CSPs and fixed-parameter tractability

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Parameterized problems

Main idea

Instead of expressing the running time as a function $T(n)$ of $n$, we express it as a function $T(n, k)$ of the input size $n$ and some parameter $k$ of the input.

In other words: we do not want to be efficient on all inputs of size $n$, only for those where $k$ is small.
Parameterized problems

Main idea
Instead of expressing the running time as a function $T(n)$ of $n$, we express it as a function $T(n, k)$ of the input size $n$ and some parameter $k$ of the input.

In other words: we do not want to be efficient on all inputs of size $n$, only for those where $k$ is small.

What can be the parameter $k$?

- The size $k$ of the solution we are looking for.
- The maximum degree of the input graph.
- The dimension of the point set in the input.
- The length of the strings in the input.
- The length of clauses in the input Boolean formula.
- ...
Parameterized complexity

Problem:  
Input:  
Question:  

**Vertex Cover**  
Graph $G$, integer $k$  
Is it possible to cover the edges with $k$ vertices?

**Independent Set**  
Graph $G$, integer $k$  
Is it possible to find $k$ independent vertices?

Complexity:  
NP-complete  
NP-complete
Parameterized complexity

Problem:  
Input: Graph $G$, integer $k$  
Question: Is it possible to cover the edges with $k$ vertices?

Complexity: NP-complete  
Brute force: $O(n^k)$ possibilities

Problem: Independent Set  
Input: Graph $G$, integer $k$  
Question: Is it possible to find $k$ independent vertices?

Complexity: NP-complete  
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Complexity: NP-complete

Brute force: $O(n^k)$ possibilities

$O(2^k n^2)$ algorithm exists 😊

No $n^{o(k)}$ algorithm known 😞
Bounded search tree method

Algorithm for VERTEX COVER:

\[ e_1 = u_1 v_1 \]
Bounded search tree method

Algorithm for \textsc{Vertex Cover}:

\[ e_1 = u_1 v_1 \]

\[ u_1 \quad v_1 \]
Bounded search tree method

Algorithm for Vertex Cover:

\[ e_1 = u_1 v_1 \]

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Bounded search tree method

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Algorithm for \textsc{Vertex Cover}:

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\[ e_2 = u_2 v_2 \]

Height of the search tree \( \leq k \) \( \Rightarrow \) at most \( 2^k \) leaves \( \Rightarrow 2^k \cdot n^{O(1)} \) time algorithm.
Fixed-parameter tractability

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Main goal of parameterized complexity: to find FPT problems.
Fixed-parameter tractability

Main definition
A parameterized problem is fixed-parameter tractable (FPT) if there is an $f(k)n^c$ time algorithm for some constant $c$.

Main goal of parameterized complexity: to find FPT problems.

Examples of NP-hard problems that are FPT:

- Finding a vertex cover of size $k$.
- Finding a path of length $k$.
- Finding $k$ disjoint triangles.
- Drawing the graph in the plane with $k$ edge crossings.
- Finding disjoint paths that connect $k$ pairs of points.
- ...
W[1]-hardness

Negative evidence similar to NP-completeness. If a problem is \textbf{W[1]-hard}, then the problem is not FPT unless FPT=\text{W}[1].

Some W[1]-hard problems:

- Finding a clique/independent set of size $k$.
- Finding a dominating set of size $k$.
- Finding $k$ pairwise disjoint sets.
- ...
Reactions to FPT

Typical graph algorithms researcher:

Hmm... Is my favorite graph problem FPT parameterized by the size of the solution/number of objects/etc.?
Reactions to FPT

Typical graph algorithms researcher:

Hmm... Is my favorite graph problem FPT parameterized by the size of the solution/number of objects/etc.?

Typical CSP researcher:

\textsc{Sat} is trivially FPT parameterized by the number of variables. So why should I care?
Parameterizing SAT

Trivial: \textsc{3Sat} is FPT parameterized by the number of variables ($2^k \cdot n^{O(1)}$ time algorithm).

