Parameterized complexity of constraint satisfaction problems

Dániel Marx

Budapest University of Technology and Economics

`dmarx@cs.bme.hu`

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Constraint satisfaction problems

Let $\mathcal{R}$ be a set Boolean of relations. An $\mathcal{R}$-formula is a conjunction of relations in $\mathcal{R}$:

$$R_1(x_1, x_4, x_5) \wedge R_2(x_2, x_1) \wedge R_1(x_3, x_3, x_3) \wedge R_3(x_5, x_1, x_4, x_1)$$

$\mathcal{R}$-SAT

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- Given: an $\mathcal{R}$-formula φ
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$\mathcal{R} = \{a \neq b\} \Rightarrow \mathcal{R}$-SAT = 2-coloring of a graph
$\mathcal{R} = \{a \vee b,\ a \vee \bar{b},\ \bar{a} \vee \bar{b}\} \Rightarrow \mathcal{R}$-SAT = 2SAT
$\mathcal{R} = \{a \vee b \vee c, a \vee b \vee \bar{c}, a \vee \bar{b} \vee \bar{c}, \bar{a} \vee \bar{b} \vee \bar{c}\} \Rightarrow \mathcal{R}$-SAT = 3SAT

Question: $\mathcal{R}$-SAT is polynomial time solvable for which $\mathcal{R}$?

It is **NP**-complete for which $\mathcal{R}$?

Schaefer's Dichotomy Theorem (1978)

For every $\mathcal{R}$, the $\mathcal{R}$-SAT problem is polynomial time solvable if one of the following holds, and **NP**-complete otherwise:

- Every relation is satisfied by the all 0 assignment
- Every relation is satisfied by the all 1 assignment
- Every relation can be expressed by a 2SAT formula
- Every relation can be expressed by a Horn formula
- Every relation can be expressed by an anti-Horn formula
- Every relation is an affine subspace over $GF(2)$

Other dichotomy results

- Approximability of MAX-SAT, MIN-UNSAT [Khanna et al., 2001]
- Approximability of MAX-ONES, MIN-ONES [Khanna et al., 2001]
- Generalization to 3 valued variables [Bulatov, 2002]
- Inverse satisfiability [Kavvadias and Sideri, 1999]
- etc.

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Our contribution: parameterized analogue of Schaefer's dichotomy theorem.

Parameterized Complexity: Summary

Two key concepts:

- A parameterized problem is **fixed-parameter tractable (FPT)** if it has an $f(k) \cdot n^c$ time algorithm, where c is independent of k.

  Example: MINIMUM VERTEX COVER is solvable in $O(2^k \cdot n)$ time.

- A **W[1]-hard** problem is unlikely to be FPT. To show that a problem L is W[1]-hard, we have to give a **parameterized reduction** from a known W[1]-hard problem to L.

  Example: MAXIMUM INDEPENDENT SET is W[1]-hard, no $n^{o(k)}$ algorithm is known.

Parameterized Problems

For a large number of **NP**-hard problems, the parameterized version is fixed-parameter tractable. For some other problems, the parameterized version is W[1]-hard.

Fixed-parameter tractable problems:

