

Algorithmic graph structure theory

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14th Max Planck Advanced Course on the Foundations of
Computer Science (ADFOCS 2013)
August 5–9, 2013
Saarbrücken, Germany

Classes of graphs

Classes of graphs can be described by

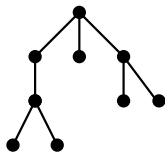
- 1 what they do not have,
(**excluded structures**)
- 2 how they look like
(**constructions and decompositions**).

In general, the second description is more useful for algorithmic purposes.

Classes of graphs

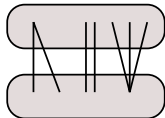
Example: Trees

- 1 Do not contain cycles (and connected)
- 2 Have a tree structure.



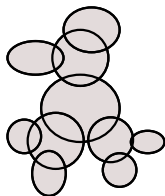
Example: Bipartite graphs

- 1 Do not contain odd cycles,
- 2 Edges going only between two classes.



Example: Chordal graphs

- 1 Do not contain induced cycles,
- 2 Clique-tree decomposition and simplicial ordering.

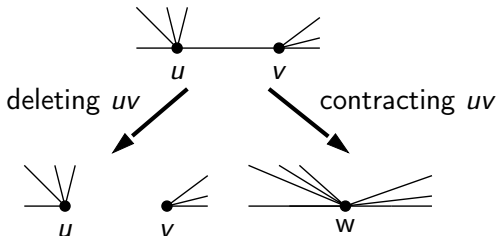


Graph Structure Theory

“Graph structure theory” usually refers to the theory developed by Robertson and Seymour on graphs excluding minors.

Definition

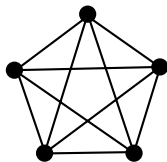
Graph H is a **minor** of G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.



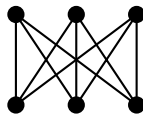
Excluding minors

Theorem [Wagner 1937]

A graph is planar if and only if it excludes K_5 and $K_{3,3}$ as a minor.



K_5

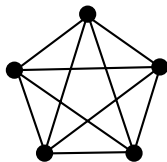


$K_{3,3}$

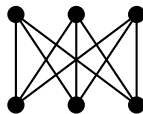
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$K_{3,3}$

- How do graphs excluding H (or H_1, \dots, H_k) look like?
- What other classes can be defined this way?

The work of Robertson and Seymour gives some kind of combinatorial answer to that and provides tools for the related algorithmic questions.

Minor closed properties

Definition

A set \mathcal{G} of graphs is **minor closed** if $G \in \mathcal{G}$ and $H \leq G$ implies $H \in \mathcal{G}$.

Examples of minor closed properties:

- planar graphs
- graphs that can be drawn on the torus
- acyclic graphs (forests)
- graphs having no cycle longer than k
- empty graphs

Examples of **not** minor closed properties:

- complete graphs
- regular graphs
- bipartite graphs

Wagner's conjecture

Let \mathcal{G} be a minor closed class of graphs. Then \mathcal{G} can be characterized by the minimal **obstructions**:

Let $H \in \mathcal{F}$ if $H \notin \mathcal{G}$, but every proper minor of H is in \mathcal{G} .

$$G \in \mathcal{G} \iff \forall H \in \mathcal{F}, H \not\leq G$$

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Theorem [Robertson and Seymour]

Every class \mathcal{G} closed under taking minors has a finite set \mathcal{F} of minimal obstructions.

Graph Minors Theorem

Well-quasi-ordering:

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Minor testing:

Theorem [Robertson and Seymour]

For every fixed graph H , there is an $O(n^3)$ time algorithm for testing whether H is a minor of the given graph G .

Corollary: For every minor closed property \mathcal{G} , there is an $O(n^3)$ time algorithm for testing whether a given graph G is in \mathcal{G} .

Graph Minors results

- The proof spans around 400 pages in the paper series “Graph Minors I–XXIII”.
- The size of the obstruction sets and the constants in the algorithms can be astronomical even for simple properties.

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Why should you know about this theory?

- The theory introduces simpler concepts and techniques that are useful on their own in many contexts.
- Some of the more complicated results can be formulated as self-contained powerful statements that can be used as a black box.

Graph Minors Theorem

Treewidth
Grid theorems
Planar graphs

Structure theorem

Minor testing

Well-quasi-ordering

Fixed-parameter tractability

Main definition

A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant c .

Main goal of parameterized complexity: to find FPT problems.

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Main goal of parameterized complexity: to find FPT problems.

Examples of NP-hard problems that are FPT:

- Finding a vertex cover of size k .
- Finding a path of length k .
- Finding k disjoint triangles.
- Drawing the graph in the plane with k edge crossings.
- Finding disjoint paths that connect k pairs of points.
- ...

Fixed-parameter tractability

- Downey and Fellows started the systematic investigation of fixed-parameter tractability and its hardness theory in the 80s.
- $n^{f(k)}$ vs. $f(k) \cdot n^c$.
- Many of the algorithmic results from graph structure theory can be formulated and appreciated using the language of fixed-parameter tractability.
- The original motivation of Downey and Fellows comes from graph structure theory!

Outline

- Treewidth
 - Definition, algorithms, properties.
 - Applications
- Graphs on surfaces
- The Graph Structure Theorem
- Minor Testing
- Well-quasi-ordering
- Other containment relations

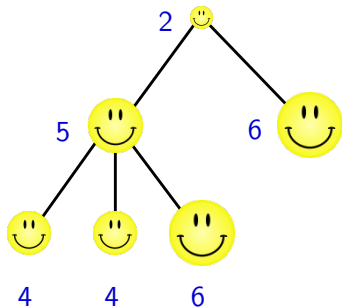
The Party Problem

PARTY PROBLEM

Problem: Invite some colleagues for a party.

Maximize: The total fun factor of the invited people.

Constraint: Everyone should be having fun.



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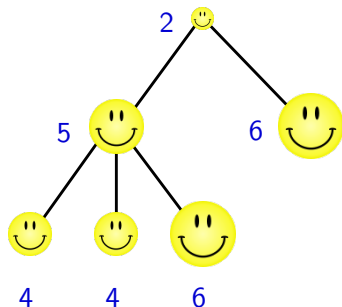
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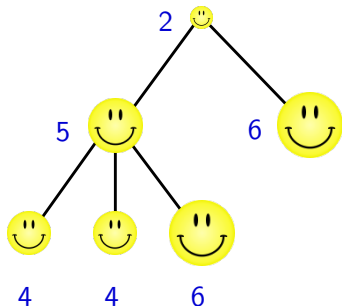
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- **Input:** A tree with weights on the vertices.
- **Task:** Find an independent set of maximum weight.

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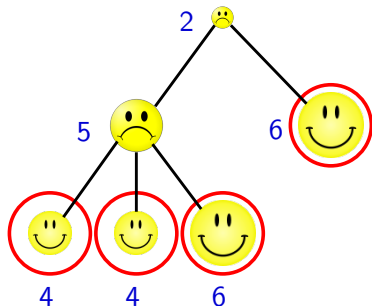
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Solving the Party Problem

Dynamic programming paradigm:

We solve a large number of subproblems that depend on each other. The answer is a single subproblem.

Subproblems:

T_v : the subtree rooted at v .

$A[v]$: max. weight of an independent set in T_v

$B[v]$: max. weight of an independent set in T_v
that does not contain v

Goal: determine $A[r]$ for the root r .

Solving the Party Problem

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Recurrence:

Assume v_1, \dots, v_k are the children of v . Use the recurrence relations

$$\begin{aligned} B[v] &= \sum_{i=1}^k A[v_i] \\ A[v] &= \max\{B[v], w(v) + \sum_{i=1}^k B[v_i]\} \end{aligned}$$

The values $A[v]$ and $B[v]$ can be calculated in a bottom-up order (the leaves are trivial).



Treewidth

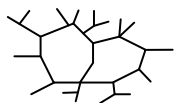
Generalizing trees

How could we define that a graph is “treelike”?

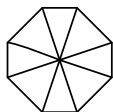
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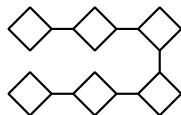
- 1 Number of cycles is bounded.



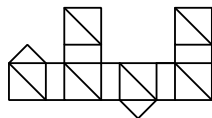
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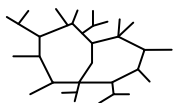


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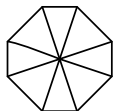
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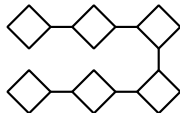
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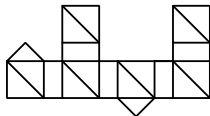
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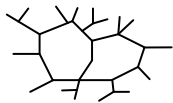


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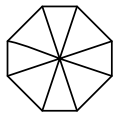


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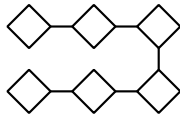
- ② Removing a bounded number of vertices makes it acyclic.



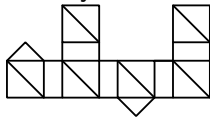
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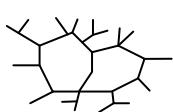


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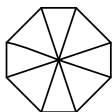
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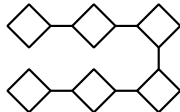
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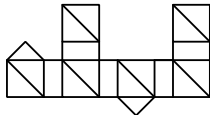
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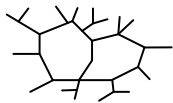


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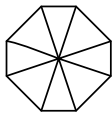


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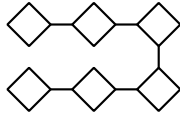
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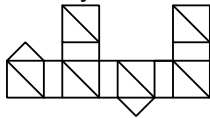
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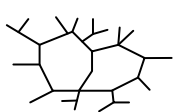


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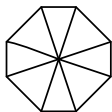


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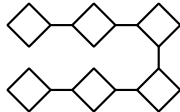
- ③ Bounded-size parts connected in a tree-like way.



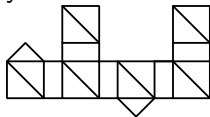
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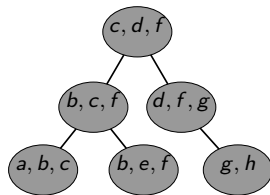
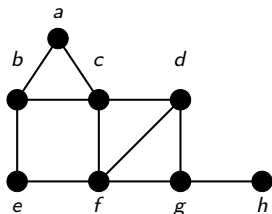


good

Treewidth — a measure of “tree-likeness”

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

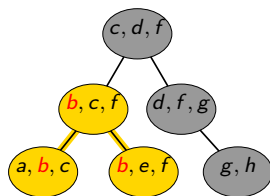
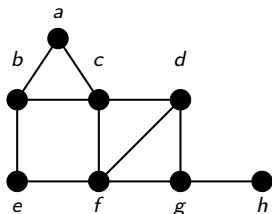
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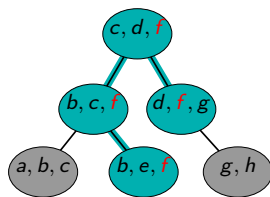
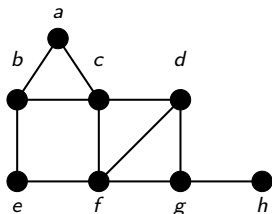
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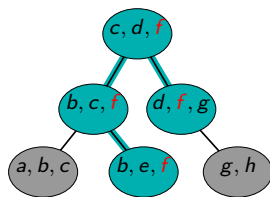
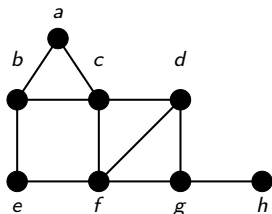
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Width of the decomposition: largest bag size -1 .

treewidth: width of the best decomposition.



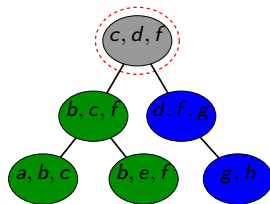
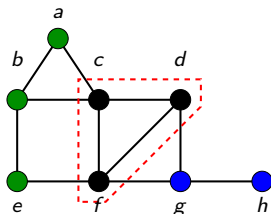
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Each bag is a separator.

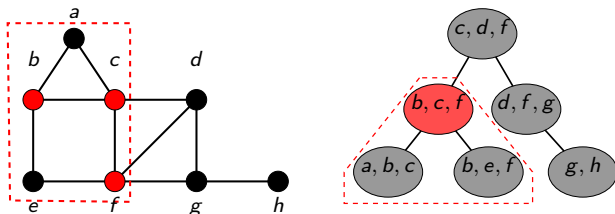
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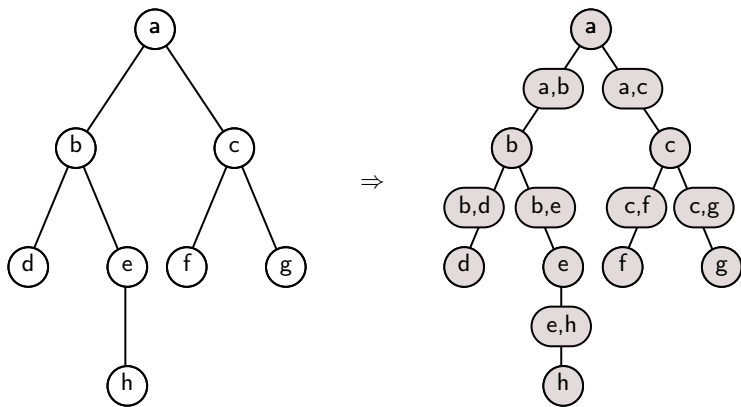
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A subtree communicates with the outside world only via the root of the subtree.

Treewidth

Fact: $\text{treewidth} = 1 \iff \text{graph is a forest}$



Exercise: A cycle cannot have a tree decomposition of width 1.

Treewidth — outline

- ① Basic algorithms
- ② Combinatorial properties
- ③ Applications

Finding tree decompositions

Hardness:

Theorem [Arnborg, Corneil, Proskurowski 1987]

It is NP-hard to determine the treewidth of a graph (given a graph G and an integer w , decide if the treewidth of G is at most w).

Fixed-parameter tractability:

Theorem [Bodlaender 1996]

There is a $2^{O(w^3)} \cdot n$ time algorithm that finds a tree decomposition of width w (if exists).

Consequence:

If we want an FPT algorithm parameterized by treewidth w of the input graph, then we can assume that a tree decomposition of width w is available.

Finding tree decompositions — approximately

Sometimes we can get better dependence on treewidth using approximation.

FPT approximation:

Theorem [Robertson and Seymour]

There is a $O(3^{3w} \cdot w \cdot n^2)$ time algorithm that finds a tree decomposition of width $4w + 1$, if the treewidth of the graph is at most w .

Polynomial-time approximation:

Theorem [Feige, Hajiaghayi, Lee 2008]

There is a polynomial-time algorithm that finds a tree decomposition of width $O(w\sqrt{\log w})$, if the treewidth of the graph is at most w .

WEIGHTED MAX INDEPENDENT SET and treewidth

Theorem

Given a tree decomposition of width w , **WEIGHTED MAX INDEPENDENT SET** can be solved in time $O(2^w \cdot w^{O(1)} \cdot n)$.

B_x : vertices appearing in node x .

V_x : vertices appearing in the subtree rooted at x .

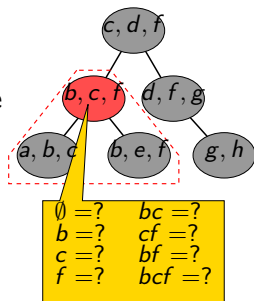
Generalizing our solution for trees:

Instead of computing 2 values $A[v]$, $B[v]$ for each **vertex** of the graph, we compute $2^{|B_x|} \leq 2^{w+1}$ values for each bag B_x .

