Minicourse on parameterized algorithms and complexity

Part 2: Iterative compression

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What iterative compression is?

Iterative compression — main idea

Recursive approach exploiting instance structure exposed by a bit oversized solution.
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Iterative compression — main idea

Recursive approach exploiting instance structure exposed by a bit oversized solution.

Solution compression:

1. First, apply some simple trick so that you can assume that a slightly too large solution is available.

2. Then exploit the structure it imposes on the input graph to construct an optimal solution.
Vertex Cover

**Input:** undirected $G$, integer $k$

**Question:** is there a subset $X \subseteq V(G)$ of size at most $k$ such that for each $uv \in E(G)$ we have $\{u, v\} \cap X \neq \emptyset$. 
Vertex Cover

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![Graph example](image-url)
We exemplify the iterative compression technique by showing $2^k n^{O(1)}$ algorithm for Vertex Cover.
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**Vertex Cover Compression**

**Input:** undirected $G$, integer $k$, vertex cover $Z \subseteq V(G)$ of size at most $2k$

**Question:** is there a vertex cover of size at most $k$?
We exemplify the iterative compression technique by showing $2^k n^{O(1)}$ algorithm for Vertex Cover.

**Vertex Cover Compression**

**Input:** undirected $G$, integer $k$, vertex cover $Z \subseteq V(G)$ of size at most $2k$

**Question:** is there a vertex cover of size at most $k$?

- Where do we get $Z$ from?
- How do we use $Z$?
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Use polynomial time 2-approximation:
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Use polynomial time 2-approximation:

- Find any inclusionwise maximal matching $M$. 

![Diagram of a graph with red edges representing the maximal matching]
Additional input: oversized solution

Where do we get $Z$ from?

Use polynomial time 2-approximation:

- Find any inclusionwise maximal matching $M$.
- If $|M| > k$, then no VC of size $\leq k$ exists.
Where do we get $Z$ from?

Use polynomial time $2$-approximation:

- Find any inclusionwise maximal matching $M$.
- If $|M| > k$, then no VC of size $\leq k$ exists.
- Otherwise, set $Z = V(M)$, we have $|Z| \leq 2k$. 

![Graph diagram]

Additional input: oversized solution
How do we use $Z$?

Guess $X \setminus Z = X \setminus Z$ (by branching into 2 cases).

Check if $Z \setminus X \setminus Z$ is independent and $|X \setminus Z \cup N(Z \setminus X \setminus Z)| \leq k$. 

$V \setminus Z$
How do we use $Z$?

- Guess $X \cap Z = X_Z$ (by branching into $2^{|Z|} \leq 4^k$ cases).
Additional input: oversized solution

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- Guess $X \cap Z = X_Z$ (by branching into $2^{|Z|} \leq 4^k$ cases).
- Check if $Z \setminus X_Z$ is independent and $|X_Z \cup N(Z \setminus X_Z)| \leq k$. 

$$N(Z \setminus X_Z) \cap (V \setminus Z)$$
Additional input: oversized solution

How do we use $Z$?

- We have obtained $2|Z| n^{O(1)} \leq 4^k n^{O(1)}$ time algorithm.
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- Can we improve the dependency on $k$ to $2^k$?
Additional input: oversized solution

How do we use $Z$?

- We have obtained $2^{|Z|} n^{O(1)} \leq 4^k n^{O(1)}$ time algorithm.
- Can we improve the dependency on $k$ to $2^k$?
- Notice that it would be enough to have $|Z| \leq k + 1$, but so far we only have $|Z| \leq 2k$. 
Vertex Cover

Vertex Cover Compression

**Input:** undirected $G$, integer $k$, vertex cover $Z \subseteq V(G)$ of size at most $k + 1$

**Question:** is there a vertex cover of size at most $k$?
Vertex Cover Compression

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**Question:** is there a vertex cover of size at most $k$?

Idea: get $Z$ from recursion!
How to get $Z$ of size at most $k + 1$?

- Assume that an instance $I = (G, k)$ without $Z$ is given.
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- Assume that an instance $I = (G, k)$ without $Z$ is given.
- Pick any $v \in V(G)$ and solve $I' = (G \setminus \{v\}, k)$ recursively.
How to get $Z$ of size at most $k + 1$?

- Assume that an instance $I = (G, k)$ without $Z$ is given.
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$(G, k, Z)$ is VC Compression instance to solve.
Bootstrapping

How to get $Z$ of size at most $k + 1$?

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- $(G, k, Z)$ is VC Compression instance to solve.

Lemma

$f(k)n^c$ time algorithm for VC Compression implies $f(k)n^{c+1}$ time algorithm for VC.
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\( f(k)n^c \) time algorithm for VC Compression implies \( f(k)n^{c+1} \) time algorithm for VC.

Reduction: Vertex Cover \( \rightarrow \) Vertex Cover Compression.
Vertex Cover - summary

**Lemma**

$f(k)n^c$ time algorithm for VC Compression implies $f(k)n^{c+1}$ time algorithm for VC.

Reduction: Vertex Cover $\rightarrow$ Vertex Cover Compression.