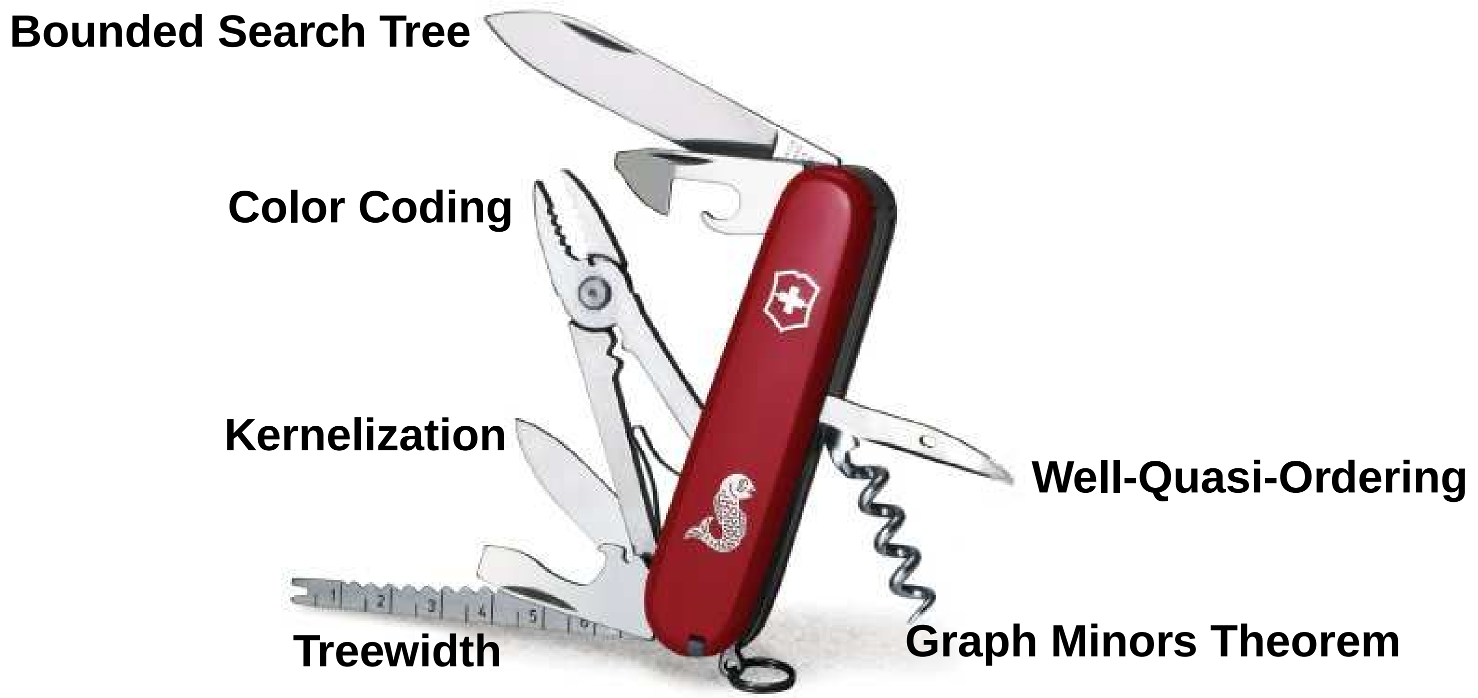
- MINIMUM VERTEX COVER
- LONGEST PATH
- DISJOINT TRIANGLES
- GRAPH GENUS
- . . .

W[1]-hard problems:

- MAXIMUM INDEPENDENT SET
- MINIMUM DOMINATING SET
- LONGEST COMMON SUBSEQUENCE
- SET PACKING
- . . .

Parameterized Complexity – Motivation

- Practical importance: efficient algorithms for small values of k.
- Powerful toolbox for designing FPT algorithms:



Parameterized dichotomy theorem

Parameterized $\mathcal{R}$-SAT

- **Input:** an $\mathcal{R}$-formula φ, an integer k
- **Parameter:** k
- **Question:** Does φ have a satisfying assignment of weight exactly k?

For which $\mathcal{R}$ is there an $f(k) \cdot n^c$ algorithm for $\mathcal{R}$-SAT?

Main theorem: For every constraint family $\mathcal{R}$, the parameterized $\mathcal{R}$-SAT problem is either fixed-parameter tractable or W[1]-complete.
(+ simple characterization of FPT cases)

Technical notes

- Are constants allowed in the formula?
  E.g., $R(x_1, 0, 1) \wedge R(1, x_2, x_3)$
- Can a variable appear multiple times in a constraint?
  E.g., $R(x_1, x_1, x_2) \wedge R(x_3, x_3, x_3)$
- Constraints that are not satisfied by the all 0 assignment can be handled easily (bounded search tree).

Weak separability

Definition: $\boldsymbol{R}$ is weakly separable if

1. the union of two disjoint satisfying assignments is also satisfying, and
2. if a satisfying assignment contains a smaller satisfying assignment, then their difference is also satisfying.

Example of 1:	Example of 2:
$\boldsymbol{R(1,1,1,1,0,0,0,0,0)=1}$	$\boldsymbol{R(1,1,1,1,1,1,0,0)=1}$
$\boldsymbol{R(0,0,0,0,1,1,0,0,0)=1}$	$\boldsymbol{R(0,0,1,1,1,1,0,0)=1}$
$\Downarrow$	$\Downarrow$
$\boldsymbol{R(1,1,1,1,1,1,0,0,0)=1}$	$\boldsymbol{R(1,1,0,0,0,0,0,0)=1}$

Main theorem: $\boldsymbol{\mathcal{R}}$-SAT is FPT if and only if every constraint is weakly separable, and W[1]-complete otherwise.