$M[x, S]$:

the max. weight of an independent set

$I \subseteq V_x$ with $I \cap B_x = S$.



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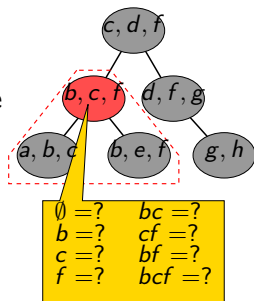
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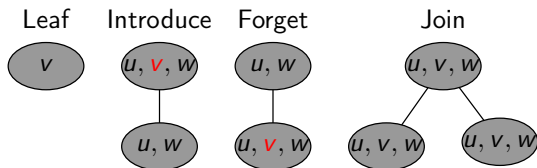
How to determine $M[x, S]$ if all the values are known for the children of x ?

Nice tree decompositions

Definition

A rooted tree decomposition is **nice** if every node x is one of the following 4 types:

- **Leaf:** no children, $|B_x| = 1$
- **Introduce:** 1 child y with $B_x = B_y \cup \{v\}$ for some vertex v
- **Forget:** 1 child y with $B_x = B_y \setminus \{v\}$ for some vertex v
- **Join:** 2 children y_1, y_2 with $B_x = B_{y_1} = B_{y_2}$



Nice tree decompositions

Definition

A rooted tree decomposition is **nice** if every node x is one of the following 4 types:

- **Leaf:** no children, $|B_x| = 1$
- **Introduce:** 1 child y with $B_x = B_y \cup \{v\}$ for some vertex v
- **Forget:** 1 child y with $B_x = B_y \setminus \{v\}$ for some vertex v
- **Join:** 2 children y_1, y_2 with $B_x = B_{y_1} = B_{y_2}$

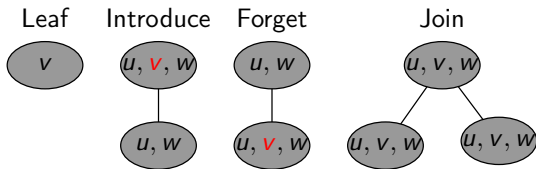
Theorem

A tree decomposition of width w and n nodes can be turned into a nice tree decomposition of width w and $O(wn)$ nodes in time $O(w^2n)$.

WEIGHTED MAX INDEPENDENT SET and nice tree decompositions

- **Leaf:** no children, $|B_x| = 1$
Trivial!
- **Introduce:** 1 child y with $B_x = B_y \cup \{v\}$ for some vertex v

$$m[x, S] = \begin{cases} m[y, S] & \text{if } v \notin S, \\ m[y, S \setminus \{v\}] + w(v) & \text{if } v \in S \text{ but } v \text{ has no} \\ & \text{neighbor in } S, \\ -\infty & \text{if } S \text{ contains } v \text{ and its neighbor.} \end{cases}$$



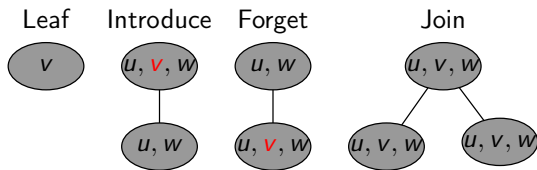
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- **Forget:** 1 child y with $B_x = B_y \setminus \{v\}$ for some vertex v

$$m[x, S] = \max\{m[y, S], m[y, S \cup \{v\}]\}$$

- **Join:** 2 children y_1, y_2 with $B_x = B_{y_1} = B_{y_2}$

$$m[x, S] = m[y_1, S] + m[y_2, S] - w(S)$$



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$$m[x, S] = m[y_1, S] + m[y_2, S] - w(S)$$

There are at most $2^{w+1} \cdot n$ subproblems $m[x, S]$ and each subproblem can be solved in $w^{O(1)}$ time (assuming the children are already solved).



Running time is $O(2^w \cdot w^{O(1)} \cdot n)$.

3-COLORING and tree decompositions

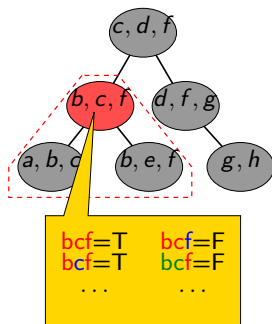
Theorem

Given a tree decomposition of width w , 3-COLORING can be solved in $O(3^w \cdot w^{O(1)} \cdot n)$.

B_x : vertices appearing in node x .

V_x : vertices appearing in the subtree rooted at x .

For every node x and coloring $c : B_x \rightarrow \{1, 2, 3\}$, we compute the Boolean value $E[x, c]$, which is true if and only if c can be extended to a proper 3-coloring of V_x .



3-COLORING and tree decompositions

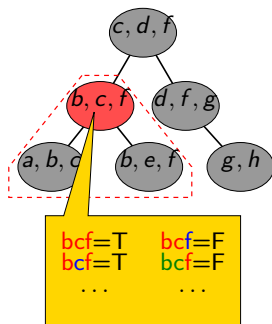
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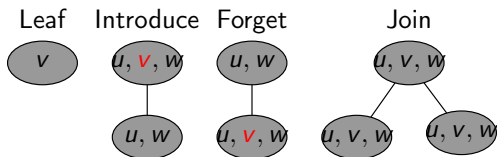
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How to determine $E[x, c]$ if all the values are known for the children of x ?

3-COLORING and nice tree decompositions

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Trivial!
- **Introduce:** 1 child y with $B_x = B_y \cup \{v\}$ for some vertex v
If $c(v) \neq c(u)$ for every neighbor u of v , then
 $E[x, c] = E[y, c']$, where c' is c restricted to B_y .
- **Forget:** 1 child y with $B_x = B_y \setminus \{v\}$ for some vertex v
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\Rightarrow Running time is $O(3^w \cdot w^{O(1)} \cdot n)$.

\Rightarrow 3-COLORING is FPT parameterized by treewidth.

Monadic Second Order Logic

Extended Monadic Second Order Logic (EMSO)

A logical language on graphs consisting of the following:

- Logical connectives $\wedge, \vee, \rightarrow, \neg, =, \neq$
- quantifiers \forall, \exists over vertex/edge variables
- predicate $\text{adj}(u, v)$: vertices u and v are adjacent
- predicate $\text{inc}(e, v)$: edge e is incident to vertex v
- quantifiers \forall, \exists over vertex/edge set variables
- \in, \subseteq for vertex/edge sets

Example:

The formula

$$\exists C \subseteq V \exists v_0 \in C \forall v \in C \exists u_1, u_2 \in C (u_1 \neq u_2 \wedge \text{adj}(u_1, v) \wedge \text{adj}(u_2, v))$$

is true on graph G if and only if ...

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is true on graph G if and only if G has a cycle.

Courcelle's Theorem

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If a graph property can be expressed in EMSO, then for every fixed $w \geq 1$, there is a linear-time algorithm for testing this property on graphs having treewidth at most w .

Note: The constant depending on w can be very large (double, triple exponential etc.), therefore a direct dynamic programming algorithm can be more efficient.

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If we can express a property in EMSO, then we immediately get that testing this property is FPT parameterized by the treewidth w of the input graph.

Can we express **3-COLORING** and **HAMILTONIAN CYCLE** in EMSO?

Using Courcelle's Theorem

3-COLORING

$$\exists C_1, C_2, C_3 \subseteq V \left(\forall v \in V (v \in C_1 \vee v \in C_2 \vee v \in C_3) \right) \wedge \left(\forall u, v \in V \text{adj}(u, v) \rightarrow (\neg(u \in C_1 \wedge v \in C_1) \wedge \neg(u \in C_2 \wedge v \in C_2) \wedge \neg(u \in C_3 \wedge v \in C_3)) \right)$$

Using Courcelle's Theorem

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HAMILTONIAN CYCLE

$$\exists H \subseteq E (\text{spanning}(H) \wedge (\forall v \in V \text{degree2}(H, v)))$$
$$\text{degree0}(H, v) := \neg \exists e \in H \text{inc}(e, v)$$
$$\text{degree1}(H, v) := \neg \text{degree0}(H, v) \wedge (\neg \exists e_1, e_2 \in H (e_1 \neq e_2 \wedge \text{inc}(e_1, v) \wedge \text{inc}(e_2, v)))$$
$$\text{degree2}(H, v) := \neg \text{degree0}(H, v) \wedge \neg \text{degree1}(H, v) \wedge (\neg \exists e_1, e_2, e_3 \in H (e_1 \neq e_2 \wedge e_2 \neq e_3 \wedge e_1 \neq e_3 \wedge \text{inc}(e_1, v) \wedge \text{inc}(e_2, v) \wedge \text{inc}(e_3, v)))$$
$$\text{spanning}(H) := \forall u, v \in V \exists P \subseteq H \forall x \in V (((x = u \vee x = v) \wedge \text{degree1}(P, x)) \vee (x \neq u \wedge x \neq v \wedge (\text{degree0}(P, x) \vee \text{degree2}(P, x))))$$

Using Courcelle's Theorem

Two ways of using Courcelle's Theorem:

- 1 The problem can be described by a single formula (e.g, 3-COLORING, HAMILTONIAN CYCLE).
 \Rightarrow Problem can be solved in time $f(w) \cdot n$ for graphs of treewidth at most w , i.e., FPT parameterized by treewidth.

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- 2 The problem can be described by a formula for each value of the parameter k .

Example: For each k , having a cycle of length exactly k can be expressed as

$$\begin{aligned} \exists v_1, \dots, v_k \in V & ((v_1 \neq v_2) \wedge (v_1 \neq v_3) \wedge \dots \wedge (v_{k-1} \neq v_k)) \\ & \wedge (\text{adj}(v_1, v_2) \wedge \text{adj}(v_2, v_3) \wedge \dots \wedge \text{adj}(v_{k-1}, v_k) \wedge \text{adj}(v_k, v_1)). \end{aligned}$$

\Rightarrow Problem can be solved in time $f(k, w) \cdot n$ for graphs of treewidth w , i.e., FPT parameterized with combined parameter k and treewidth w .

SUBGRAPH ISOMORPHISM

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Input: graphs H and G

Find: a copy of H in G as subgraph.

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For each H , we can construct a formula ϕ_H that expresses “ G has a subgraph isomorphic to H ” (similarly to the k -cycle on the previous slide).

\Rightarrow By Courcelle's Theorem, SUBGRAPH ISOMORPHISM can be solved in time $f(H, w) \cdot n$ if G has treewidth at most w .

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Input: graphs H and G

Find: a copy of H in G as subgraph.

Since there is only a finite number of simple graphs on k vertices, **SUBGRAPH ISOMORPHISM** can be solved in time $f(k, w) \cdot n$ if H has k vertices and G has treewidth at most w .

Theorem

SUBGRAPH ISOMORPHISM is FPT parameterized by combined parameter $k := |V(H)|$ and the treewidth w of G .

MSO on words

Theorem [Büchi, Elgot, Trakhtenbrot 1960]

If a language $L \subseteq \Sigma^*$ can be defined by an MSO formula ϕ using the relation $<$, then L is regular.

Example: a^*bc^* is defined by

$$\exists x : P_b(x) \wedge (\forall y : (y < x) \rightarrow P_a(y)) \wedge (\forall y : (x < y) \rightarrow P_c(y)).$$

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We prove a more general statement for formulas $\phi(w, X_1, \dots, X_k)$ and words over $\Sigma \cup \{0, 1\}^k$, where X_i is a subset of symbols of w .

Induction over the structure of ϕ :

- FSM for $\neg\phi(w)$, given FSM for $\phi(w)$.
- FSM for $\phi_1(w) \wedge \phi_2(w)$, given FSMs for $\phi_1(w)$ and $\phi_2(w)$.
- FSM for $\exists X\phi(w, X)$, given FSM for $\phi(w, X)$.
- etc.

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Proving Courcelle's Theorem:

- Generalize from words to trees.
- A width- k tree decomposition can be interpreted as a tree over an alphabet of size $f(k)$.
- Formula \Rightarrow tree automata.

Algorithms — overview

- Algorithms exploit the fact that a subtree communicates with the rest of the graph via a single bag.
- Key point: defining the subproblems.
- Courcelle's Theorem makes this process automatic for many problems.
- There are notable problems that are easy for trees, but hard for bounded-treewidth graphs.

Treewidth — outline

- 1 Basic algorithms
- 2 Combinatorial properties
- 3 Applications

Properties of treewidth

Fact: Treewidth does not increase if we delete edges, delete vertices, or contract edges.

\Rightarrow If F is a **minor** of G , then the treewidth of F is at most the treewidth of G .

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Fact: The treewidth of the k -clique is $k - 1$.

Fact: For every $k \geq 2$, the treewidth of the $k \times k$ grid is exactly k .



The Cops and Robber game

Game: k cops try to capture a robber in the graph.

- In each step, the cops can move from vertex to vertex arbitrarily with helicopters.
- The robber moves infinitely fast on the edges, and sees where the cops will land.

Theorem [Seymour and Thomas 1993]

$k+1$ cops can win the game \iff the treewidth of the graph is at most k .

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Consequence 1: Algorithms

The winner of the game can be determined in time $n^{O(k)}$ using standard techniques (there are at most n^k positions for the cops)



For every fixed k , it can be checked in polynomial-time if treewidth is at most k .

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Consequence 2: Lower bounds

Exercise 1:

Show that the treewidth of the $k \times k$ grid is at least $k - 1$.

(E.g., robber can win against $k - 1$ cops.)

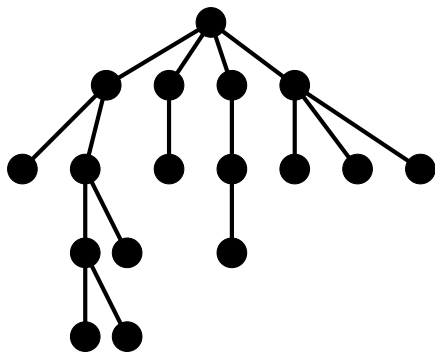
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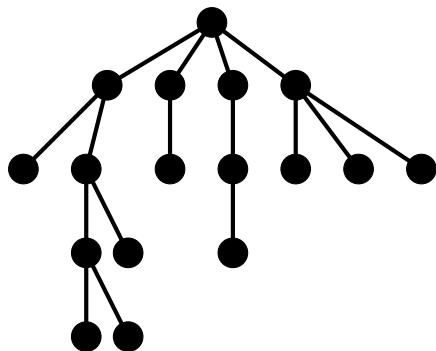
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Example: 2 cops have a winning strategy in a tree.