Trivial: \textsc{3Sat} is FPT parameterized by the number of clauses ($2^{3k} \cdot n^{O(1)}$ time algorithm).

What about \textsc{Sat} parameterized by the number $k$ of clauses?
Parameterizing $\text{SAT}$

**Trivial:** $3\text{SAT}$ is FPT parameterized by the number of variables ($2^k \cdot n^{O(1)}$ time algorithm).

**Trivial:** $3\text{SAT}$ is FPT parameterized by the number of clauses ($2^{3k} \cdot n^{O(1)}$ time algorithm).

What about $\text{SAT}$ parameterized by the number $k$ of clauses?

Algorithm 1: Problem kernel

- If a clause has more than $k$ literals: can be ignored, removing it does not make the problem any easier.
- If every clause has at most $k$ literals: there are at most $k^2$ variables, use brute force.
Parameterizing SAT

**Trivial:** \(3\text{SAT}\) is FPT parameterized by the number of variables \((2^k \cdot n^{O(1)}\) time algorithm\).

**Trivial:** \(3\text{SAT}\) is FPT parameterized by the number of clauses \((2^{3k} \cdot n^{O(1)}\) time algorithm\).

What about \(\text{SAT}\) parameterized by the number \(k\) of clauses?

Algorithm 2: Bounded search tree

- Pick a variable occurring both positively and negatively, branch on setting it to 0 or 1.
- In both branches, the number of clauses strictly decreases \(\Rightarrow\) search tree of size \(2^k\).
Max Sat

- **Max Sat**: Given a formula, satisfy at least $k$ clauses.
- Polynomial for fixed $k$: guess the $k$ clauses, use the previous algorithm to check if they are satisfiable.
- Is the problem FPT?
Max Sat

- **Max Sat**: Given a formula, satisfy at least $k$ clauses.
- Polynomial for fixed $k$: guess the $k$ clauses, use the previous algorithm to check if they are satisfiable.
- Is the problem FPT?
  - YES: If there are at least $2k$ clauses, a random assignment satisfies $k$ clauses on average. Otherwise, use the previous algorithm.

This is not very insightful, can we say anything more interesting?
Above average MAX SAT

\( m/2 \) satisfiable clauses are guaranteed. But can we satisfy \( m/2 + k \) clauses?
Above average **MAX SAT**

$m/2$ satisfiable clauses are guaranteed. But can we satisfy $m/2 + k$ clauses?

- Above average **MAX SAT** (satisfy $m/2 + k$ clauses) is FPT [Mahajan and Raman 1999]
- Above average **MAX $r$-SAT** (satisfy $(1 - 1/2^r)m + k$ clauses) is FPT [Alon et al. 2010]
- Satisfying $\sum_{i=1}^{m}(1 - 1/2^{r_i}) + k$ clauses is NP-hard for $k = 2$ [Crowston et al. 2012]
- Above average **MAX $r$-LIN-2** (satisfy $m/2 + k$ linear equations) is FPT [Gutin et al. 2010]
- Permutation CSPs such as **MAXIMUM ACYCLIC SUBGRAPH** and **BETWEENNESS** [Gutin et al. 2010].
- ...
Boolean constraint satisfaction problems

Let $\Gamma$ be a set of **Boolean** relations. An $\Gamma$-formula is a conjunction of relations in $\Gamma$:

$$R_1(x_1, x_4, x_5) \land R_2(x_2, x_1) \land R_1(x_3, x_3, x_3) \land R_3(x_5, x_1, x_4, x_1)$$

SAT($\Gamma$)

- Given: an $\Gamma$-formula $\varphi$
- Find: a variable assignment satisfying $\varphi$

It is NP-complete for which $\Gamma$?
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**SAT($\Gamma$)**
- Given: an $\Gamma$-formula $\varphi$
- Find: a variable assignment satisfying $\varphi$

