Interval Deletion is Fixed-Parameter Tractable

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We study the minimum *interval deletion* problem, which asks for the removal of a set of at most k vertices to make a graph of n vertices into an interval graph. We present a parameterized algorithm of runtime $10^k \cdot n^{O(1)}$ for this problem, that is, we show the problem is fixed-parameter tractable.

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1. INTRODUCTION

A graph is an interval graph if its vertices can be assigned to intervals of the real line such that there is an edge between two vertices if and only if their corresponding intervals intersect. Interval graphs are the natural models for DNA chains in biology and many other applications, among which the most cited ones include jobs scheduling in industrial engineering [Bar-Noy et al. 2001] and seriation in archeology [Kendall 1969]. Motivated by pure contemplation of combinatorics and practical problems of biology respectively, Hajós [1957] and Benzer [1959] independently initiated the study of interval graphs.

Interval graphs are a proper subset of chordal graphs. After more than half century of intensive investigation, the properties and the recognition of interval and chordal graphs are well understood [Booth and Lueker 1976]. More generally, many NP-hard problems (coloring, maximum independent set, etc.) are known to be polynomial-time solvable when restricted to interval or chordal graphs. Therefore, one would like to generalize these results to graphs that do not belong to these classes, but close to them in the sense that they have only a few "erroneous"/"missing" edges or vertices. As a first step in understanding such generalizations, one would like to know how far the given graph is from the class and to find the erroneous/missing elements. This leads us naturally to the area of graph modification problems, where given a graph G, the task is to apply a minimum number of operations on G to make it a member of some prescribed graph class $\mathcal{F}$. Depending on the operations we allow, we can consider, e.g., completion (edge-addition), edge-deletion, and vertex-deletion

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versions of these problems. Let us point out that, when $\mathcal{F}$ is hereditary, the vertex deletion version can be considered as the most robust variant, which in some sense encompasses both edge addition and edge deletion: if G can be made a member of $\mathcal{F}$ by k_1 edge additions and k_2 edge deletions, then it can be also made a member of $\mathcal{F}$ by deleting at most $k_1 + k_2$ vertices (e.g., by deleting one endpoint of each added/deleted edge).

Unfortunately, most of these graph modification problems are computationally hard: for example, a classical result of Lewis and Yannakakis [1980] shows that the vertex deletion problem is NP-hard for *every* nontrivial and hereditary class $\mathcal{F}$, and according to Lund and Yannakakis [1993], they are also MAX SNP-hard. Therefore, early work of Kaplan et al. [1999] and Cai [1996] focused on the fixed-parameter tractability of graph modification problems. Recall that a problem, parameterized by k, is *fixed-parameter tractable (FPT)* if there is an algorithm with runtime $f(k)\cdot n^{O(1)}$, where f is a computable function depending only on k [Downey and Fellows 2013]. In the special case when the desired graph class $\mathcal{F}$ can be characterized by a finite number of forbidden (induced) subgraphs, then fixed-parameter tractability of such a problem follows from a basic bounded search tree algorithm [Cai 1996]. However, many important graph classes, such as forests, bipartite graphs, and chordal graphs have minimal obstructions of arbitrarily large size (cycles, odd cycles, and holes, respectively). It is much more challenging to obtain fixed-parameter tractability results for such classes, see results on, e.g., bipartite graphs [Reed et al. 2004; Kawarabayashi and Reed 2010], planar graphs [Marx and Schlotter 2012; Kawarabayashi 2009], acyclic graphs [Cao et al. 2010; Chen et al. 2008], and minor-closed classes [Adler et al. 2008; Fomin et al. 2012].

For interval graphs, the fixed-parameter tractability of the completion problem was raised as an open question by Kaplan et al. [1999] in 1994, to which a positive answer with a $k^{2k}\cdot n^{O(1)}$-time algorithm was given by Villanger et al. [2009] in 2007. In this paper, we answer the complementary question on vertex deletion:

THEOREM 1.1 (**Main result**). *There is a $10^k \cdot n^{O(1)}$-time algorithm for deciding whether or not there is a set of at most k vertices whose deletion makes an n-vertex graph G an interval graph.*

Related work. Let us put our result into context. Interval graphs form a subclass of chordal graphs, which are graphs containing no induced cycle of length greater than 3 (also called *holes*). In other words, the minimal obstruction for being a chordal graph might be holes of arbitrary length, hence infinitely many of them. Even so, CHORDAL COMPLETION (to make a graph chordal by the addition of at most k edges) can still be solved by a bounded search tree algorithm by observing that a large hole immediately implies a negative answer to the problem [Kaplan et al. 1999; Cai 1996]. No such simple argument works for CHORDAL DELETION (to make the graph chordal by removing at most k edges/vertices) and its fixed-parameter tractability was procured by a completely different and much more complicated approach [Marx 2010].