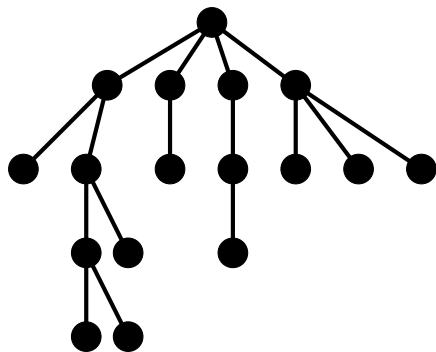
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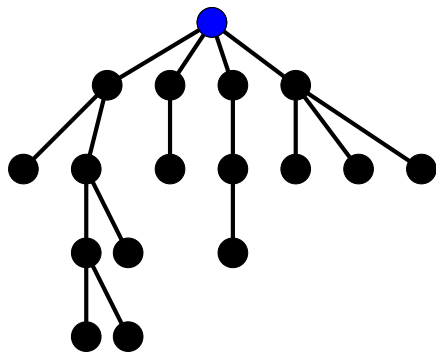
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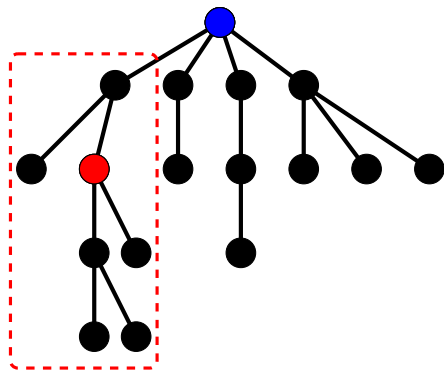
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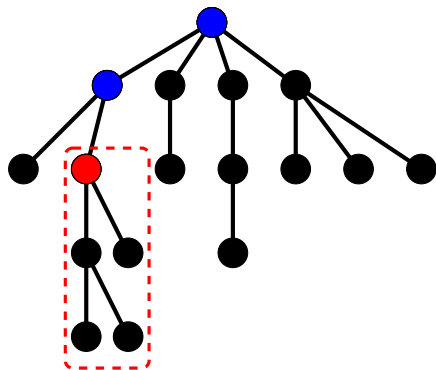
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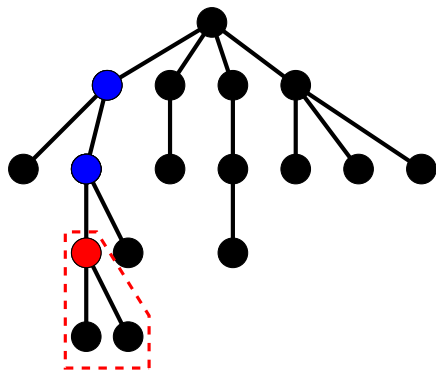
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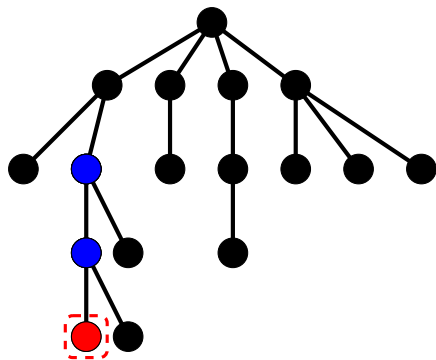
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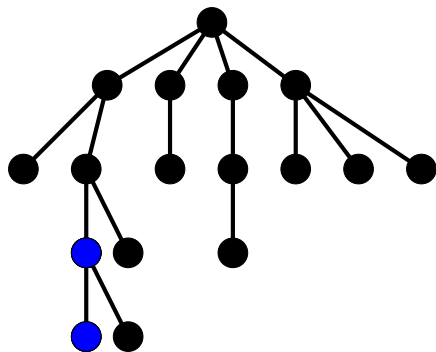
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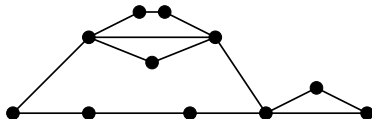


A perfect structure theorem

Theorem

The following are equivalent:

- G does not have a K_4 minor.
- G has treewidth ≤ 2 .
- G is subgraph of a series-parallel graph.



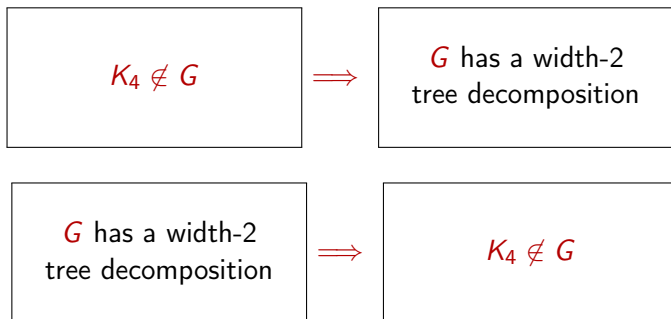
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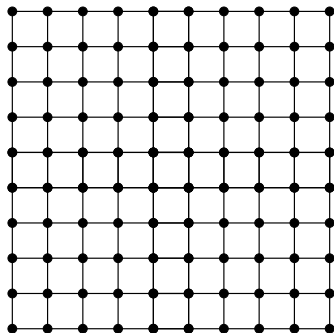
A perfect structure theorem:



Excluded Grid Theorem

Excluded Grid Theorem [Diestel et al. 1999]

If the treewidth of G is at least $k^{4k^2(k+2)}$, then G has a $k \times k$ grid minor.



(A $k^{O(1)}$ bound was just announced [Chekuri and Chuznoy 2013]!)

Excluded Grid Theorem

Excluded Grid Theorem [Diestel et al. 1999]

If the treewidth of G is at least $k^{4k^2(k+2)}$, then G has a $k \times k$ grid minor.

Observation: Every planar graph is the minor of a sufficiently large grid.

Consequence

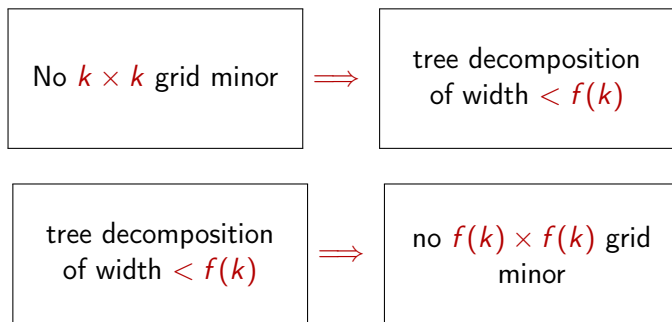
If H is planar, then every H -minor free graph has treewidth at most $f(H)$.

Excluded Grid Theorem

Excluded Grid Theorem [Diestel et al. 1999]

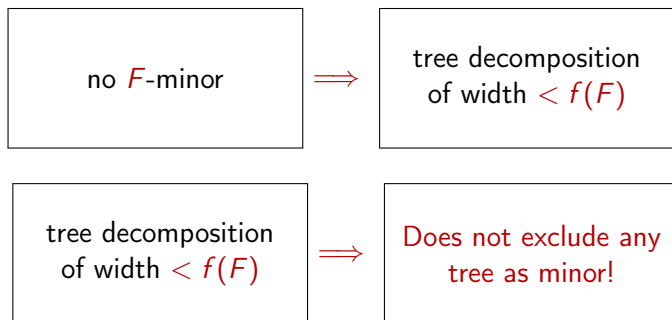
If the treewidth of G is at least $k^{4k^2(k+2)}$, then G has a $k \times k$ grid minor.

A large grid minor is a “witness” that treewidth is large, but the relation is approximate:



Excluding trees

As every forest is planar, the following holds for every forest F :



This is not a good (approximate) structure theorem.

Excluding trees

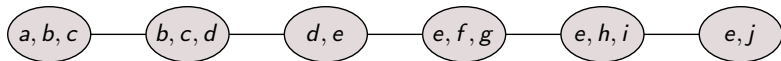
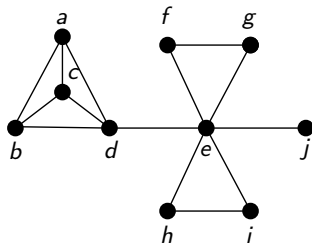
Path decomposition: the tree of bags is a path.

Pathwidth: defined analogously to treewidth.

Example: A complete binary tree on k levels has pathwidth $k - 1$.

Theorem [Diestel 1995]

If F is a forest, then every F -minor free graph has pathwidth at most $|V(F)| - 2$.



Excluding trees

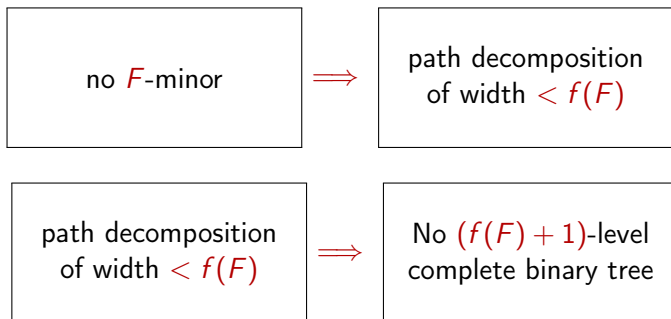
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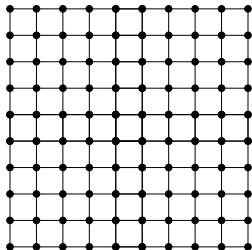


Planar Excluded Grid Theorem

For planar graphs, we get linear instead of exponential dependence:

Theorem [Robertson, Seymour, Thomas 1994]

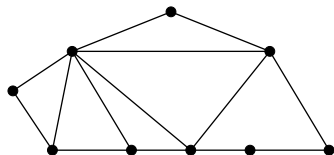
Every **planar graph** with treewidth at least $5k$ has a $k \times k$ grid minor.



Outerplanar graphs

Definition

A planar graph is **outerplanar** if it has a planar embedding where every vertex is on the infinite face.



Fact

Every outerplanar graph has treewidth at most 2.

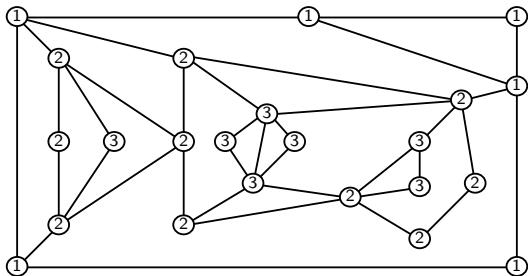
⇒ Every outerplanar graph is subgraph of a series-parallel graph.

k -outerplanar graphs

Given a planar embedding, we can define **layers** by iteratively removing the vertices on the infinite face.

Definition

A planar graph is **k -outerplanar** if it has a planar embedding having at most k layers.



Fact

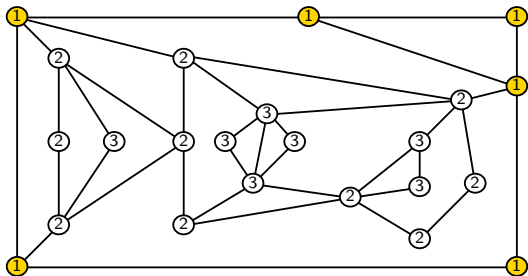
Every k -outerplanar graph has treewidth at most $3k + 1$.

k -outerplanar graphs

Given a planar embedding, we can define **layers** by iteratively removing the vertices on the infinite face.

Definition

A planar graph is **k -outerplanar** if it has a planar embedding having at most k layers.



Fact

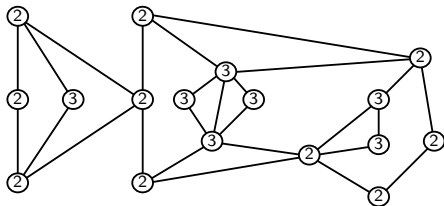
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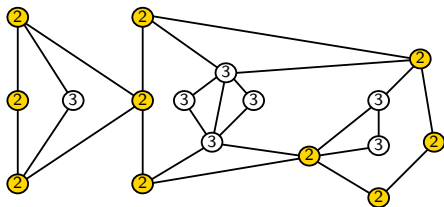
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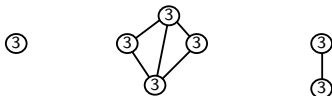
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Treewidth — outline

- ① Basic algorithms
- ② Combinatorial properties
- ③ Applications
 - The shifting technique
 - Bidimensionality
 - Complexity of CSP

Approximation schemes

Definition

A **polynomial-time approximation scheme (PTAS)** for a problem P is an algorithm that takes an instance of P and a rational number $\epsilon > 0$,

- always finds a $(1 + \epsilon)$ -approximate solution,
- the running time is polynomial in n for every fixed $\epsilon > 0$.

Typical running times: $2^{1/\epsilon} \cdot n$, $n^{1/\epsilon}$, $(n/\epsilon)^2$, n^{1/ϵ^2} .

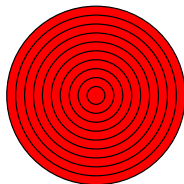
Some classical problems that have a PTAS:

- **INDEPENDENT SET** for planar graphs
- **TSP** in the Euclidean plane
- **STEINER TREE** in planar graphs
- **KNAPSACK**

Baker's shifting strategy for PTAS

Theorem

There is a $2^{O(1/\epsilon)} \cdot n$ time PTAS for **INDEPENDENT SET** for planar graphs.

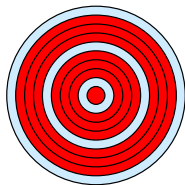


- Let $D := 1/\epsilon$. For a fixed $0 \leq s < D$, delete every layer L_i with $i = s \pmod{D}$

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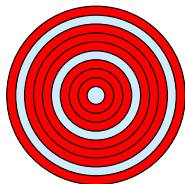


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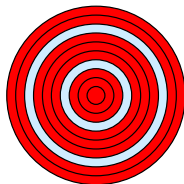


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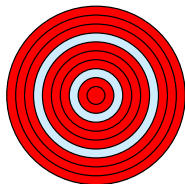


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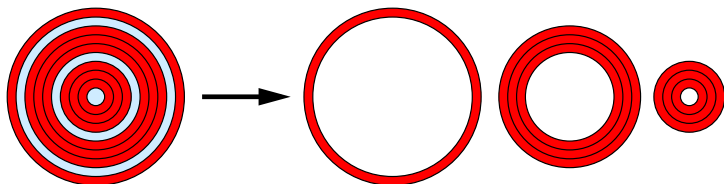


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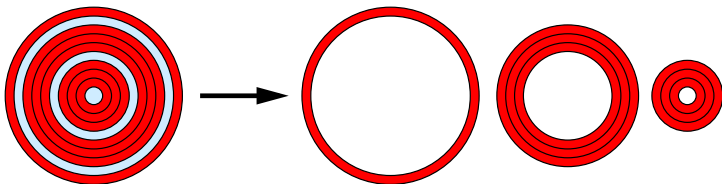


- Let $D := 1/\epsilon$. For a fixed $0 \leq s < D$, delete every layer L_i with $i = s \pmod{D}$
- The resulting graph is D -outerplanar, hence it has treewidth at most $3D + 1 = O(1/\epsilon)$.
- Using the $2^{O(\text{tw})} \cdot n$ time algorithm for **INDEPENDENT SET**, the problem on the D -outerplanar graph can be solved in time $2^{O(1/\epsilon)} \cdot n$.

Baker's shifting strategy for PTAS

Theorem

There is a $2^{O(1/\epsilon)} \cdot n$ time PTAS for **INDEPENDENT SET** for planar graphs.



We do this for every $0 \leq s < D$:
for at least one value of s , we delete
at most $1/D = \epsilon$ fraction of the solution



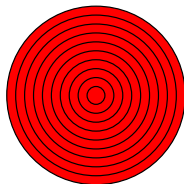
We get a $(1 + \epsilon)$ -approximate solution.

Baker's shifting strategy for FPT

SUBGRAPH ISOMORPHISM

Input: graphs H and G

Find: a copy of H in G as subgraph.

