Subexponential parameterized algorithms on planar graphs via low-treewidth pattern covering

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(joint work with Fedor Fomin,
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Square root phenomenon

Most NP-hard problems (e.g., 3-Coloring, Independent Set, Hamiltonian Cycle, Steiner Tree, etc.) remain NP-hard on planar graphs,[1] but often get easier on planar graphs in the sense that the running time is still exponential, but significantly smaller:

$$\begin{aligned} 2^{O(n)} &\Rightarrow 2^{O(\sqrt{n})} \\ n^{O(k)} &\Rightarrow n^{O(\sqrt{k})} \\ 2^{O(k)} \cdot n^{O(1)} &\Rightarrow 2^{O(\sqrt{k})} \cdot n^{O(1)} \end{aligned}$$

This talk: a new technique for such algorithms.

[1] Notable exception: Max Cut is in P for planar graphs.

Subgraph Isomorphism

Subgraph Isomorphism

Input: Graphs H and G

Decide: Does G has a subgraph isomorphic to H?

Standard dynamic programming:

Fact

If connected graph H has k vertices and maximum degree Δ, G has treewidth w, then Subgraph Isomorphism can be solved

- in time $2^{O(k)} \cdot k^{O(w)} \cdot n$ or
- in time $k^{O(\Delta w)} \cdot n$.

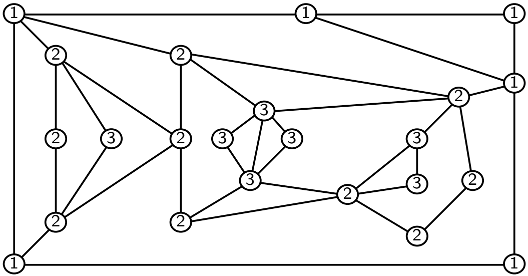
Remark: Robust algorithm, can be easily generalized to colored, directed, weighted etc. versions.

k-outerplanar graphs

Given a planar embedding, we can define **layers** by iteratively removing the vertices on the infinite face.

Definition

A planar graph is k-**outerplanar** if it has a planar embedding having at most k layers.



Fact

Every k-outerplanar graph has treewidth at most $3k + 1$.

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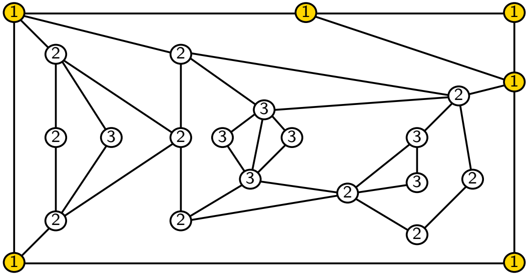

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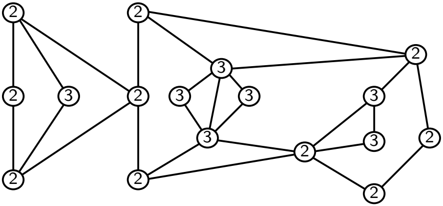

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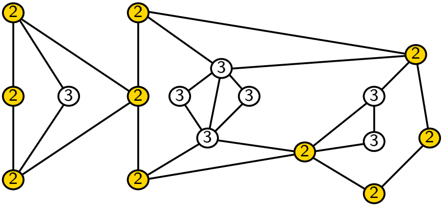

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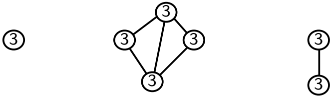

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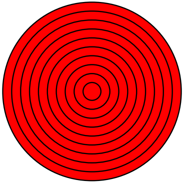
Baker's shifting strategy



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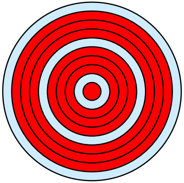

- For a fixed $0 \leq s < k+1$, delete every layer L_i with $i = s \pmod{k+1}$

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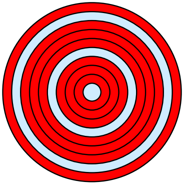