Vertex Cover Compression can be solved in time $2^{|Z|}n^{O(1)}$, which leads to $2^k n^{O(1)}$ algorithm for VC.
Outline

1. Iterative compression - introduction.
2. Learning by example - vertex cover.
3. Learning by example - FVS in tournament.
4. Generic steps of the method.
5. $5^k n^{O(1)}$ algorithm for FVS.
6. $3^k n^{O(1)}$ algorithm for OCT - sketch.
FVS in tournaments

Feedback Vertex Set (FVS) in Tournaments

**Input**: a tournament (oriented clique) $T$, integer $k$

**Question**: is there a subset $X \subseteq V(T)$ of size at most $k$, such that $T \setminus X$ is acyclic
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This lemma implies a simple $3^k n^{O(1)}$ branching algorithm.
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- This lemma implies a simple $3^k n^{O(1)}$ branching algorithm.
- By using iterative compression we will see how to improve the running time to $2^k n^{O(1)}$. 
FVS in tournaments

Start with the recursive trick, reducing the problem to its compression version.
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Feedback Vertex Set (FVS) in Tournaments Compression

**Input**: a tournament (oriented clique) $T$, integer $k$
- a FVS $Z \subseteq V(T)$ of size at most $k + 1$

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**Lemma**

$f(k)n^c$ time algorithm for FVST Compression implies $f(k)n^{c+1}$ time algorithm for FVST.
FVS in tournaments

Pf: this time we use induction (loop) - alternative to recursion.

Let \( V(T) = \{v_1, \ldots, v_n\} \).
FVS in tournaments

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- Let $V(T) = \{v_1, \ldots, v_n\}$.
- We want to solve $FVST(T[V_i], k)$ for $i = 1, \ldots, n$, where $V_i = \{v_1, \ldots, v_i\}$. 
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We want to solve $FVST(T[V_i], k)$ for $i = 1, \ldots, n$, where $V_i = \{v_1, \ldots, v_i\}$.

Set $X_1 = \emptyset$, which is a solution for $FVST(T[v_1], k)$. 
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- Let $V(T) = \{v_1, \ldots, v_n\}$.
- We want to solve $FVST(T[V_i], k)$ for $i = 1, \ldots, n$,
  where $V_i = \{v_1, \ldots, v_i\}$.
- Set $X_1 = \emptyset$, which is a solution for $FVST(T[v_1], k)$.
- For $2 \leq i \leq n$ do
  - $Z_i = X_{i-1} \cup \{v_i\}$,
  - let $X_i$ be a solution to $FVST\ Compression(T[V_i], k, Z_i)$.
  - if no solution found for $T[V_i]$, then return NO.
Feedback Vertex Set (FVS) in Tournaments Compression

**Input:** a tournament (oriented clique) $T$, integer $k$

a FVS $Z \subseteq V(T)$ of size at most $k + 1$

**Question:** is there a subset $X \subseteq V(T)$ of size at most $k$, such that $T \setminus X$ is acyclic

By guessing a partition $Z = X_Z \cup W$, we get to the disjoint version.
FVS in tournaments

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Disjoint FVS in Tournaments Compression

**Input:** a tournament (oriented clique) $T$, integer $k$

a FVS $W \subseteq V(T)$ of size at most $k + 1$

**Question:** is there a subset $X \subseteq V(T)$ of size at most $k$,
disjoint with $W$, such that $T \setminus X$ is acyclic
### Disjoint FVS in Tournaments Compression

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### Lemma

Poly time algorithm for Disjoint FVST Compression implies $2^k n^{O(1)}$ time algorithm for FVST Compression.
Observation

For an acyclic tournament, there is a single topological ordering.
Disjoint FVS in tournaments

Simple reduction rules:

Reduction 1
If $T[W]$ is not acyclic, then answer NO.

Let $A = V(T) \setminus W$ (removable set).

Reduction 2
If for $v \in A$ the graph $T[W \cup \{v\}]$ contains a cycle, then remove $v$ and reduce $k$ by one.
Simple reduction rules:

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Disjoint FVS in tournaments

\[ A = V(T) \setminus W \]
Disjoint FVS in tournaments

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Disjoint FVS in tournaments

Consequently Disjoint FVST Compression may be reduced to finding longest nondecreasing subsequence.
General framework

Iterative compression schema:
- By using induction we can assume that a solution $Z \subseteq V(G)$, $|Z| \leq k + 1$ is given as part of input.
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- Branch into $2^{|Z|}$ cases, guessing what part of $Z$ should be in a solution.
General framework

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- By using induction we can assume that a solution $Z \subseteq V(G)$, $|Z| \leq k + 1$ is given as part of input.
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- Solve a disjoint version of the problem, where given a solution $W \subseteq V(G)$ we look for $X \subseteq V(G) \setminus W$ of size at most $k$. 
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- Solve a disjoint version of the problem, where given a solution $W \subseteq V(G)$ we look for $X \subseteq V(G) \setminus W$ of size at most $k$.
- $c^k n^{O(1)}$ time algorithm for the disjoint version implies $(2c)^k n^{O(1)}$ time algorithm for the general problem.
General framework