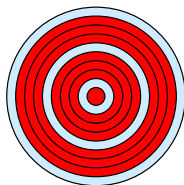


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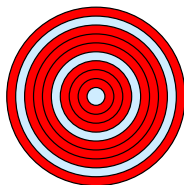
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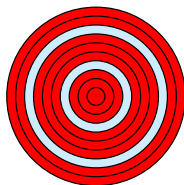
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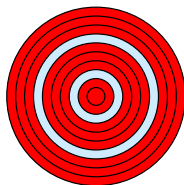
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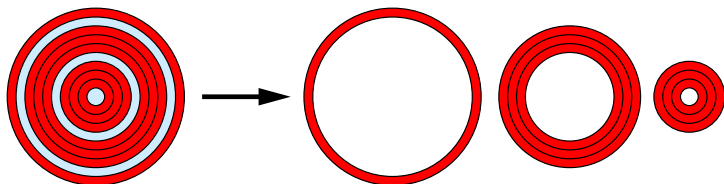
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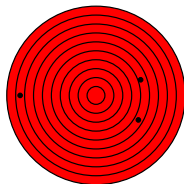
- For a fixed $0 \leq s < k + 1$, delete every layer L_i with $i = s \pmod{k + 1}$
- The resulting graph is k -outerplanar, hence it has treewidth at most $3k + 1$.
- Using the $f(k, tw) \cdot n$ time algorithm for **SUBGRAPH ISOMORPHISM**, the problem can be solved in time $f(k, 3k + 1) \cdot n$.

Baker's shifting strategy for FPT

SUBGRAPH ISOMORPHISM

Input: graphs H and G

Find: a copy of H in G as subgraph.



We do this for every $0 \leq s < k + 1$:
for at least one value of s , we do not delete
any of the k vertices of the solution



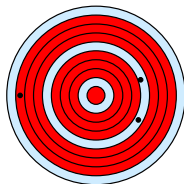
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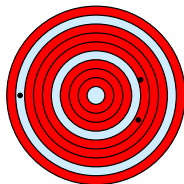
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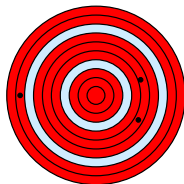
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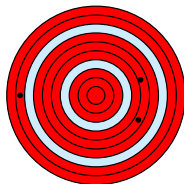
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Baker's shifting strategy for FPT

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Theorem

SUBGRAPH ISOMORPHISM for planar graphs is FPT parameterized by $k := |V(H)|$.

Baker's shifting strategy for FPT

- The technique is very general, works for many problems on planar graphs:
 - INDEPENDENT SET
 - VERTEX COVER
 - DOMINATING SET
 - ...
- More generally: First-Order Logic problems.
- But for some of these problems, much better techniques are known (see the following slides).

Bidimensionality

A powerful framework for efficient algorithms on planar graphs.

Setup:

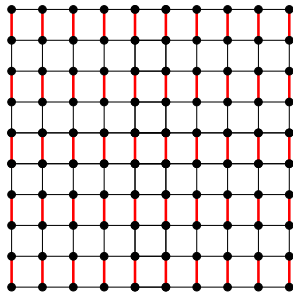
- Let $x(G)$ be some graph invariant (i.e., an integer associated with each graph).
- Given G and k , we want to decide if $x(G) \leq k$ (or $x(G) \geq k$).
- Typical examples:
 - Maximum independent set size.
 - Minimum vertex cover size.
 - Length of the longest path.
 - Minimum dominating set size.
 - Minimum feedback vertex set size.

Bidimensionality [Demaine, Fomin, Hajiaghayi, Thilikos 2005]

For many natural invariants, we can do this in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$ on planar graphs.

Bidimensionality for VERTEX COVER

- Observation:** If the treewidth of a planar graph G is at least $5\sqrt{2k}$
- \Rightarrow It has a $\sqrt{2k} \times \sqrt{2k}$ grid minor (Planar Excluded Grid Theorem)
 - \Rightarrow The grid has a matching of size k
 - \Rightarrow Vertex cover size is at least k in the grid.
 - \Rightarrow Vertex cover size is at least k in G .

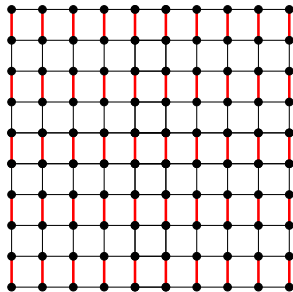


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 - \Rightarrow Vertex cover size is at least k in G .

We use this observation to solve VERTEX COVER on planar graphs:

- Set $w := 5\sqrt{2k}$.
- Find a 4-approximate tree decomposition.
 - If treewidth is at least w : we answer “vertex cover is $\geq k$.”
 - If we get a tree decomposition of width $4w$, then we can solve the problem in time $2^{O(w)} \cdot n^{O(1)} = 2^{O(\sqrt{k})} \cdot n^{O(1)}$.

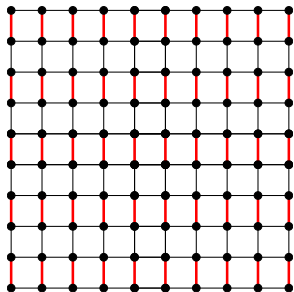


Bidimensionality

Definition

A graph invariant $x(G)$ is **minor-bidimensional** if

- $x(G') \leq x(G)$ for every minor G' of G , and
- If G_k is the $k \times k$ grid, then $x(G_k) \geq ck^2$ (for some constant $c > 0$).



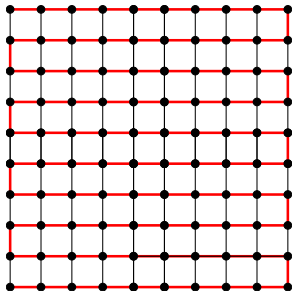
Examples: **minimum vertex cover**, length of the longest path, feedback vertex set are minor-bidimensional.

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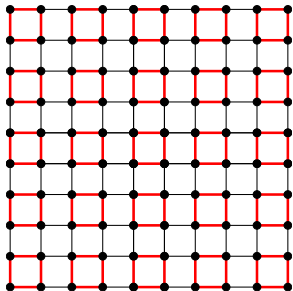
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Examples: minimum vertex cover, length of the longest path, **feedback vertex set** are minor-bidimensional.

Bidimensionality (cont.)

We can answer “ $x(G) \geq k$?” for a minor-bidimensional invariant the following way:

- Set $w := c\sqrt{k}$ for an appropriate constant c .
- Use the 4-approximation tree decomposition algorithm.
 - If treewidth is at least w : $x(G)$ is at least k .
 - If we get a tree decomposition of width $4w$, then we can solve the problem using dynamic programming on the tree decomposition.

Running time:

- If we can solve the problem on tree decomposition of width w in time $2^{O(w)} \cdot n^{O(1)}$, then the running time is $2^{O(\sqrt{k})} \cdot n^{O(1)}$.
- If we can solve the problem on tree decomposition of width w in time $w^{O(w)} \cdot n^{O(1)}$, then the running time is $2^{O(\sqrt{k} \log k)} \cdot n^{O(1)}$.

Contraction bidimensionality

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Exercise: DOMINATING SET is **not** minor-bidimensional.

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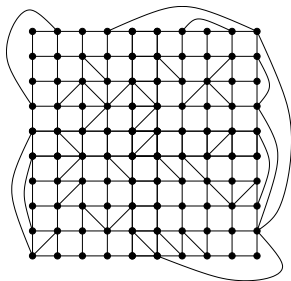
Exercise: DOMINATING SET is **not** minor-bidimensional.

We fix the problem by allowing only contractions but not edge/vertex deletions.

Contraction bidimensionality

Theorem

Every **planar graph** with treewidth at least $5k$ can be contracted to a **partially triangulated** $k \times k$ grid.

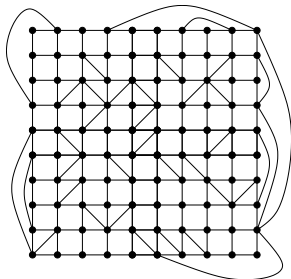


Contraction bidimensionality

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A graph invariant $x(G)$ is **contraction-bidimensional** if

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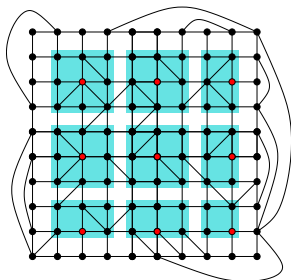


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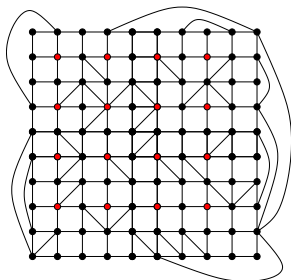
Example: **minimum dominating set**, maximum independent set are contraction-bidimensional.

Contraction bidimensionality

Definition

A graph invariant $x(G)$ is **contraction-bidimensional** if

- $x(G') \leq x(G)$ for every **contraction** G' of G , and
- If G_k is a $k \times k$ **partially triangulated grid**, then $x(G_k) \geq ck^2$ (for some constant $c > 0$).



Example: minimum dominating set, **maximum independent set** are contraction-bidimensional.

Bidimensionality for DOMINATING SET

The size of a minimum dominating set is a **contraction bidimensional** invariant: we need at least $(k - 2)^2/9$ vertices to dominate all the internal vertices of a partially triangulated $k \times k$ grid (since a vertex can dominate at most 9 internal vertices).

Theorem

Given a tree decomposition of width w , DOMINATING SET can be solved in time $3^w \cdot w^{O(1)} \cdot n^{O(1)}$.

Solving DOMINATING SET on planar graphs:

- Set $w := 5(3\sqrt{k} + 2)$.
- Use the 4-approximation tree decomposition algorithm.
 - If treewidth is at least w : we answer 'dominating set is $\geq k$ '.
 - If we get a tree decomposition of width $4w$, then we can solve the problem in time $3^{4w} \cdot n^{O(1)} = 2^{O(\sqrt{k})} \cdot n^{O(1)}$.

Constraint Satisfaction Problems (CSP)

A CSP instance is given by describing the

- variables,
- domain of the variables,
- constraints on the variables.

Task: Find an assignment that satisfies every constraint.

$$I = C_1(x_1, x_2, x_3) \wedge C_2(x_2, x_4) \wedge C_3(x_1, x_3, x_4)$$

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Examples:

- **3SAT**: 2-element domain, every constraint is ternary
- **VERTEX COLORING**: domain is the set of colors, binary constraints
- **k-CLIQUE** (in graph G): k variables, domain is the vertices of G , $\binom{k}{2}$ binary constraints

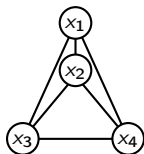
Graphs and hypergraphs related to CSP

Gaifman/primal graph: vertices are the variables, two variables are adjacent if they appear in a common constraint.

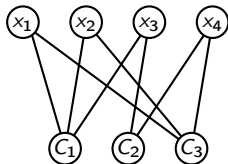
Incidence graph: bipartite graph, vertices are the variables and constraints.

Hypergraph: vertices are the variables, constraints are the hyperedges.

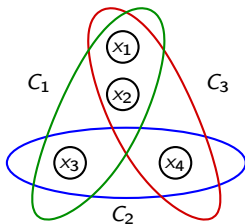
$$I = C_1(x_2, x_1, x_3) \wedge C_2(x_4, x_3) \wedge C_3(x_1, x_4, x_2)$$



Primal graph



Incidence graph



Hypergraph

Treewidth and CSP

Theorem [Freuder 1990]

For every fixed k , CSP can be solved in polynomial time if the primal graph of the instance has treewidth at most k .

Proof sketch:

- Find a tree decomposition of width k (linear-time for fixed k).
- For each bag, enumerate every assignment of the bag that satisfies every constraint fully contained in the bag. Each bag has at most $k + 1$ variables, thus there are at most $|D|^{k+1}$ such assignments for each bag.
- Use bottom-up DP to find a satisfying assignment.
- Each constraint induces a clique in the primal graph, thus each constraint is fully contained in one of the bags.
- Running time of DP is polynomial in $|D|^{k+1}$ and the number of variables.

Dichotomy for binary CSP

Binary CSP: Every constraint is of arity 2.

We know that binary $\text{CSP}(\mathcal{G})$ is polynomial-time solvable for every class \mathcal{G} of graphs with bounded treewidth. Are there other polynomial cases?

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Theorem [Grohe-Schwentick-Segoufin 2001]

Let \mathcal{G} be a recursively enumerable class of **graphs**. Assuming $\text{FPT} \neq \text{W}[1]$, the following are equivalent:

- Binary $\text{CSP}(\mathcal{G})$ is polynomial-time solvable.
- Binary $\text{CSP}(\mathcal{G})$ is FPT.
- \mathcal{G} has bounded treewidth.

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- Binary $\text{CSP}(\mathcal{G})$ is FPT.
- \mathcal{G} has bounded treewidth.

Note: $\text{FPT} \neq \text{W}[1]$ is a standard complexity assumption.

Note: Fixed-parameter tractability does not give us more power here than polynomial-time solvability.

Proof outline

Suppose that \mathcal{G} has unbounded treewidth, but $\text{CSP}(\mathcal{G})$ is FPT.

- Assuming $\text{FPT} \neq \text{W}[1]$, there is no $f(k)n^c$ time algorithm for k -CLIQUE. But we can solve k -CLIQUE the following way:
- Formulate k -CLIQUE as a binary CSP instance on the $k \times k$ grid.
- Find a $G_k \in \mathcal{G}$ containing a $k \times k$ minor (there is such a G_k by the Excluded Grid Theorem).
- Reduce CSP on the $k \times k$ grid to CSP with graph G_k , which is an instance of $\text{CSP}(\mathcal{G})$.
- Use the assumed algorithm for $\text{CSP}(\mathcal{G})$.
- The running time is $f(k)n^c$: the nonpolynomial factors in the running time depend only on k (finding G_k , size of G_k , solving $\text{CSP}(\mathcal{G})$)
 $\Rightarrow k$ -CLIQUE is FPT, contradicting the hypothesis $\text{FPT} \neq \text{W}[1]$.

Treewidth — overview

- Algorithms
 - Dynamic programming
 - Courcelle's Theorem
- Properties
 - Characterization by the Cops and Robber game.
 - Excluding a grid, excluding a tree.
 - k -outerplanar graphs.
- Applications
 - Shifting technique for PTAS and FPT.
 - Minor/contraction bidimensionality.
 - Excluded Grid Theorem in the classification of $\text{CSP}(\mathcal{G})$.

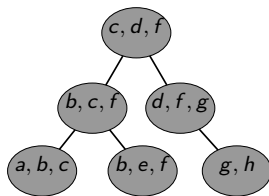
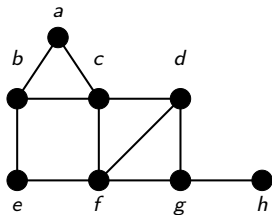
Treewidth

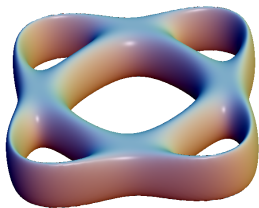
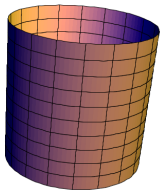
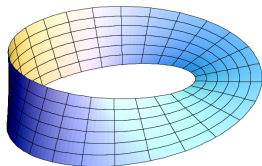
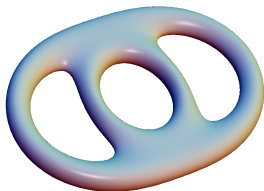
Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

- 1 If u and v are neighbors, then there is a bag containing both of them.
- 2 For every v , the bags containing v form a connected subtree.

Width of the decomposition: largest bag size -1 .

treewidth: width of the best decomposition.