$\Gamma = \{a \neq b\} \Rightarrow$ SAT($\Gamma$) = 2-coloring of a graph
$\Gamma = \{a \lor b, a \lor \bar{b}, \bar{a} \lor \bar{b}\} \Rightarrow$ SAT($\Gamma$) = 2SAT
$\Gamma = \{a \lor b \lor c, a \lor b \lor \bar{c}, a \lor \bar{b} \lor \bar{c}, \bar{a} \lor \bar{b} \lor \bar{c}\} \Rightarrow$ SAT($\Gamma$) = 3SAT

**Question:** SAT($\Gamma$) is polynomial time solvable for which $\Gamma$? It is NP-complete for which $\Gamma$?
### Schaefer’s Dichotomy Theorem (1978)

**Theorem [Schaefer 1978]**

For every $\Gamma$, the $SAT(\Gamma)$ problem is polynomial-time solvable if one of the following holds, and NP-complete otherwise:

- Every relation is satisfied by the all 0 assignment
- Every relation is satisfied by the all 1 assignment
- Every relation can be expressed by a 2SAT formula
- Every relation can be expressed by a Horn formula
- Every relation can be expressed by an anti-Horn formula
- Every relation is an affine subspace over GF(2)

This is surprising for two reasons: this family does not contain NP-intermediate problems and the boundary of polynomial-time and NP-hard problems can be cleanly characterized.
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Other dichotomy results

- Approximability of **Max-Sat, Min-Unsat** [Khanna et al. 2001]
- Approximability of **MaxOnes-Sat, MinOnes-Sat** [Khanna et al. 2001]
- Generalization to 3-valued variables [Bulatov 2002]
- Inverse satisfiability [Kavvadias and Sideri, 1999]
- etc.
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- etc.

Celebrated open question: generalize Schaefer’s result to relations over variables with non-Boolean, but fixed domain.

\textbf{CSP}(\Gamma): similar to \textbf{SAT}(\Gamma), but with non-Boolean domain.

\textbf{Conjecture} [Feder and Vardi 1998]

Let \(\Gamma\) be a finite set of relations over an arbitrary fixed domain.
Then \textbf{CSP}(\Gamma) is either polynomial-time solvable or NP-complete.
Weighted problems

Parameterizing by the weight (\(=\) number of 1s) of the solution.

- **Minimum Ones-Sat** (\(\Gamma\)) :
  Find a satisfying assignment with weight at most \(k\)

- **Exact Ones-Sat** (\(\Gamma\)) :
  Find a satisfying assignment with weight exactly \(k\)

- **Maximum Ones-Sat** (\(\Gamma\)) :
  Find a satisfying assignment with weight at least \(k\)

The first two problems can be always solved in \(n^{O(k)}\) time, and the third one as well if \(\text{Sat}(\Gamma)\) is in P.

**Goal:** Characterize which languages \(\Gamma\) make these problems FPT.
**Theorem [Marx 2004]**

\( \text{ExactOnes-Sat}(\Gamma) \) is FPT if \( \Gamma \) is weakly separable and \( W[1] \)-hard otherwise.

Examples of weakly separable constraints:

- affine constraints
- “0 or 5 out of 8”

Examples of not weakly separable constraints:

- \((\neg x \lor \neg y)\)
- \(x \rightarrow y\)
- “0 or 4 out of 8”
Larger domains

What is the generalization of \textsc{ExactOnes-Sat}(\Gamma) to larger domains?

1. Find a solution with exactly $k$ nonzero values (zeros constraint).
2. Find a solution where nonzero value $i$ appears exactly $k_i$ times (cardinality constraint).

**Theorem** [Bulatov and M. 2011]

For every $\Gamma$ closed under substituting constants, CSP($\Gamma$) with zeros constraint is FPT or W[1]-hard.