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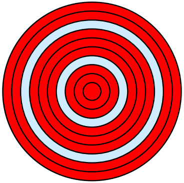

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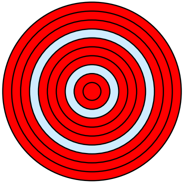

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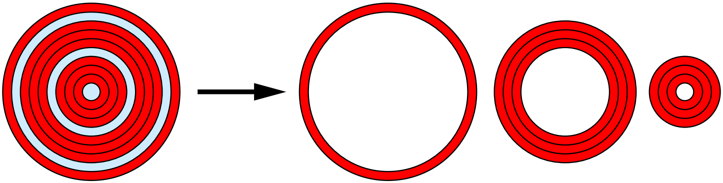

- For a fixed $0 \leq s < k+1$, delete every layer L_i with $i = s \pmod{k+1}$
- The resulting graph is k-outerplanar, hence it has treewidth at most $w := 3k+1$.
- Using the $2^{O(k)} \cdot k^{O(w)} \cdot n$ time algorithm for Subgraph Isomorphism, the problem can be solved in time $k^{O(k)} \cdot n = 2^{O(k \log k)} \cdot n$.

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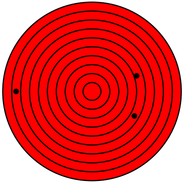

We do this for every $0 \leq s < k+1$:
for at least one value of s, we do not delete
any of the k vertices of the solution

$\Downarrow$

We find a copy of H in G if there is one.

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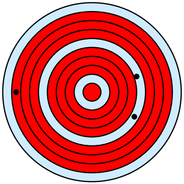

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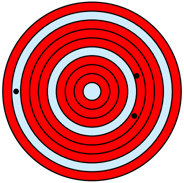

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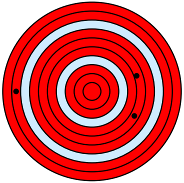

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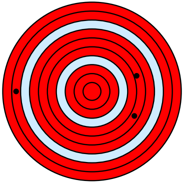

Theorem

Subgraph Isomorphism for planar graphs can be solved in time $2^{O(k \log k)} \cdot n$ for $k := |V(H)|$.

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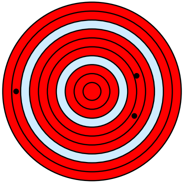

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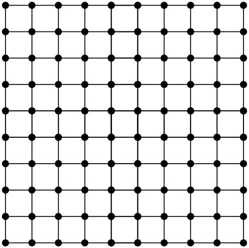
Next: Improved algorithm for the special case k-Path via bidimensionality.

Planar Excluded Grid Theorem

Theorem [Robertson, Seymour, Thomas 1994]

Every planar graph with treewidth at least $5k$ has a $k \times k$ grid minor.



Note: for general graphs, treewidth at least k^{19} or so guarantees a $k \times k$ grid minor!

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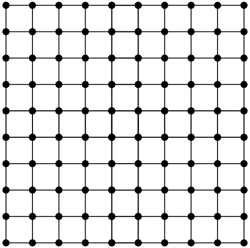
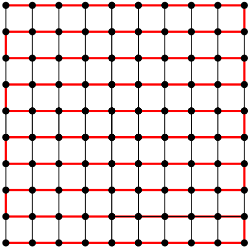


Consequence: every n-vertex planar graph has treewidth $O(\sqrt{n})$.

Subexponential algorithm for k-Path

Observation: If the treewidth of a planar graph G is at least $5\sqrt{k}$
$\Rightarrow$ It has a $\sqrt{k} \times \sqrt{k}$ grid minor (Planar Excluded Grid Theorem)
$\Rightarrow$ The grid has a path of length at least k.
$\Rightarrow$ G has a path of length at least k.



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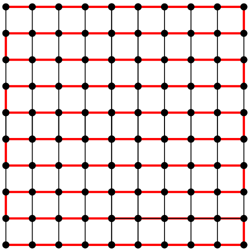
We use this observation to find a path of length at least k on planar graphs:

- Set $w := 5\sqrt{k}$.
- Find an $O(1)$-approximate tree decomposition.
  - If treewidth is at least w: we answer "there is a path of length at least k."
  - If we get a tree decomposition of width $O(w)$, then we can solve the problem in time $k^{O(\Delta w)} \cdot n^{O(1)} = 2^{O(\sqrt{k}\log k)} \cdot n^{O(1)}$.



