setminus A_i$’s are pairwise disjoint or
- a family $S_1, \ldots, S_{\sqrt{k}}$ of disjoint “small” $S - T$ seps. with $|S_i| \leq 2k + 2$.

We will use the duality with $C = 2(k + 1)$ and $\ell = \sqrt{k}$. 
Using duality

The correct viewpoint:
Using the duality between the outside and the inside.

But where is this “inside”?
Conclusions

- Subexponential parameterized algorithms for finding bounded-degree connected subgraphs.
- Connectedness of the pattern $H$ seems essential (but easy to generalize to bounded number of connected components).
- Can be generalized to bounded local treewidth, $H$-minor-free in progress.
- Other classes of graphs: polynomial growth property.