(E.g., if $R(x_1, x_2, x_3, x_4) \in \Gamma$, then $R(x_1, 3, x_3, 0) \in \Gamma$.)
Larger domains

The following two problems are equivalent:

- **CSP(Γ)** with cardinality constraint, where Γ contains only the relation \( R = \{00, 10, 02\} \).
- **Biclique**: Find a complete bipartite graph with \( k \) vertices on each side. The fixed-parameter tractability of **Biclique** is a notorious open problem (conjectured to be hard).
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So the best we can get at this point:

**Theorem [Bulatov and M. 2011]**

For every Γ closed under substituting constants, CSP(Γ) with cardinality constraint is FPT or Biclique-hard.
\textbf{\textsc{MinOnes-Sat}}(\(\Gamma\))

The bounded-search tree algorithm for \textsc{Vertex Cover} can be generalized to \textsc{MinOnes-Sat}.

\textbf{Observation}

\textsc{MinOnes-Sat}(\(\Gamma\)) is FPT for every finite \(\Gamma\).
The bounded-search tree algorithm for **Vertex Cover** can be generalized to **MinOnes-Sat**.

**Observation**

**MinOnes-Sat**($\Gamma$) is FPT for every finite $\Gamma$.

But can we solve the problem simply by preprocessing?

**Definition**

A polynomial kernel is a polynomial-time reduction creating an equivalent instance whose size is polynomial in $k$.

**Goal**: Characterize the languages $\Gamma$ for which **MinOnes-Sat**($\Gamma$) has a polynomial kernel.

**Example**: the special case **$d$-Hitting Set** (where $\Gamma$ contains only $R = x_1 \lor \cdots \lor x_d$) has a polynomial kernel.
Sunflower lemma

**Definition**
Sets $S_1, S_2, \ldots, S_k$ form a **sunflower** if the sets $S_i \setminus (S_1 \cap S_2 \cap \cdots \cap S_k)$ are disjoint.

**Lemma** [Erdős and Rado, 1960]
If the size of a set system is greater than $(p - 1)^d \cdot d!$ and it contains only sets of size at most $d$, then the system contains a sunflower with $p$ petals.
Sunflowers and \( d \)-Hitting Set

\( d \)-Hitting Set
Given a collection \( S \) of sets of size at most \( d \) and an integer \( k \), find a set \( S \) of \( k \) elements that intersects every set of \( S \).

Reduction Rule
Suppose more than \( k + 1 \) sets form a sunflower.

- If the sets are disjoint \( \Rightarrow \) No solution.
- Otherwise, keep only \( k + 1 \) of the sets.
Kernelization for general $\text{MinOnes-Sat}(\Gamma)$ generalizes the sunflower reduction, and requires that $\Gamma$ is “mergeable.”

**Theorem** [Kratsch and Wahlström 2010]

1. If $\text{MinOnes-Sat}(\Gamma)$ is polynomial-time solvable or $\Gamma$ is mergeable, then $\text{MinOnes-Sat}(\Gamma)$ has a polynomial kernelization.

2. If $\text{MinOnes-Sat}(\Gamma)$ is NP-hard and $\Gamma$ is not mergeable, then $\text{MinOnes-Sat}(\Gamma)$ does not have a polynomial kernel, unless the polynomial hierarchy collapses.
Dichotomy for kernelization

Similar results for other problems:

Theorem [Kratsch, M., Wahlström 2010]

- If $\Gamma$ has property $X$, then $\text{MAXONES-SAT}(\Gamma)$ has a polynomial kernel, and otherwise no (unless the polynomial hierarchy collapses).
- If $\Gamma$ has property $Y$, then $\text{EXACTONES-SAT}(\Gamma)$ has a polynomial kernel, and otherwise no (unless the polynomial hierarchy collapses).
Local search

Walk in the solution space by iteratively replacing the current solution with a better solution in the local neighborhood.