Lower bounds based on ETH

Lower bound technology introduced by Impagliazzo, Paturi, and Zane:

Exponential-Time Hypothesis

There is no $2^{o(n)}$-time algorithm for n-variable $\mathrm{3SAT}$.

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Textbook reduction from 3SAT to PLANAR HAMILTONIAN PATH:

3SAT formula ϕ		Planar graph G'
n variables	$\Rightarrow$	$O((n+m)^2)$ vertices
m clauses		$O((n+m)^2)$ edges

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Corollary

Assuming ETH, there is no $2^{o(\sqrt{n})}$ algorithm for Planar Hamiltonian Path on an n-vertex planar graph G.

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Corollary

Assuming ETH, there is no $2^{o(\sqrt{k})} \cdot n^{O(1)}$ algorithm for k-Path on an n-vertex planar graph G.

Our $2^{O(\sqrt{k}\log k)} \cdot n^{O(1)}$ algorithm is essentially best possible.

Other problems:

Good news:

- Same algorithm works for finding a cycle of length at least k.

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- Same algorithm works for finding a cycle of length at least k.

Bad news:

- Does not work for finding a cycle of length **exactly** k.
- Does not work for finding an $s - t$ path of length at least/exactly k.
- Does not work for finding a minimum weight k-path.
- Does not work for finding a directed k-path.
- ...

Main combinatorial result

Theorem

There is a randomized polynomial-time algorithm that, given a planar graph G and an integer k, computes an induced subgraph G' such that

1. G' has treewidth $O(\sqrt{k} \cdot \mathrm{polylog}(k))$ and
2. for any **connected** subgraph $H \subseteq G$ with at most k vertices, we have $H \subseteq G'$ with probability at least $(2^{O(\sqrt{k} \cdot \mathrm{polylog}(k))} \cdot n^{O(1)})^{-1}$.

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Thus the Subgraph Isomorphism problem for connected H can be solved by restriction to G'.

Theorem

Subgraph Isomorphism for planar graphs can be solved in time $2^{O(\Delta\sqrt{k} \cdot \text{polylog}(k))} \cdot n^{O(1)}$ if H is connected with maximum degree Δ.

Remark: Robust algorithm, can be easily generalized to colored, directed, weighted etc. versions.

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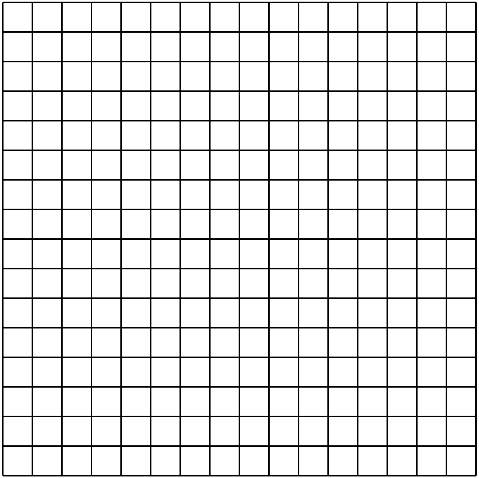
Theorem [Bodlaender, Nederlof, van der Zanden 2016]

Assuming ETH, there is no $2^{o(k/\log k)} \cdot n^{O(1)}$ time algorithm for planar Subgraph Isomorphism, even when

- H is a forest of maximum degree 3, or
- H is a tree with only one vertex having degree larger than 3.