Surfaces

Surfaces

A topological surface is a nonempty second countable Hausdorff topological space in which every point has an open neighborhood homeomorphic to some open subset of the Euclidean plane E^2 .

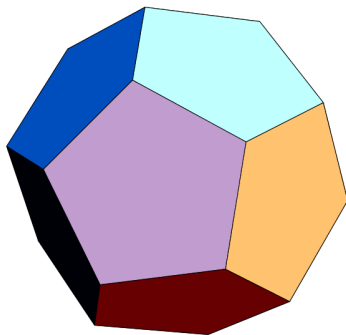
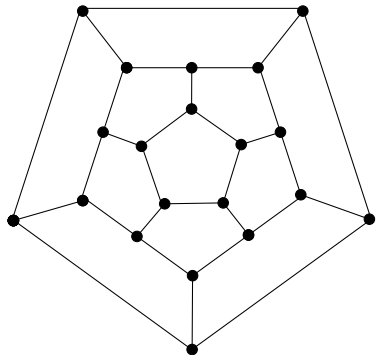
Intuitively: something thin floating in space.

Our viewpoint: which graphs can be drawn on the different surfaces, thus we do not distinguish surfaces that are *homeomorphic*.

Planar graphs

The following are equivalent:

- Graph G can be drawn on the plane.
- Graph G can be drawn inside a disc.
- Graph G can be drawn on the sphere.



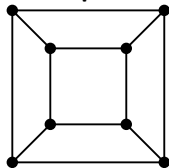
Euler's Formula

Theorem

If G is a connected simple graph drawn in the plane with v vertices, e edges, and f faces, then

$$v + f = e + 2.$$

Example:



$$v = 8$$

$$f = 6$$

$$e = 12$$

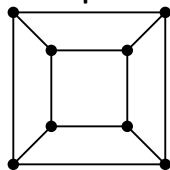
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Consequence: $e \leq 3v - 6$

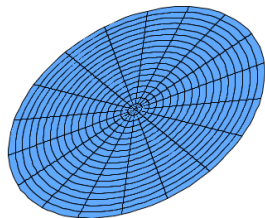
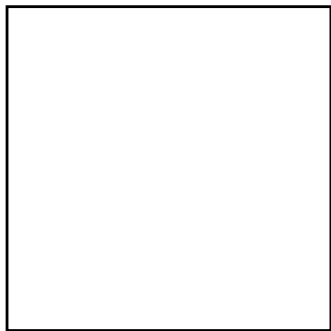
Proof: $2e \geq 3f$ (every face has at least 3 edges)

$$e = v + f - 2 \leq v + \frac{2}{3}e - 2$$

$$\frac{1}{3}e \leq v - 2$$

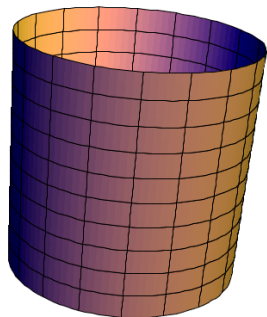
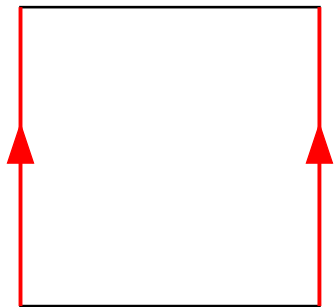
$$e \leq 3v - 6$$

Examples of surfaces: disk



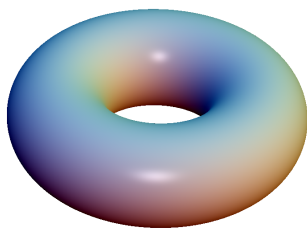
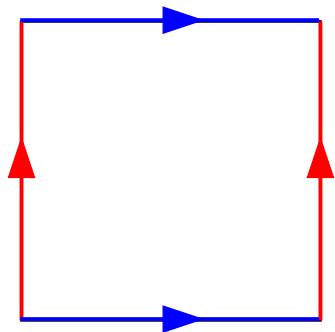
Rectangle with boundary: same as disk.

Examples of surfaces: cylinder



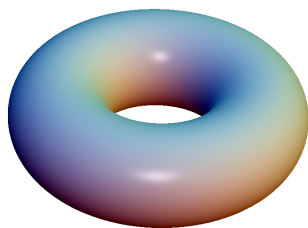
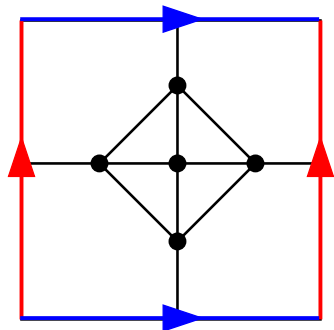
Gluing together the vertical sides creates a cylinder.

Examples of surfaces: torus



Gluing together both the two horizontal and the two vertical sides creates a torus.

Examples of surfaces: torus

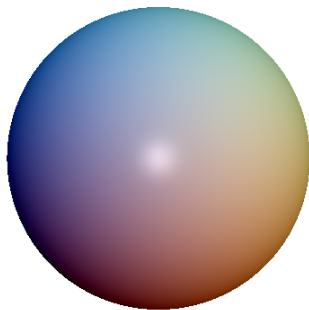
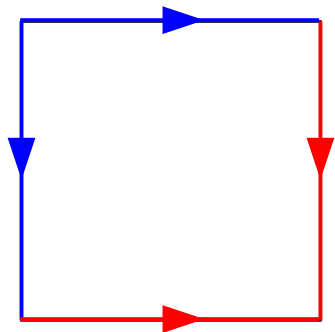


Gluing together both the two horizontal and the two vertical sides creates a torus.

K_5 can be drawn on the torus.

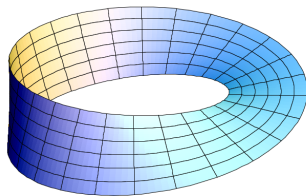
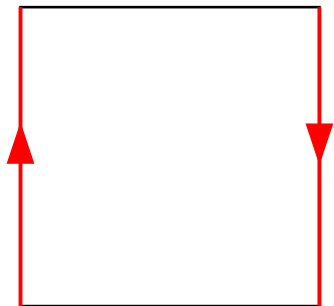
Exercise: draw K_7 on the torus.

Examples of surfaces: sphere



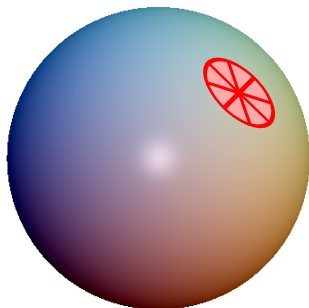
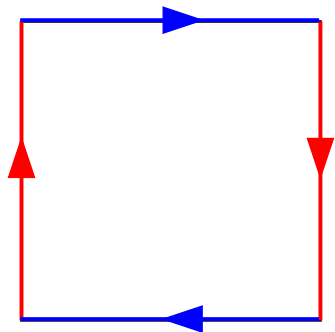
Gluing together top with left and bottom with right creates a sphere.

Examples of surfaces: Möbius strip



Gluing together the vertical sides in twisted way creates a Möbius strip.

Examples of surfaces: real projective plane



Gluing together both the horizontal and vertical sides in a twisted way creates a real projective plane, which is a sphere with a cross cap.

Examples of surfaces: Klein bottle

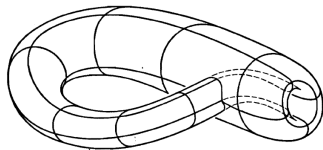
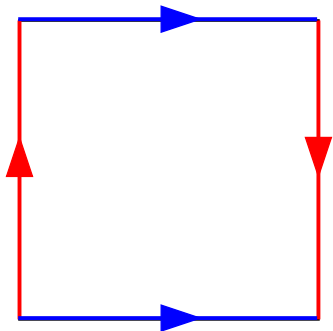
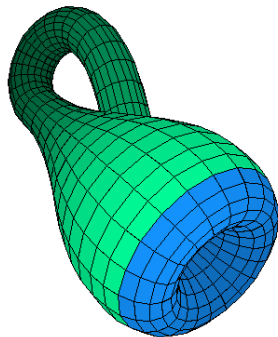
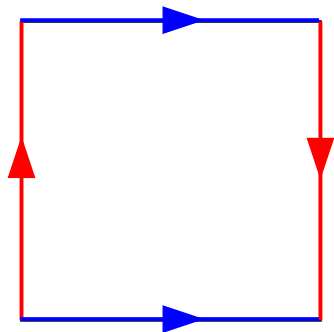


FIG. 297

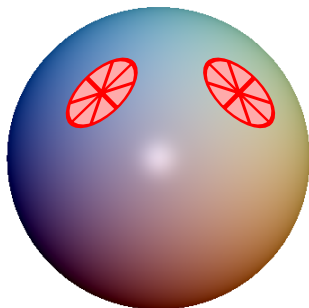
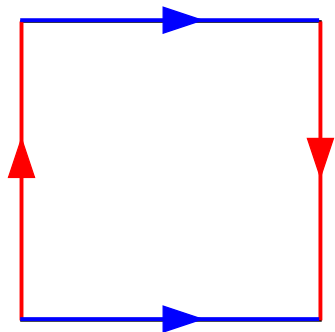
Gluing together both the horizontal sides in a normal and the vertical sides in a twisted way creates a Klein bottle, which is a sphere with two cross caps.

Examples of surfaces: Klein bottle



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Examples of surfaces: Klein bottle

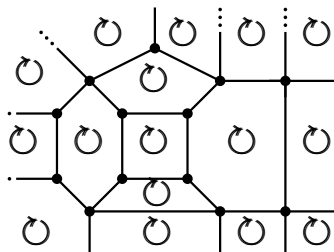


Gluing together both the horizontal sides in a normal and the vertical sides in a twisted way creates a Klein bottle, which is a sphere with two cross caps.

Orientable vs. nonorientable

Definition

A surface Σ is **orientable** if whenever a graph is drawn on Σ such that every face is a disk, then each face can be assigned an orientation such that two faces sharing an edge give the opposite orientation to that edge.



- The sphere and the torus are orientable.
- The Möbius strip and the Klein bottle are nonorientable.

Surfaces with boundaries

Some surfaces have boundaries:



- The cylinder and the Möbius strip have boundaries.
- The sphere, torus, Klein bottle are **closed surfaces**.

Surfaces with boundaries

Some surfaces have boundaries:



- The cylinder and the Möbius strip have boundaries.
- The sphere, torus, Klein bottle are **closed surfaces**.

Every surface with boundaries can be obtained from a closed surface by removing some number of disks.

As removing disks does not change which graphs can be embedded, we consider only closed surfaces from now.

Classification of closed surfaces

Theorem [Brahana 1921]

Every closed surface is equivalent either to

- a sphere with $k \geq 0$ handles (orientable surfaces), or
- or to a sphere with $k \geq 1$ crosscaps (nonorientable surfaces).

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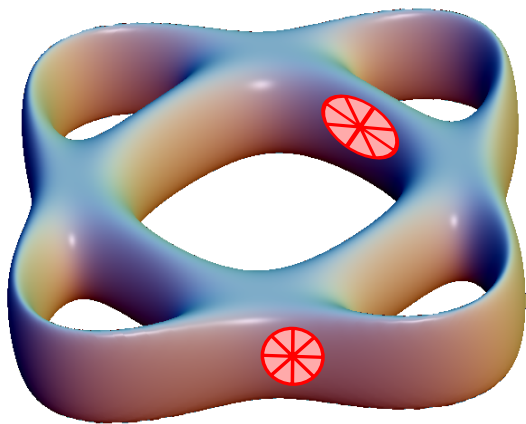
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- or to a sphere with $k \geq 1$ crosscaps (nonorientable surfaces).

Alternative version:

Theorem [Brahana 1921]

Every closed surface is equivalent to a sphere with $k \geq 0$ handles and 0, 1, or 2 crosscaps attached to it.

5 handles, 2 crosscaps



Euler's formula

Theorem

Let G be a connected simple graph drawn on a closed surface Σ such that every face is a disk. If G has v vertices, e edges, and f faces, then

$$v + f = e + 2 - \text{eg}(\Sigma),$$

where the **Euler genus** $\text{eg}(\Sigma)$ is

- $2k$ if Σ is a sphere with k handles, and
- k if Σ is a sphere with k crosscaps.

Consequence: $e \leq 3v - 6 + 3\text{eg}(\Sigma)$

Bounded-genus graphs have bounded average degree.

Algorithms for bounded-genus graphs

Can we generalize the powerful techniques from planar graphs to surfaces?

- Shifting strategy for approximation schemes/parameterized algorithms

Crucial tool: bounding the treewidth of k -outerplanar graphs.

- Subexponential algorithms for minor/contraction-bidimensional problems.

Crucial tool: grid theorems.

Grid theorems

Theorem

Every **planar graph** with treewidth at least $5k$ has a $k \times k$ grid minor.

Theorem [Demaine, Fomin, Hajiaghayi, Thilikos 2005]

If G is a graph drawn on Σ and has treewidth at least $c(\text{eg}(\Sigma) + 1) \cdot k$, then G has a $k \times k$ grid minor.

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Subexponential parameterized algorithms for e.g., k -**VERTEX COVER** go through:

- either the graph has a $\Omega(\sqrt{k}) \times \Omega(\sqrt{k})$ grid minor and then the vertex cover size is at least k , or
- treewidth is $O(\sqrt{k}(\text{eg}(\Sigma) + 1))$ and we can solve the problem in time $2^{O(\sqrt{k}(\text{eg}(\Sigma)+1))} \cdot n^{O(1)}$.

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Similar (more complicated) generalizations for contraction-bidimensional problems.

Local treewidth

The shifting technique relied on the fact that the treewidth of k -outerplanar graphs have bounded treewidth.

Definition

A class \mathcal{G} of graphs has **bounded local treewidth** if there is a function f such that $\text{tw}(G) \leq f(\text{diam}(G))$ for every $G \in \mathcal{G}$.

Bounded genus implies bounded local treewidth:

Theorem

The class \mathcal{G}_Σ of graphs embeddable into Σ has bounded local treewidth with $\text{tw}(G) \leq 3\text{eg}(\Sigma)\text{diam}(G)$.

Local treewidth

$B_v[d]$: the **ball** containing vertices at distance $\leq d$ from v .

$R_v[x, y]$ the **ring** containing vertices at distance $x \leq d \leq y$ from v .

Lemma

Let \mathcal{G} be a **minor-closed** class of graphs having **bounded local treewidth**. Then the treewidth of $R_v[x, y]$ can be bounded by a function of $y - x + 1$.

Local treewidth

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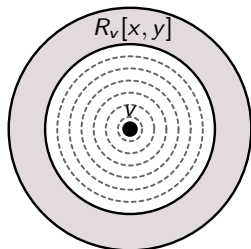
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Let \mathcal{G} be a **minor-closed** class of graphs having **bounded local treewidth**. Then the treewidth of $R_v[x, y]$ can be bounded by a function of $y - x + 1$.

Proof:

- Contract $B_v[x - 1]$.
- Ring $R_v[x, y]$ appears now at distance $y - x + 1$ from v .
- The ring appears in a graph of treewidth $f(y - x + 1)$.



Local treewidth

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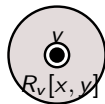
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PTAS using bounded local treewidth

Theorem

If \mathcal{G} is minor-closed and has bounded local treewidth, then INDEPENDENT SET has a PTAS on \mathcal{G} .