Problem:
Local search can stop at a local optimum (no better solution in the local neighborhood).

More sophisticated variants: simulated annealing, tabu search, etc.
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Local neighborhood

The local neighborhood is defined in a problem-specific way:

- For TSP, the neighbors are obtained by swapping 2 cities or replacing 2 edges.
- For a problem with 0-1 variables, the neighbors are obtained by flipping a single variable.
- For subgraph problems, the neighbors are obtained by adding/removing one edge.

More generally: reordering $k$ cities, flipping $k$ variables, etc.
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Larger neighborhood (larger $k$):

- algorithm is less likely to get stuck in a local optimum,
- it is more difficult to check if there is a better solution in the neighborhood.
Searching the neighborhood

**Question:** Is there an efficient way of finding a better solution in the $k$-neighborhood?

We study the complexity of the following problem:

$k$-step Local Search

**Input:** instance $I$, solution $x$, integer $k$

**Find:** A solution $x'$ with $\text{dist}(x, x') \leq k$ that is “better” than $x$. 

Remark 1: If the optimization problem is hard, then it is unlikely that this local search problem is polynomial-time solvable: otherwise we would be able to find an optimum solution.

Remark 2: Size of the $k$-neighborhood is usually $n^{O(k)} \Rightarrow$ local search is polynomial-time solvable for every fixed $k$, but this is not practical for larger $k$. 

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$k$-step Local Search

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If yes, then local search algorithms can consider larger neighborhoods, improving their efficiency.

**Important:** $k$ is the number of allowed changes and not the size of the solution. Relevant even if solution size is large.
**k-step Local Search**

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**Examples:**

- Local search is easy: it is FPT to find a larger independent set in a planar graph with at most $k$ exchanges [Fellows et al. 2008].
- Local search is hard: it is W[1]-hard to check if it is possible to obtain a shorter TSP tour by replacing at most $k$ arcs [M. 2008].
Local search for SAT

Simple satisfiability:

**Theorem [Dantsin et al. 2002]**

Finding a satisfying assignment in the $k$-neighborhood for $q$-SAT is FPT.
Local search for $\text{SAT}$

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Finding a satisfying assignment in the $k$-neighborhood for $q$-$\text{SAT}$ is FPT.

An optimization problem:

**Theorem [Szeider 2011]**
Finding a better assignment in the $k$-neighborhood for $\text{Max 2-SAT}$ is $\text{W}[1]$-hard.
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A family of problems:

**Theorem** [Krokhin and M. 2008]
Dichotomy results for MinOnes-Sat($\Gamma$).
Strict vs. permissive

Something strange: for some problems (e.g., Vertex Cover on bipartite graphs), local search is hard, even though the problem is polynomial-time solvable.
Strict vs. permissive

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### Strict $k$-step Local Search

**Input:** instance $I$, solution $x$, integer $k$

**Find:** A solution $x'$ with $\text{dist}(x, x') \leq k$ that is “better” than $x$. 

**Permissive $k$-step Local Search**
Strict vs. permissive

Something strange: for some problems (e.g., Vertex Cover on bipartite graphs), local search is hard, even though the problem is polynomial-time solvable.

**Strict $k$-step Local Search**

- **Input:** instance $I$, solution $x$, integer $k$
- **Find:** A solution $x'$ with $\text{dist}(x, x') \leq k$ that is “better” than $x$.

**Permissive $k$-step Local Search**

- **Input:** instance $I$, solution $x$, integer $k$
- **Find:** Any solution $x'$ “better” than $x$, if there is such a solution at distance at most $k$. 
Constraint Satisfaction Problems (CSP)

A CSP instance is given by describing the
- variables,
- domain of the variables,
- constraints on the variables.