Example: grids



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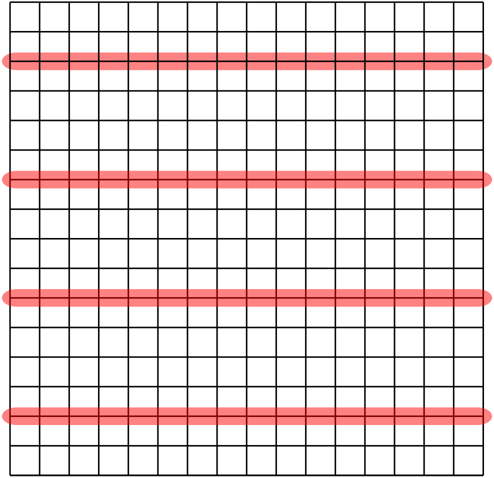

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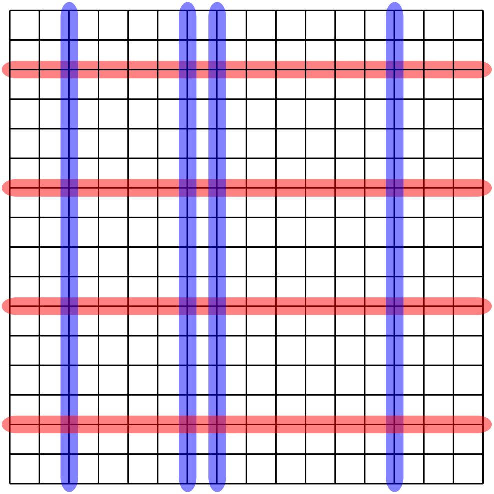

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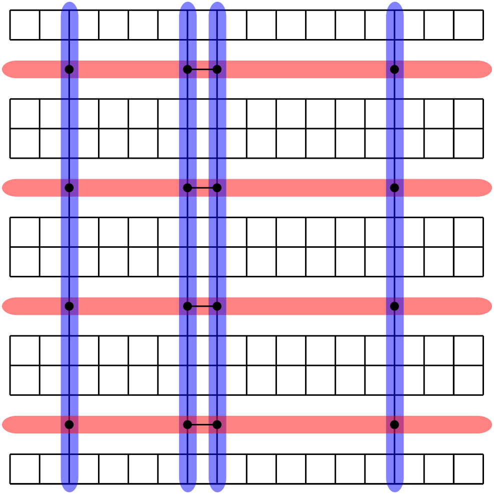

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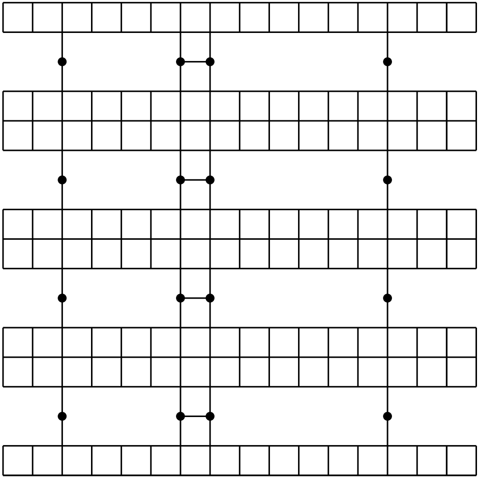

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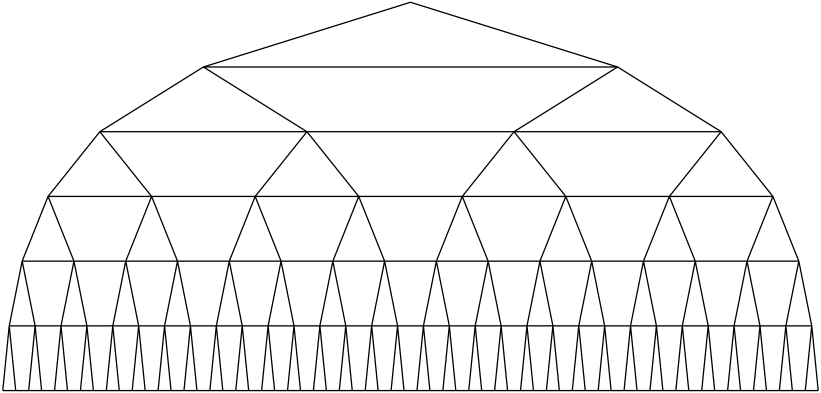