Weak separability: examples

The constraint EVEN is weakly separable:

Property 1:

$$R(\overbrace{1,1,1,1}^{\text{even}},0,0,0,0,0)=1$$
$$R(0,0,0,0,\underbrace{1,1}_{\text{even}},0,0,0)=1$$
$$\Downarrow$$
$$R(\underbrace{1,1,1,1,1,1}_{\text{even}},0,0,0)=1$$

Property 2:

$$R(\overbrace{1,1,1,1,1,1}^{\text{even}},0,0)=1$$
$$R(0,0,\underbrace{1,1,1,1}_{\text{even}},0,0)=1$$
$$\Downarrow$$
$$R(\underbrace{1,1}_{\text{even}},0,0,0,0,0,0)=1$$

More generally: every **affine** constraint is weakly separable.

Parameterized vs. classical

The easy and hard cases are different in the classical and the parameterized version:

Constraint	**Classical**	**Parameterized**
$x \vee y$	in P	FPT (VERTEX COVER)
$\bar{x} \vee \bar{y}$	in P	W[1]-complete (MAXIMUM INDEPENDENT SET)
affine	in P	FPT
2-in-3	NP-complete	FPT

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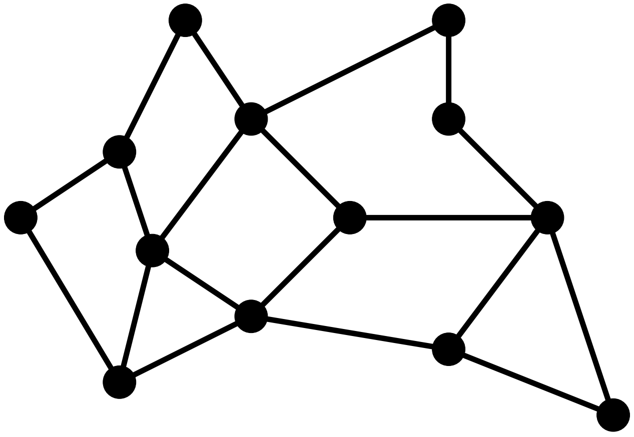
Sketch of proof begins...

Bounded number of occurrences

Primal graph: Vertices are the variables, two variables are connected if they appear in some clause together.



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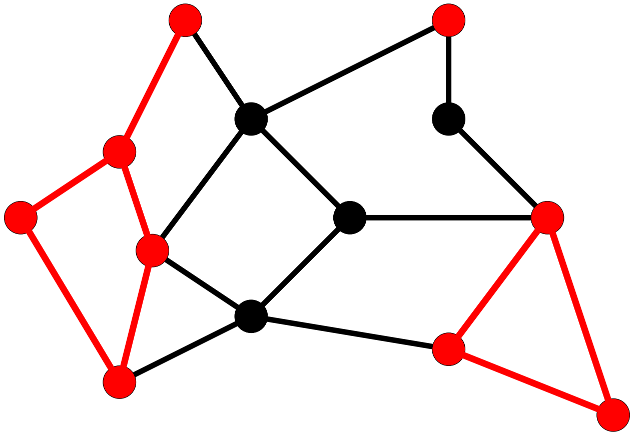


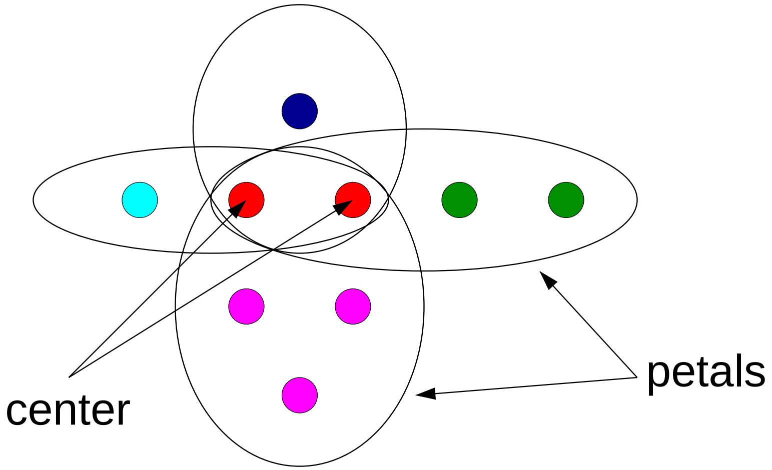
Every satisfying assignment is composed of **connected satisfying assignments**.