- Repeat the following for $i = 0, \dots, D$, where $D = \lceil 1/\epsilon \rceil$.
- Pick a vertex v and remove every vertex at distance $jD + i$ for $j = 0, 1, \dots$.
- The graph falls apart into disjoint rings $R_v[0, i - 1]$, $R_v[i + 1, D + i - 1]$, $R_v[D + i + 1, 2D + i - 1]$, \dots .
- Thus treewidth is $f(D)$, i.e., can be bounded as function of ϵ .
- Problem can be solved in time $f(1/\epsilon) \cdot n$.
- At least one choice of i removes at most an ϵ fraction of the optimum solution.

Bounded degree

A potential source of confusion

- 3-regular graphs have bounded local treewidth: if diameter is d , then there are at most $\sum_{i=0}^d 3^i = (3^{d+1} - 1)/2$ vertices, hence treewidth is bounded by a function of d .
- **INDEPENDENT SET** is APX-hard on 3-regular graphs, thus it has no PTAS unless $P = NP$.

Have we just proved $P = NP$?

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Theorem

If \mathcal{G} is a **minor-closed** and has **bounded local treewidth**, then **INDEPENDENT SET** has a PTAS on \mathcal{G} .

Local treewidth is useful only for **minor-closed** classes!

Local treewidth

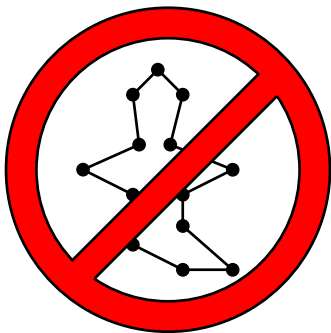
Theorem [Frick and Grohe 2001]

If a graph property can be expressed in first-order logic and \mathcal{G} is a class of graphs with bounded local treewidth, then there is a linear-time algorithm for testing this property on members of \mathcal{G} .

Note: we do not need here that \mathcal{G} is closed under taking minors.

Shows, e.g., that **SUBGRAPH ISOMORPHISM** is FPT on planar graphs or on bounded-degree graphs.

Exercise: Can this result be generalized to EMSO instead of first-order logic?

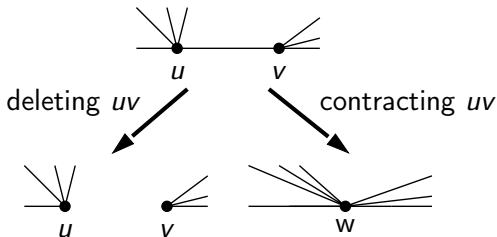


Excluding minors

Minors

Definition

Graph H is a **minor** G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.

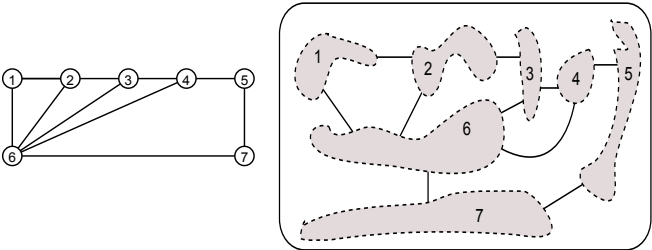


Minors

Equivalent definition

Graph H is a **minor** of G if there is a mapping ϕ (the minor model) that maps each vertex of H to a connected subset of G such that

- $\phi(u)$ and $\phi(v)$ are disjoint if $u \neq v$, and
- if $uv \in E(G)$, then there is an edge between $\phi(u)$ and $\phi(v)$.

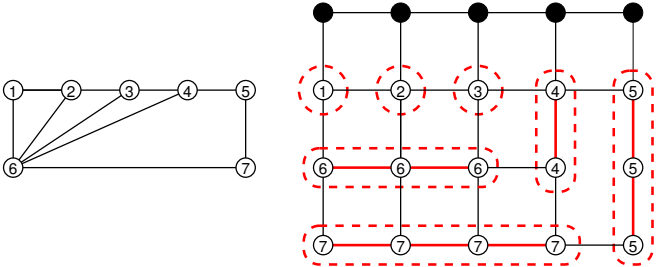


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Excluding minors

Connection to surfaces:

- Graphs excluding K_5 - and $K_{3,3}$ -minors are planar.
- Graphs that can be drawn on a fixed surface (e.g., torus) can be characterized by a finite list of excluded minors.

Is it true for every H that H -minor free graphs can be drawn on a fixed surface?

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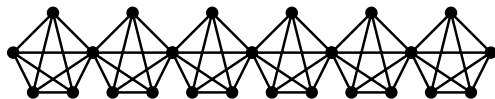
Is it true for every H that H -minor free graphs can be drawn on a fixed surface?

NO (clique sums), **NO** (apices), **NO** (vortices)

YES (in a sense — Robertson-Seymour Structure Theorem)

Excluding minors

The following graph does not have a K_6 -minor, but its genus can be large:

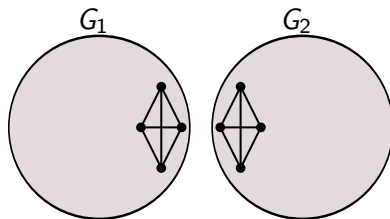


Connecting bounded-genus graphs can increase genus without creating a clique minor.

Clique sums

Definition

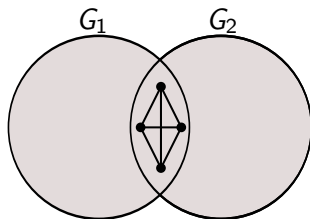
Let G_1 and G_2 be two graphs with two cliques $K_1 \subseteq V(G_1)$ and $K_2 \subseteq V(G_2)$ of the same size. Graph G is a **clique sum** of G_1 and G_2 if it can be obtained by identifying K_1 and K_2 , and then removing some of the edges of the clique.



Clique sums

Definition

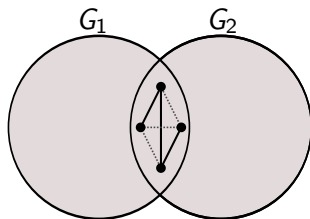
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Clique sums

Observation

If $K_k \not\leq G_1, G_2$ and G is a clique sum of G_1 and G_2 , then $K_k \not\leq G$.

Thus we can build K_k -minor-free graphs by repeated clique sums.

Clique sums

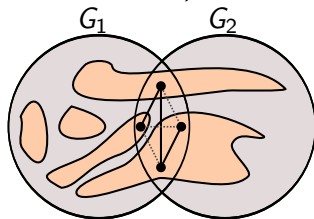
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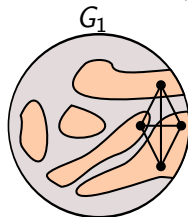
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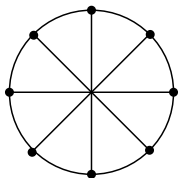
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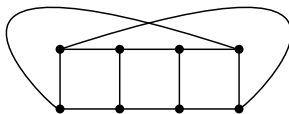
Excluding K_5

Theorem [Wagner 1937]

A graph is K_5 -minor-free if and only if it can be built from planar graphs and V_8 by repeated clique sums.



V_8

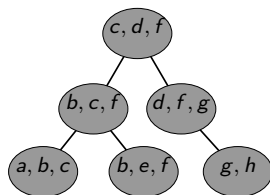
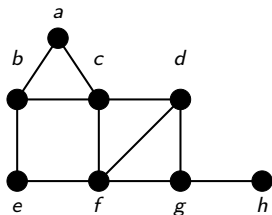


V_8

Tree decomposition

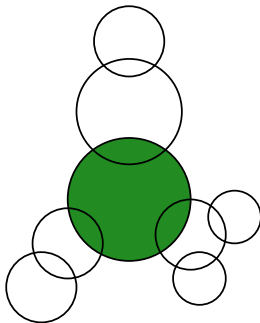
Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

- 1 If u and v are neighbors, then there is a bag containing both of them.
- 2 For every v , the bags containing v form a connected subtree.



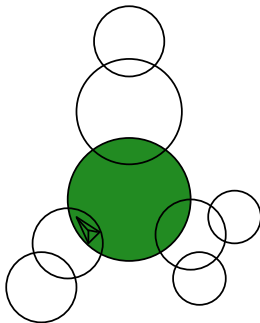
Torso

Torso of a bag: we make the intersections with the adjacent bags cliques.



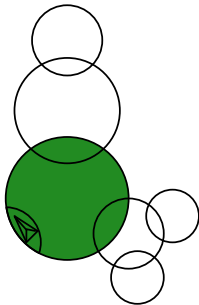
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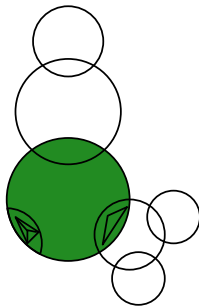
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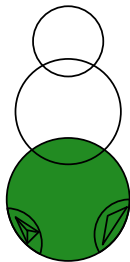
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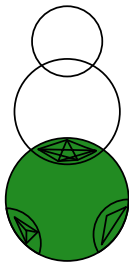
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Excluding K_5 — restated

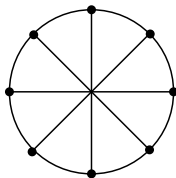
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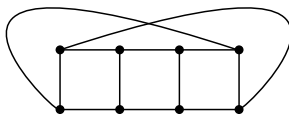
Equivalently:

Theorem [Wagner 1937]

A graph is K_5 -minor-free if and only if it has a tree decomposition where every torso is either a planar graph or the graph V_8 .



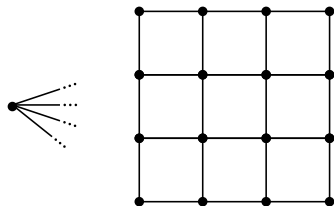
V_8



V_8

Apex vertices

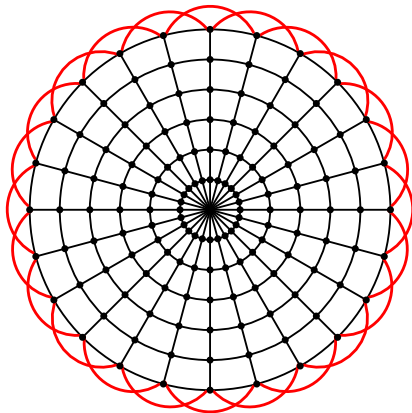
The graph formed from a grid by attaching a universal vertex is K_6 -minor-free, but has large genus.



- A planar graph + k extra vertices has no K_{k+5} -minor.
- Instead of bounded genus graphs, our building blocks should be “bounded genus graphs + a bounded number of apex vertices connected arbitrarily.”

Vortices

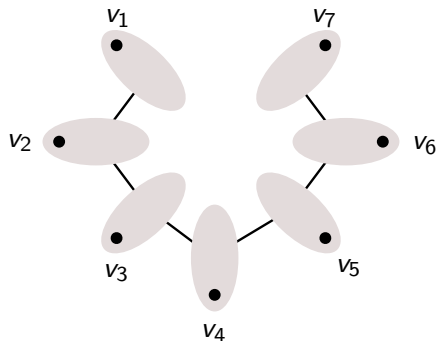
One can show that the following graph has large genus, but cannot have a K_8 -minor.



Removing a few apex vertices or decomposing by clique sums do not help.

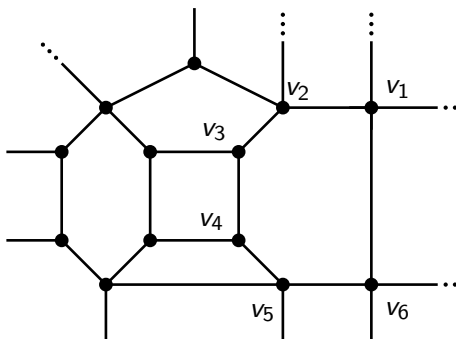
Vortices

- A **vortex** of width k and perimeter v_1, \dots, v_n is a graph F that has a width- k path decomposition B_1, \dots, B_n such that $v_i \in B_i$.
- Let G be embedded in Σ and let D be a disk intersecting G only in vertices v_1, \dots, v_n . Attaching a vortex on D means taking the union of G and a vortex on v_1, \dots, v_n (the vortex intersects G only in these vertices).



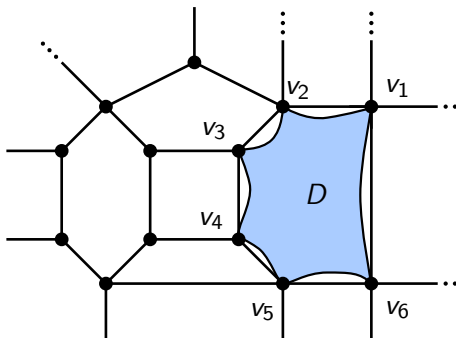
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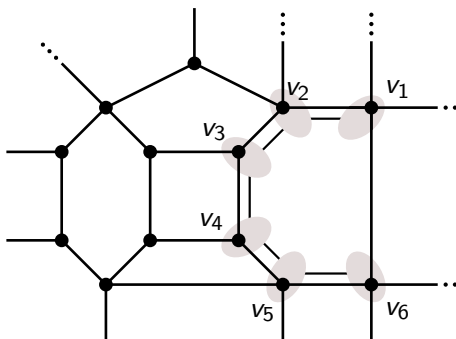
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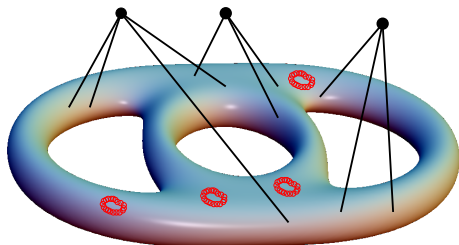


k -almost embeddable

Definition

Graph G is k -almost embeddable in surface Σ if

- there is a set X of at most k apex vertices and
- a graph G_0 embedded in Σ , such that
- $G \setminus X$ can be obtained from G_0 by attaching vortices of width k on disjoint disks D_1, \dots, D_k .



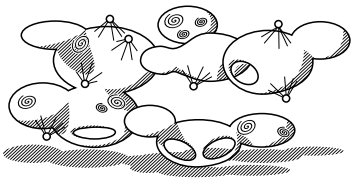
Graph Structure Theorem

Theorem [Robertson-Seymour]

For every graph H , there is an integer k and a surface Σ such that every H -minor-free graph has a tree decomposition where every torso is k -almost embeddable in Σ .

Originally stated only combinatorially, algorithmic versions are known.

- Running time was improved from $n^{f(H)}$ to $f(H) \cdot n^{O(1)}$.
- Algorithm finds also an apex set of size at most k for each torso.



H-Minor-Free



Bounded Genus



Planar

[figure by Felix Reidl]

Excluding cliques

What do we get by excluding small cliques?

- K_3 -minor free: every torso is size ≤ 2 (trees).
- K_4 -minor free: every torso is size ≤ 3 (series-parallel graphs).
- K_5 -minor free: every torso is planar or V_8 .
- K_k -minor free for $k \geq 6$: every torso is k -almost embeddable in some surface Σ_k .

Algorithmic applications

Theorem [Demaine, Hajiaghayi, Kawarabayashi 2005]

For every graph H , there is a constant c_H such that for any $k \geq 1$, every H -minor-free graph G can be partitioned into $k + 1$ vertex sets V_1, \dots, V_{k+1} such that $G \setminus V_i$ has treewidth at most $c_H \cdot k$ for any i . Furthermore, such a partition can be found in polynomial time.

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PTAS is immediate for e.g., **INDEPENDENT SET**:

- Set $k := \lceil 1/\epsilon \rceil$ and find the partition.
- For every $i = 1, \dots, k + 1$, compute the solution optimally for $G \setminus V_i$.
- There is one i for which the solution is $k/(k + 1) \geq 1 - \epsilon$ times the optimum.