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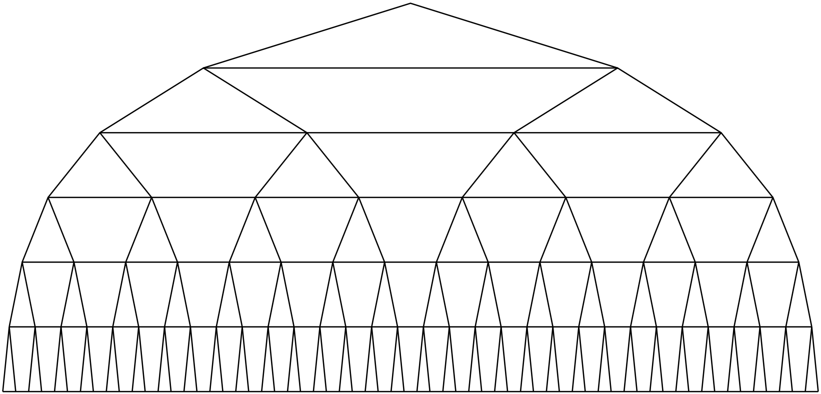


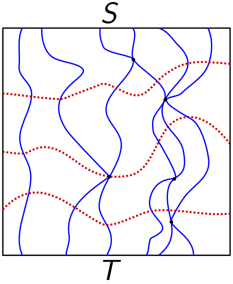
Ball-growing argument: there is an index $\sqrt{k} \leq i \leq O(\sqrt{k} \log k)$ such that the first i rows in total contain $\sqrt{k}$ times more vertices of the solution than row i.

Duality result

Theorem

Given a graph G and two sets of vertices S and T there is either

- a family $P_1, \ldots, P_C$ of "almost-disjoint" $S-T$ paths such that $\exists A_i \subseteq P_i$ with $\sum |A_i| \leq \ell C$ and $P_i \setminus A_i$'s are pairwise disjoint or
- a family $S_1, \ldots, S_\ell$ of disjoint "small" $S-T$ separators with $|S_i| \leq C$.



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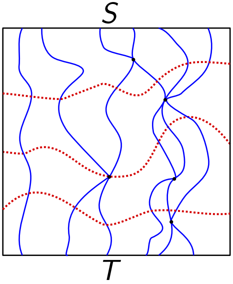

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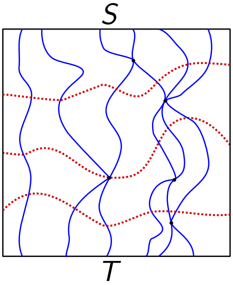

We will use the duality with $C = 2(k+1)$ and $\ell = \sqrt{k}$

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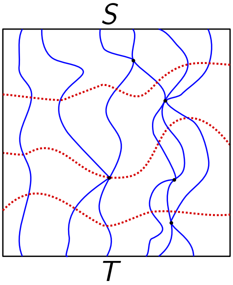
- a family $S_1, \ldots, S_{\sqrt{k}}$ of disjoint "small" $S-T$ seps. with $|S_i| \leq 2k+2$.



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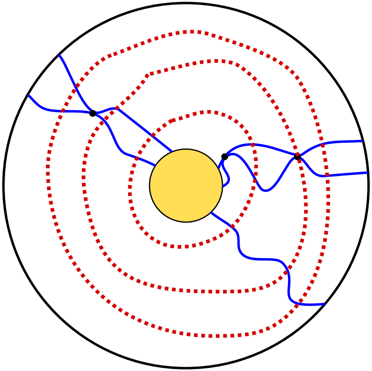
Using duality

The correct viewpoint:
Using the duality between the outside and the inside.



But where is this “inside”?

Conclusions

- Subexponential parameterized algorithms for finding bounded-degree connected subgraphs.
- Connectedness of the pattern H seems essential (but easy to generalize to bounded number of connected components).
- Can be generalized to bounded local treewidth, H-minor-free in progress.
- Other classes of graphs: polynomial growth property.