Lemma: There are at most $(rd)^{k^2} \cdot n$ connected satisfying assignments of size at most k. (r is the maximum arity, d is the maximum no. of occurrences)

Algorithm: Use color coding to put together the connected assignments to obtain a size k assignment.

The sunflower lemma

Definition: Sets S_1, S_2, $\ldots$, S_k form a **sunflower** if the sets $S_i \setminus (S_1 \cap S_2 \cap \cdots \cap S_k)$ are disjoint.



Lemma (Erdős and Rado, 1960): If the size of a set system is greater than $(p-1)^\ell \cdot \ell!$ and it contains only sets of size at most ℓ, then the system contains a sunflower with p petals.

Sunflower of clauses

Definition: A **sunflower** is a set of k clauses such that for every i

- either the same variable appears at position i in every clause,
- or every clause "owns" its ith variable.

$$R(x_1, x_2, x_3, x_4, x_5, x_6)$$
$$R(x_1, x_2, x_3, x_7, x_8, x_9)$$
$$R(x_1, x_2, x_3, x_{10}, x_{11}, x_{12})$$
$$R(x_1, x_2, x_3, x_{13}, x_{14}, x_{15})$$

Lemma: If a variable occurs more than $c_{\mathcal{R}}(k)$ times in an $\mathcal{R}$-formula, then the formula contains a sunflower of clauses with more than k petals.

Plucking the sunflower

For weakly separable constraints, the formula can be reduced if there is a sunflower with $k+1$ petals. Example:

$$k+1 \begin{cases} \text{EVEN}(x_1, x_2, x_3, x_4, x_5, x_6) \\ \text{EVEN}(x_1, x_2, x_3, x_7, x_8, x_9) \\ \text{EVEN}(x_1, x_2, x_3, x_{10}, x_{11}, x_{12}) \\ \text{EVEN}(x_1, x_2, x_3, x_{13}, x_{14}, x_{15}) \end{cases}$$

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$$\Downarrow$$

$$\text{EVEN}(x_1, x_2, x_3)$$

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$$\text{EVEN}(x_7, x_8, x_9)$$

$$\text{EVEN}(x_{10}, x_{11}, x_{12})$$

$$\text{EVEN}(x_{13}, x_{14}, x_{15})$$

The algorithm

- If there is a variable that occurs more than $c_{\mathcal{R}}(k)$ times:
  - Find a sunflower with $k+1$ petals
  - Pluck the sunflower $\Rightarrow$ shorter formula
- If every variable occurs at most $c_{\mathcal{R}}(k)$ times:
  - Apply the bounded occurrence algorithm

Running time: $2^{k^{r+2} \cdot 2^{2^{O(r)}}} \cdot n \log n$, where r is the maximum arity in the constraint family $\mathcal{R}$.

Hardness results: case 1

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$$R(x,x,x,y,y,0,0,0) = 1 \iff \bar{x} \vee \bar{y}$$

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$$\Downarrow$$

MAXIMUM INDEPENDENT SET

$$R(x,x,x,y,y,0,0,0) = 1 \iff \bar{x} \vee \bar{y}$$

$\Rightarrow$ can be expressed!

Hardness results: case 2

Definition: R is weakly separable if

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2. if a satisfying assignment contains a smaller satisfying assignment, then their difference is also satisfying.

If property 2 is violated:

$$R(0,0,0,0,0,0,0,0)=1$$

$$R(1,1,1,1,1,0,0,0)=1$$

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Lemma: The problem is W[1]-complete for the constraint $\longrightarrow$.

Summary

- Parameterized version of $\mathcal{R}$-SAT
- FPT or W[1]-complete depending on weak separability
- Bounded occurences: color coding using connected solutions
- Reduction using the sunflower lemma
- Linear time solvable on planar formulae

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Thank you for your attention!
Questions?