Finding minors

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H -minor testing

Input: graph G

Find: a model of H in G .

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Theorem

H -minor testing for **planar** H can be solved in time $f(H) \cdot n^{O(1)}$.

Proof:

- If G has treewidth $\geq g(H)$, then it contains a large grid minor, hence contains H .
- If G has treewidth $< g(H)$, then e.g., Courcelle's Theorem can be invoked to check if G contains an H -minor.

Finding rooted minors

Theorem [Robertson and Seymour]

H -minor testing can be solved in time $f(H) \cdot n^3$.

Robertson and Seymour actually solved a more general problem:

Rooted H -minor testing

Input: graph G , a vertex $\rho(v) \in V(G)$ for every $v \in V(H)$.

Find: a model of H in G where the image of v contains $\rho(v)$.

A very useful special case (let H be a matching with k edges):

k -Disjoint Paths

Input: graph G with vertices $(s_1, t_1), \dots, (s_k, t_k)$.

Find: vertex-disjoint paths P_1, \dots, P_k
where P_i connects s_i and t_i .

Algorithm for minor testing

A vertex $v \in V(G)$ is **irrelevant** if its removal does not change the answer to $H \leq G$.

Ingredients of minor testing by [Robertson and Seymour]

- 1 Solve the problem on bounded-treewidth graphs.
- 2 If treewidth is large, either find an **irrelevant** vertex or the model of a large clique minor.
- 3 If we have a large clique minor, then either we are done (if the clique minor is “close” to the roots), or a vertex of the clique minor is **irrelevant**.

By iteratively removing irrelevant vertices, eventually we arrive to a graph of bounded treewidth.

Planar k -Disjoint Paths

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Theorem [Adler et al. 2011]

The k -DISJOINT PATHS problem on planar graphs can be solved in time $2^{2^{O(k)}} \cdot n^{O(1)}$.

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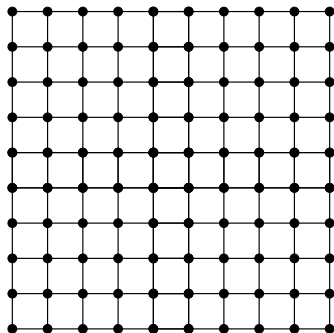
Main argument:

- either treewidth is $2^{O(k)}$ and we can use standard algorithmic techniques of bounded treewidth graphs, or
- treewidth is $2^{\Omega(k)}$ and we can find an **irrelevant vertex** whose deletion does not change the problem.

Planar k -Disjoint Paths

Theorem

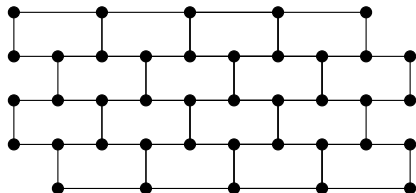
Every planar graph with treewidth at least $5k$ has a $k \times k$ grid minor.



Planar k -Disjoint Paths

Theorem

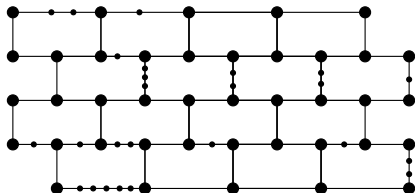
If treewidth of a planar graph is $\Omega(k)$, then it contains the subdivision of a $k \times k$ wall.



Planar k -Disjoint Paths

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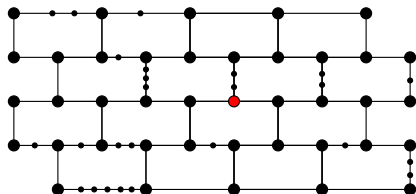
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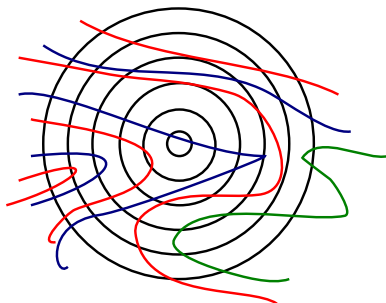
Lemma [Adler et al. 2011]

If a $2^{O(k)} \times 2^{O(k)}$ wall of a planar graph does not enclose any terminals, then the middle vertex of the wall is irrelevant to the k -disjoint paths problem.

Irrelevant vertices

Lemma [Adler et al. 2011]

If there are $2^{O(k)}$ concentric cycles in a planar graph not enclosing any terminals, then the innermost cycle is irrelevant to the k -disjoint paths problem.



Any solution can be rerouted to avoid the innermost cycle.



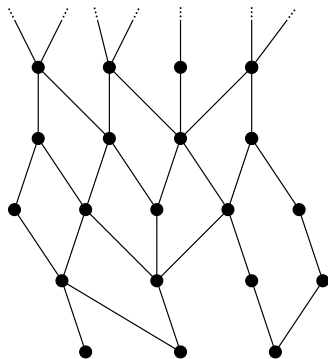
Well-quasi-ordering

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Definition

A partial order is a **well-quasi-ordering** if

- 1 There is no infinite antichain.
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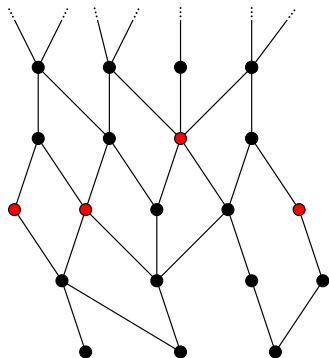


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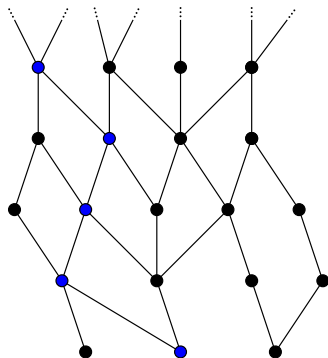


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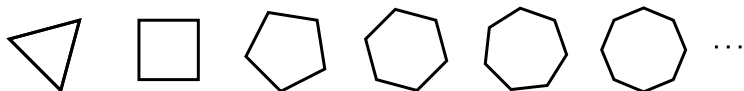
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Example: the subgraph relation \subseteq is **not** a well-quasi-ordering:



Well-quasi-ordering

Graph Minors Theorem

The minor relation \leq is a well-quasi-ordering on finite graphs.

Some equivalent reformulations:

Corollary

If \mathcal{G} is minor closed, then $\overline{\mathcal{G}}$ has a finite number of minimal elements.

Corollary

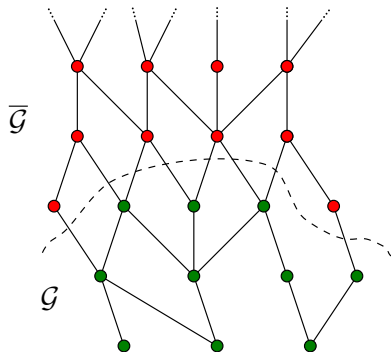
If \mathcal{G} is minor closed, then \mathcal{G} has a finite obstruction set $\mathcal{F} = \{H_1, \dots, H_k\}$, i.e.,

$$G \in \mathcal{G} \iff \forall H \in \mathcal{F}, H \not\leq G$$

Well-quasi-ordering

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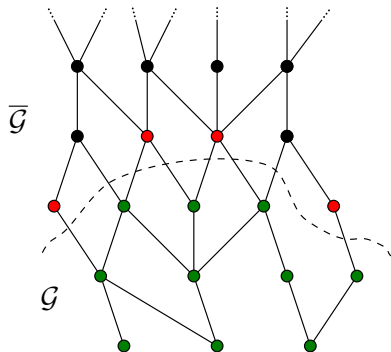
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Nonconstructive algorithms

Corollary

If \mathcal{G} is minor closed, then \mathcal{G} has a finite obstruction set $\mathcal{F} = \{H_1, \dots, H_k\}$, i.e.,

$$G \in \mathcal{G} \iff \forall H \in \mathcal{F}, H \not\leq G$$

As we have a $O(n^3)$ minor test algorithm for every $H_i \in \mathcal{H}$:

Theorem

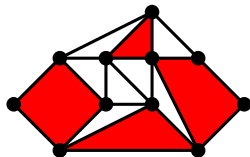
If \mathcal{G} is minor closed, then there is a $O(n^3)$ time algorithm for recognizing graphs in \mathcal{G} .

Examples:

- graphs that can be drawn on a torus (double torus etc.) form a minor-closed class: there is a $O(n^3)$ algorithm.
- graphs that have a linkless embedding in 3-space form a minor closed class: there is a $O(n^3)$ algorithm.

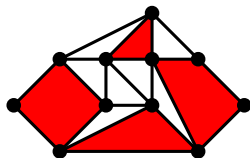
Applications

PLANAR FACE COVER: Given a graph G and an integer k , find an embedding of planar graph G such that there are k faces that cover all the vertices.



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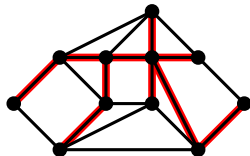


For every fixed k , the class \mathcal{G}_k of graphs of yes-instances is minor closed.

For every fixed k , there is a $O(n^3)$ time algorithm for **PLANAR FACE COVER**.

Applications

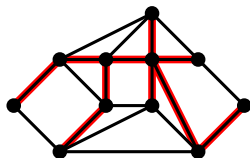
k -LEAF SPANNING TREE: Given a graph G and an integer k , find a spanning tree with **at least k** leaves.



Technical modification: Is there such a spanning tree for at least one component of G ?

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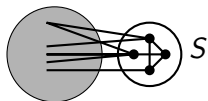


For every fixed k , k -LEAF SPANNING TREE can be solved in time $O(n^3)$.

$\mathcal{G} + k$ vertices

Definition

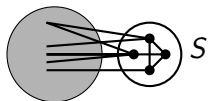
If \mathcal{G} is a graph property, then $\mathcal{G} + kv$ contains graph G if there is a set $S \subseteq V(G)$ of k vertices such that $G \setminus S \in \mathcal{G}$.



$\mathcal{G} + k$ vertices

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Observation

If \mathcal{G} is minor closed, then $\mathcal{G} + kv$ is minor closed for every fixed k .

\Rightarrow It is (nonuniform) FPT to decide if G can be transformed into a member of \mathcal{G} by deleting k vertices.

$\mathcal{G} + k$ vertices

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\Rightarrow It is (nonuniform) FPT to decide if G can be transformed into a member of \mathcal{G} by deleting k vertices.

- If $\mathcal{G} =$ forests $\Rightarrow \mathcal{G} + kv =$ graphs that can be made acyclic by the deletion of k vertices
 \Rightarrow FEEDBACK VERTEX SET is FPT.
- If $\mathcal{G} =$ planar graphs $\Rightarrow \mathcal{G} + kv =$ graphs that can be made planar by the deletion of k vertices (k -apex graphs)
 $\Rightarrow k$ -APEX GRAPH is FPT.
- If $\mathcal{G} =$ empty graphs $\Rightarrow \mathcal{G} + kv =$ graphs with vertex cover number at most k
 \Rightarrow VERTEX COVER is FPT.

Nonconstructive algorithms

- The running time is beyond horrible.
- Quick tool for obtaining very general results.
- For many concrete problems, simpler and more efficient algorithms were found.
- Nonuniform FPT: a separate algorithm for every fixed k , rather than a single $f(k) \cdot n^{O(1)}$ algorithm.



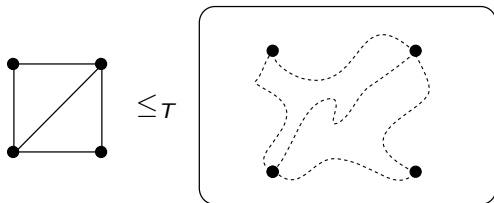
Other containment relations

Topological subgraphs

Definition

Subdivision of a graph: replacing each edge by a path of length 1 or more.

Graph H is a **topological subgraph** of G (or **topological minor** of G , or $H \leq_T G$) if a subdivision of H is a subgraph of G .

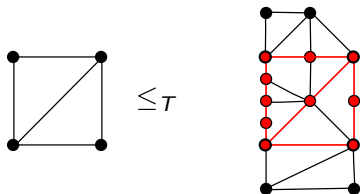


Topological subgraphs

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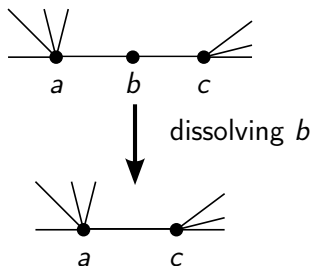
Topological subgraphs

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Equivalently, $H \leq_T G$ means that H can be obtained from G by removing vertices, removing edges, and dissolving degree two vertices.



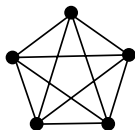
A classical result

Theorem [Kuratowski 1930]

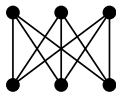
A graph G is planar if and only if $K_5 \not\leq_T G$ and $K_{3,3} \not\leq_T G$.

Theorem [Wagner 1937]

A graph G is planar if and only if $K_5 \not\leq G$ and $K_{3,3} \not\leq G$.



K_5

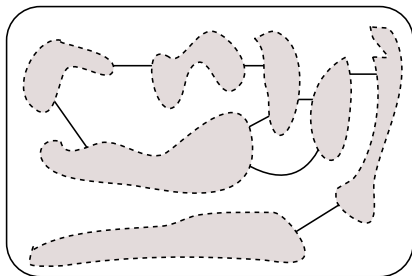


$K_{3,3}$

Minors vs. subdivisions

Simple fact

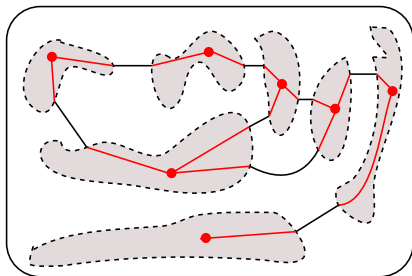
If H has max. degree ≤ 3 , then $H \leq G \iff H \leq_T G$.



Minors vs. subdivisions

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Minors vs. subdivisions

Simple fact

For every H , there is a finite set \mathcal{H} such that

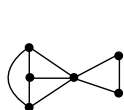
$$H \leq G \iff \exists H' \in \mathcal{H} : H' \leq_T G.$$

Minors vs. subdivisions

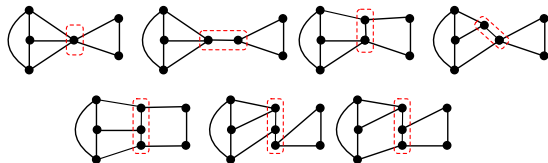
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H



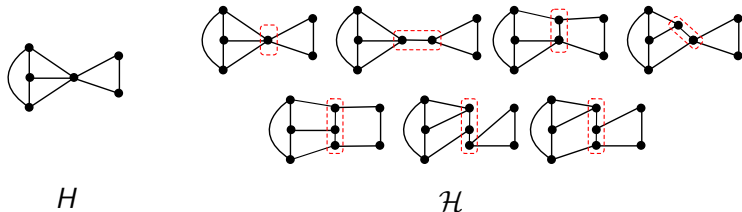
\mathcal{H}

Minors vs. subdivisions

Simple fact

For every H , there is a finite set \mathcal{H} such that

$$H \leq G \iff \exists H' \in \mathcal{H} : H' \leq_T G.$$



- Every class that can be defined by excluding minors can be defined by excluding topological subgraphs.
- But the converse is not true: excluding a $K_{1,4}$ topological subgraph means that max. degree is < 4 .