Lemma

\[ c^k n^{O(1)} \text{ time algorithm for the disjoint version implies } (c + 1)^k n^{O(1)} \text{ time algorithm for the general problem.} \]
General framework

**Lemma**

$c^k n^{O(1)}$ time algorithm for the disjoint version implies $(c + 1)^k n^{O(1)}$ time algorithm for the general problem.

$$
\sum_{X \subseteq Z} c^{k-|X|} = \sum_{i=0}^{k+1} \binom{k + 1}{i} c^{k-i} 1^i = (c + 1)^{k+1}/c
$$
Remarks:

- To make induction work, we need to find a solution (answering YES/NO is not enough).
General framework

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  - To make induction work, we need to find a solution (answering YES/NO is not enough).
  - By default iterative compression adds $n$ factor to the running time.

Ex: show that for VC and FVST this factor can be reduced to $O(k)$ (hint: use $O(1)$-approximation).

Some natural problems are not vertex deletion closed. Ex: reduce Connected Vertex Cover (CVC) to CVC-Compression.
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- Ex: reduce Connected Vertex Cover (CVC) to CVC-Compression.
Feedback Vertex Set (FVS)

**Input:** undirected $G$, integer $k$

**Question:** is there a subset $X \subseteq V(G)$ of size at most $k$, such that $G \setminus X$ is a forest
FVS is vertex deletion closed, so we can apply iterative compression schema and solving the following problem in time $c^k n^{O(1)}$ leads to $(c + 1)^k n^{O(1)}$ time algorithm for FVS.
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Disjoint FVS Compression

**Input:** undirected $G$, integer $k$

- a FVS $W \subseteq V(G)$ of size at most $k + 1$

**Question:** is there a subset $X \subseteq V(G)$ of size at most $k$, disjoint with $W$, such that $G \setminus X$ is a forest
Reduction 0

If $G[W]$ contains a cycle, return NO.
FVS - reduction rules

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If $G[W]$ contains a cycle, return NO.

We want $v$ to have $\geq 2$ incident edges going to $W$. 
Reduction 1

Remove all degree at most 1 vertices from $G$. 
Reduction 2

If there is $v \in A$ with $\deg(v) = 2$ and at least one neighbor in $A$, then add an edge between neighbours of $v$ (even if there was one) and remove $v$. 

![Diagram showing the reduction rule](image-url)
Any leaf $v$ in $A$ has now at least two edges to $W$. 
FVS - one more reduction rule

If for $v \in A = V(G) \setminus W$ the graph $G[W \cup \{v\}]$ contains a cycle, then remove $v$ and decrease $k$ by one.
FVS - one more reduction rule

Reduction 3
If for \( v \in A = V(G) \setminus W \) the graph \( G[W \cup \{v\}] \) contains a cycle, then remove \( v \) and decrease \( k \) by one.
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FVS branching
FVS branching

Formally, we branch into instances:

- \((G \setminus \{v\}, k - 1, W)\),
- \((G, k, W \cup \{v\})\).
FVS branching

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Observation

A potential \(\pi(I) = k + \#cc(G[W])\) decreases in each branch.
Formally, we branch into instances:

- \((G \setminus \{v\}, k - 1, W)\),
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**Observation**

A potential \(\pi(I) = k + \#cc(G[W])\) decreases in each branch.

**Lemma**

Disjoint FVS Compression can be solved in time \(4^k n^{O(1)}\), consequently there is \(5^k n^{O(1)}\) time algorithm for FVS.
Odd Cycle Transversal (OCT)

**Input:** undirected $G$, integer $k$

**Question:** is there a subset $X \subseteq V(G)$ of size at most $k$, such that $G \setminus X$ is bipartite
The heart of the solution for OCT by iterative compression is the following problem, which can be solved in polynomial time!

**Annotated Bipartite Coloring**

**Input:** bipartite $G = (V_1, V_2, E)$, integer $k$, a partial coloring $f_0 : V(G) \rightarrow \{1, 2, ?\}$

**Question:** is there a subset $X \subseteq V(G)$ of size at most $k$, and a proper coloring $f$ of $G \setminus X$ consistent with $f_0$. 
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- $V_1$
- $V_2$
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each blue vertex is either removed or recolored wrt $V_1 \cup V_2$, 

\[ V_1 \]

\[ V_2 \]
each blue vertex is either removed or recolored wrt $V_1 \cup V_2$, each green vertex is removed or maintains color wrt $V_1 \cup V_2$,.
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each green vertex is removed or maintains color wrt $V_1 \cup V_2$, 
for each $e \in E(G \setminus X)$ either both vertices are recolored, or none,
each blue vertex is either removed or recolored wrt $V_1 \cup V_2$,  
each green vertex is removed or maintains color wrt $V_1 \cup V_2$,  
for each $e \in E(G \setminus X)$ either both vertices are recolored, or none,  
algorithm: find min cut between green and blue!
Summary

**Iterative compression**

Recursive approach exploiting instance structure exposed by a bit oversized solution.

We have seen it applied to:

- Vertex Cover,
- FVS in Tournaments,
- FVS,
- OCT (sketch).