**Task:** Find an assignment that satisfies every constraint.

\[ I = C_1(x_1, x_2, x_3) \land C_2(x_2, x_4) \land C_3(x_1, x_3, x_4) \]
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Examples:
- \textbf{3Sat}: 2-element domain, every constraint is ternary
- \textbf{Vertex Coloring}: domain is the set of colors, binary constraints
- \textbf{k-Clique} (in graph \( G \)): \( k \) variables, domain is the vertices of \( G \), \( \binom{k}{2} \) binary constraints
Graphs and hypergraphs related to CSP

**Gaifman/primal graph:** vertices are the variables, two variables are adjacent if they appear in a common constraint.

**Incidence graph:** bipartite graph, vertices are the variables and constraints.

**Hypergraph:** vertices are the variables, constraints are the hyperedges.

\[ I = C_1(x_2, x_1, x_3) \land C_2(x_4, x_3) \land C_3(x_1, x_4, x_2) \]
Theorem [Freuder 1990]
For every fixed $k$, CSP can be solved in polynomial time if the primal graph of the instance has treewidth at most $k$.

Note: The running time is $|D|^{O(k)}$, which is not FPT parameterized by treewidth.
Theorem [Freuder 1990]

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We know that binary CSP($G$) is polynomial-time solvable for every class $G$ of graphs with bounded treewidth. Are there other polynomial cases?
Question: Which graph properties lead to polynomial-time solvable CSP instances?

Systematic study:

- Binary CSP: Every constraint is of arity 2.
- $\text{CSP}(G)$: problem restricted to binary CSP instances with primal graph in $G$.
- Which classes $G$ make $\text{CSP}(G)$ polynomial-time solvable?
- E.g., if $G$ is the set of trees, then it is easy, if $G$ is the set of 3-regular graphs, then it is $W[1]$-hard parameterized by the number of variables (hence unlikely to be polynomial-time solvable).
Dichotomy for binary CSP

Complete answer for every class $G$:

**Theorem [Grohe-Schwentick-Segoufin 2001]**

Let $G$ be a computable class of graphs.

1. If $G$ has bounded treewidth, then $\text{CSP}(G)$ is polynomial-time solvable.

2. If $G$ has unbounded treewidth, then $\text{CSP}(G)$ is $\text{W}[1]$-hard parameterized by number of variables.

**Note:** In (2), $\text{CSP}(G)$ is not necessarily NP-hard.
Dichotomy for binary CSP

Complete answer for every class $G$:

Theorem [Grohe-Schwentick-Segoufin 2001]

Let $G$ be a recursively enumerable class of graphs. Assuming $\text{FPT} \neq \text{W}[1]$, the following are equivalent:

- Binary $\text{CSP}(G)$ is polynomial-time solvable.
- Binary $\text{CSP}(G)$ is FPT parameterized by the number of variables.
- $G$ has bounded treewidth.

Note: Fixed-parameter tractability does not give us more power here than polynomial-time solvability!
Combination of parameters

CSP can be parameterized by many (combination of) parameters.

Examples:

- CSP is W[1]-hard parameterized by the treewidth of the primal graph.
- CSP is FPT parameterized by the treewidth of the primal graph and the domain size.
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Examples:

- CSP is $W[1]$-hard parameterized by the treewidth of the primal graph.
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[Samer and Szeider 2010] considered 11 parameters and determined the complexity of CSP by any subset of these parameters.

- $\text{tw}$: treewidth of primal graph
- $\text{tw}^d$: tw of dual graph
- $\text{tw}^*$: tw of incidence graph
- $\text{vars}$: number of variables
- $\text{dom}$: domain size
- $\text{cons}$: number of constraints
- $\text{arity}$: maximum arity
- $\text{dep}$: largest relation size
- $\text{deg}$: largest variable occurrence
- $\text{ovl}$: largest overlap between scopes
- $\text{diff}$: largest difference between scopes
Summary

- Fixed-parameter tractability results for SAT and CSPs do exist.
- Choice of parameter is not obvious.
- Above average parameterization.
- Local search.
- Parameters related to the graph of the constraints.