Finding topological subgraphs

Deciding $H \leq_T G$:

- Guess the image of each $v \in V(H)$ in G .
- Solve the k -DISJOINT PATHS where $k = |E(H)|$ and the paths correspond to the edges of H in G .

Corollary

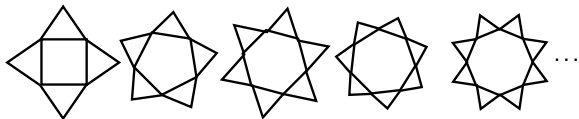
We can decide in $n^{f(H)}$ time if $H \leq_T G$.

Theorem [Grohe, Kawarabayashi, M., Wollan 2011]

We can decide in $f(H) \cdot n^3$ time if $H \leq_T G$.

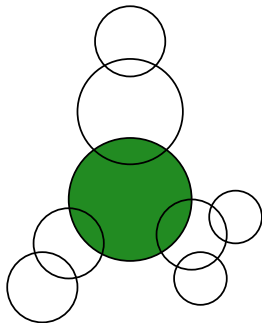
Well-quasi-ordering (lack of)

The relation \leq_T is **not** a well-quasi-ordering.



Torso

Torso of a bag: we make the intersections with the adjacent bags cliques.



Torso

Torso of a bag: we make the intersections with the adjacent bags cliques.



Structure theorems

Theorem [Grohe and M. 2012]

For every H , there is an integer $k \geq 1$ such that every H -subdivision free graph has a tree decomposition where the torso of every bag is either

- K_k -minor free or
- has degree at most k with the exception of at most k vertices (“almost bounded degree”).

Note: there is an $f(H) \cdot n^{O(1)}$ time algorithm for computing such a decomposition.

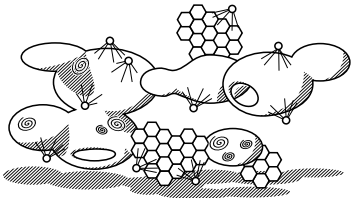
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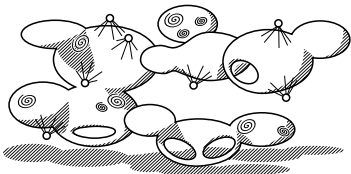
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H-Topological- Minor-Free



H-Minor-Free



Bounded Genus



Planar

[figure by Felix Reidl]

Algorithmic applications

Theorem [Grohe and M. 2012]

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- has degree at most k with the exception of at most k vertices (“almost bounded degree”).

General message:

If a problem can be solved both

- on (almost-) embeddable graphs and
- on (almost-) bounded degree graphs,

then these results can be raised to

- H -subdivision free graphs

without too much extra effort.

Partial Dominating Set

PARTIAL DOMINATING SET

Input: graph G , integer k

Find: a set S of at most k vertices whose closed neighborhood has maximum size

Theorem

PARTIAL DOMINATING SET can be solved in time $f(H, k) \cdot n^{O(1)}$ on H -subdivision free graphs.

Partial Dominating Set

Sketch:

- **PARTIAL DOMINATING SET** can be solved in linear-time on bounded-degree graphs (the closed neighborhood has bounded size).
- **PARTIAL DOMINATING SET** can be solved in linear-time on planar graphs (standard layering/treewidth arguments).
- With some extra work, we can generalize this to almost-bounded degree and almost-embeddable graphs.
- The structure theorem together with bottom-up dynamic programming gives an algorithm for H -subdivision free graphs.

Graph Isomorphism

Theorem [Luks 1982] [Babai, Luks 1983]

For every fixed d , GRAPH ISOMORPHISM can be solved in polynomial time on graphs with maximum degree d .

Theorem [Ponomarenko 1988]

For every fixed H , GRAPH ISOMORPHISM can be solved in polynomial time on H -minor free graphs.

Theorem [Grohe and M. 2012]

For every fixed H , GRAPH ISOMORPHISM can be solved in polynomial-time on H -subdivision free graphs.

Note:

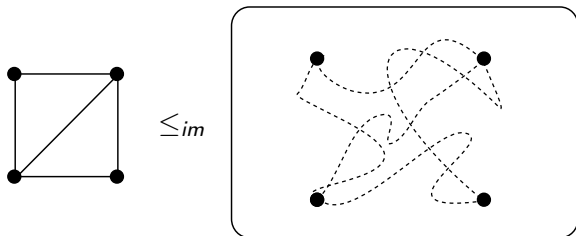
- Running time is $n^{f(H)}$, not FPT parameterized by H .
- Requires a more general “invariant acyclic tree-like decomposition.”

Immersions

Definition

Graph H has an **immersion** in G ($H \leq_{im} G$) if there is a mapping ϕ such that

- For every $v \in V(H)$, $\phi(v)$ is a distinct vertex in G .
- For every $xy \in E(H)$, $\phi(xy)$ is a path between $\phi(x)$ and $\phi(y)$, and all these paths are edge disjoint.



Note: $H \leq_T G$ implies $H \leq_{im} G$.

Immersion

Finding immersions:

Theorem [Grohe, Kawarabayashi, M., Wollan 2011]

We can decide in $f(H) \cdot n^3$ time if $H \leq_{im} G$.

Well-quasi-ordering:

Robertson and Seymour

The immersion relation \leq_{im} is a well-quasi-ordering on finite graphs.

What about a structure theorem?

Excluding immersions

As excluding K_k -immersions implies excluding K_k topological subgraphs, we get:

Theorem [Grohe and M. 2012]

For every H , there is an integer $k \geq 1$ such that every H -immersion free graph has a tree decomposition where the torso of every bag is either

- k -almost embeddable in a surface of genus at most k or
- has degree at most k with the exception of at most k vertices (“almost bounded degree”).

Excluding immersions

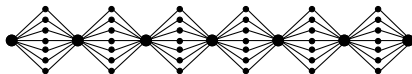
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However, embeddability does not seem to be relevant for immersions: the following graph has large clique immersions.



Excluding immersions

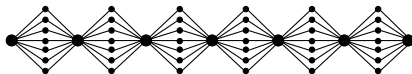
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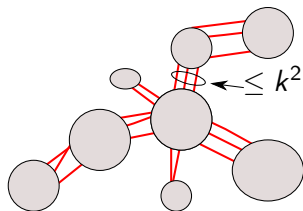
Can we omit the first case?

Excluding immersions

Theorem [Wollan]

If K_k has no immersion in G , then G has a “tree-cut decomposition” of adhesion at most k^2 such that each “torso” has at most k vertices of degree at least k^2 .

Tree cut decomposition: a partition of the vertex set in tree-like way.

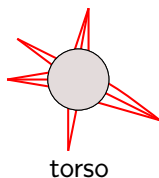


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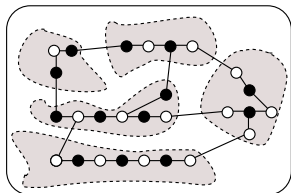


Odd minors

Definition

Graph H is an **odd minor** of G ($H \leq_{\text{odd}} G$) if G has a 2-coloring and there is a mapping ϕ that maps each vertex of H to a tree of G such that

- $\phi(u)$ and $\phi(v)$ are disjoint if $u \neq v$,
- every edge of $\phi(u)$ is bichromatic,
- if $uv \in E(H)$, then there is a monochromatic edge between $\phi(u)$ and $\phi(v)$.



Example: K_3 is an odd minor of G if and only if G is not bipartite.

Odd minors

Finding odd minors:

Theorem [Kawarabayashi, Reed, Wollan 2011]

There is an $f(H) \cdot n^{O(1)}$ time algorithm for finding an odd H -minor.

Structure theorem:

Theorem [Demaine, Hajiaghayi, Kawarabayashi 2010]

For every H , there is an integer $k \geq 1$ such that every odd H -minor free graph has a tree decomposition where the torso of every bag is

- k -almost embeddable in a surface of genus at most k or
- bipartite after deleting at most k vertices (“almost bipartite”)

Consequence:

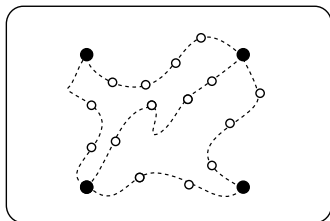
Theorem [Demaine, Hajiaghayi, Kawarabayashi 2010]

For every fixed H , there is a polynomial-time 2-approximation algorithm for chromatic number on odd H -minor free graphs.

Odd subdivisions

Definition

Odd subdivision of a graph: replacing each edge by a path of odd length (1 or more).



If G contains an odd H -subdivision, then $H \leq_T G$ and $H \leq_{\text{odd}} G$.

Odd subdivisions

A structure theorem for excluding odd H -subdivision should be stronger than

- the structure theorem for excluded subdivisions (k -almost embeddable, almost bounded degree) and
- the structure theorem for excluded odd minors (k -almost embeddable, almost bipartite).

Odd subdivisions

A structure theorem for excluding odd H -subdivision should be stronger than

- the structure theorem for excluded subdivisions (k -almost embeddable, almost bounded degree) and
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Theorem [Kawarabayashi 2013]

For every H , there is an integer $k \geq 1$ such that every odd H -subdivision free graph has a tree decomposition where the torso of every bag is either

- k -almost embeddable in a surface of genus at most k ,
- has degree at most k with the exception of at most k vertices (“almost bounded degree”), or
- bipartite after deleting at most k vertices (“almost bipartite”).

What did we learn, Palmer?

- Algorithms for bounded treewidth graphs: tedious, but elementary.
(dynamic programming, Courcelle's Theorem)
- Applications of bounded treewidth algorithms.
(the shifting technique, bidimensionality, grid theorems)
- Generalization to bounded genus graphs.
- The structure theorem.
- Minor testing and well-quasi-ordering.

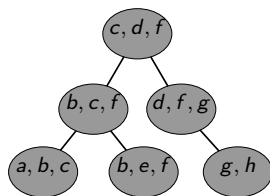
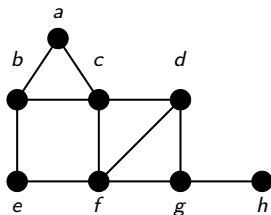
Treewidth

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

- 1 If u and v are neighbors, then there is a bag containing both of them.
- 2 For every v , the bags containing v form a connected subtree.

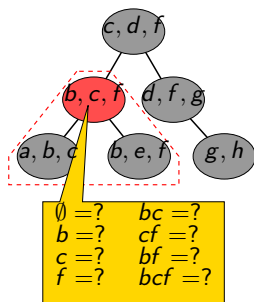
Width of the decomposition: largest bag size -1 .

treewidth: width of the best decomposition.

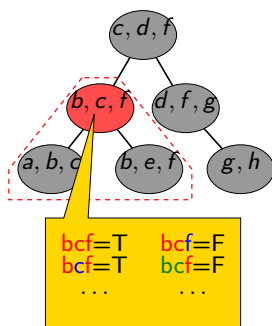


Dynamic programming

WEIGHTED MAX INDEPENDENT SET



3-COLORING



Courcelle's Theorem

Courcelle's Theorem

If a graph property can be expressed in EMSO, then for every fixed $w \geq 1$, there is a linear-time algorithm for testing this property on graphs having treewidth at most w .

3-COLORING

$$\exists C_1, C_2, C_3 \subseteq V (\forall v \in V (v \in C_1 \vee v \in C_2 \vee v \in C_3)) \wedge (\forall u, v \in V \text{adj}(u, v) \rightarrow (\neg(u \in C_1 \wedge v \in C_1) \wedge \neg(u \in C_2 \wedge v \in C_2) \wedge \neg(u \in C_3 \wedge v \in C_3)))$$

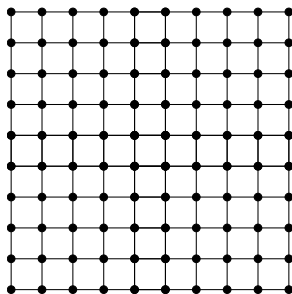
Excluded Grid Theorem

Excluded Grid Theorem [Diestel et al. 1999]

If the treewidth of G is at least $k^{4k^2(k+2)}$, then G has a $k \times k$ grid minor.

Theorem [Robertson, Seymour, Thomas 1994]

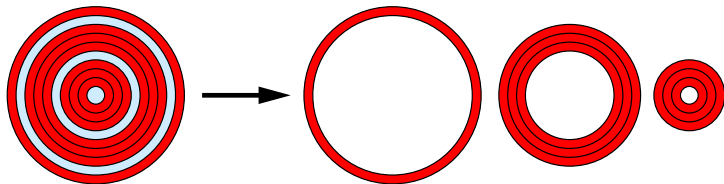
Every **planar graph** with treewidth at least $5k$ has a $k \times k$ grid minor.



Shifting strategy

Theorem

There is a $2^{O(1/\epsilon)} \cdot n$ time PTAS for **INDEPENDENT SET** for planar graphs.



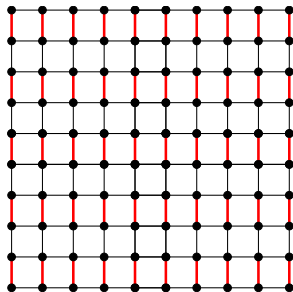
Bidimensionality

Bidimensionality [Demaine, Fomin, Hajiaghayi, Thilikos 2005]

For many natural invariants, we can do this in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$ on planar graphs.

Win/win situation:

- If treewidth is $\Omega(\sqrt{k}) \Rightarrow$ There is a $\Omega(\sqrt{k}) \times \Omega(\sqrt{k})$ grid minor \Rightarrow value of the invariant is at least k .
- If treewidth is $O(\sqrt{k})$, then we can solve the problem in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$.



Surfaces



genus 1



genus 3

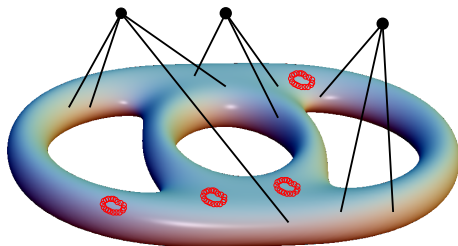


unbounded genus

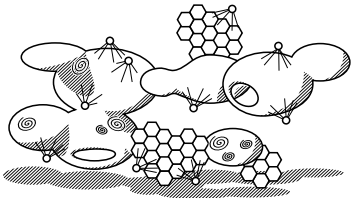
Graph Structure Theorem

Theorem [Robertson-Seymour]

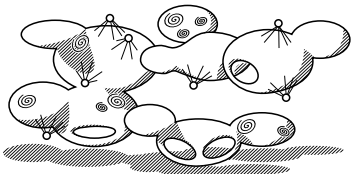
For every graph H , there is an integer k and a surface Σ such that every H -minor-free graph has a tree decomposition where every torso is k -almost embeddable in Σ .



k -almost embeddable



H-Topological- Minor-Free



H-Minor-Free



Bounded Genus



Planar

[figure by Felix Reidl]

Graph Minors Theorem

Well-quasi-ordering:

Theorem [Robertson and Seymour]

Every class \mathcal{G} closed under taking minors has a finite set \mathcal{F} of minimal obstructions.

Minor testing:

Theorem [Robertson and Seymour]

For every fixed graph H , there is an $O(n^3)$ time algorithm for testing whether H is a minor of the given graph G .

Corollary: For every minor closed property \mathcal{G} , there is an $O(n^3)$ time algorithm for testing whether a given graph G is in \mathcal{G} .