It is known that a graph is an interval graph if and only if it is chordal and does not contain a structure called "asteroidal triple" (AT for short), i.e., three vertices such that each pair of them is connected by a path avoiding neighbors of the third one [Lekkerkerker and Boland 1962]. Therefore, in the graph modification problems related to interval graphs, one has to destroy not only all holes, but all ATs as well. The algorithm of Villanger et al. [2009] for the INTERVAL COMPLETION problem first destroys all holes by the same bounded search tree technique as in CHORDAL COMPLETION. This step is followed by a delicate analysis of the ATs and a complicated branching step to break them in the resulting chordal graph.

A subclass of interval graphs that received attention is the class of unit interval graphs: graphs that can be represented by intervals of unit length. Interestingly, this class coincides with proper interval graphs, which are those graphs that have a representation with no interval containing another one. It is known that unit interval graphs can be characterized as not having holes and three other specific forbidden subgraphs, thus graph modification problems related to unit interval graphs [Kaplan et al. 1999; van 't Hof and Villanger 2013] are very different from those related to interval graphs, where the minimal obstructions include an infinite family of ATs.

Our techniques. Even though both CHORDAL DELETION and INTERVAL COMPLETION seem related to INTERVAL DELETION, our algorithm is completely different from the published algorithms for these two problems. The algorithm of Marx [2010] for CHORDAL DELETION is based on iterative compression, identifying irrelevant vertices in large cliques, and the use of Courcelle's Theorem on a bounded treewidth graph; none of these techniques appears in the present paper.

Villanger et al. [2009] used a simple bounded search tree algorithm to try every minimal way of completing all the holes; therefore, one can assume that the input graph is chordal. ATs in a chordal graph are known to have the property of being *shallow*, and in a minimal witness of an AT, every vertex of the triple is *simplicial*. This means that the algorithm of [Villanger et al. 2009] can focus on completing such ATs. On the other hand, there is no similar upper bound known on the number of minimal ways of breaking all holes by removing vertices, and it is unlikely to exist. Therefore, in a sense, INTERVAL DELETION is inherently harder than INTERVAL COMPLETION: in the former problem, we have to deal with two types of forbidden structures, holes and shallow ATs, while in the second problem, only shallow ATs concern us. Indeed, we spend significant effort in the present paper to make the graph chordal; the main part of the proof is understanding how holes interact and what the minimal ways of breaking them are.

The main technical idea to handle holes is developing a reduction rule based on the modular decomposition of the graph and analyzing the structural properties of reduced graphs. It turns out that the holes remaining in a reduced graph interact in a very special way (each hole is fully contained in the closed neighborhood of any other hole). This property allows us to prove that the number of minimal ways of breaking the holes is polynomially bounded, and thus a simple branching step can reduce the problem to the case when the graph is chordal. As another consequence of our reduction rule, we can prove that this chordal graph already has a structure close to interval graphs (it has a clique tree that is a caterpillar). We can show that in such a chordal graph, ATs interact in a well-behaved way and we can find a set of 10 vertices such that there always exists a minimum solution that contains at least one of these 10 vertices. Therefore, we can complete our algorithm by branching on the deletion of one of these vertices.

Motivation. The motivation for the graph modification problem studied in this paper is twofold: theoretical and coming from applications. Many classical graph-theoretic problems can be formulated as graph deletion to special graph classes. For instance, VERTEX COVER, FEEDBACK VERTEX SET, CLUSTER VERTEX DELETION, and ODD CYCLE TRANSVERSAL can be viewed as vertex deletion problems where the class $\mathcal{F}$ is the class of all empty graphs, forests, cluster graphs (i.e., disjoint union of cliques), and bipartite graphs, respectively. Thus, the study of graph modification problems related to important graph classes can be seen as a natural extension of the study of classical combinatorial problems. In light of the importance of interval graphs, it is not surprising that there are natural combinatorial problems that can be formulated as, or computationally reduced to INTERVAL DELETION, and then our algorithm for INTERVAL DELETION can be applied. For instance, Narayanaswamy and Subashini [2013] recently used Theorem 1.1 as a subroutine to solve the maximum CONSECUTIVE ONES SUB-MATRIX problem and the minimum CONVEX BIPARTITE DELETION problem.

As a historical coincidence, interval graph modification problems are motivated not only from the aforementioned theoretical studies, but because they have wide applications. One central problem in molecular biology is to reconstruct the relative positions of clones along the target DNA based on their pairwise overlap information obtained via experimental methods. These data are naturally formulated as a graph, where each clone is a vertex, and two clones are adjacent iff they overlap. The graph should be an interval graph provided the relations are *perfect*, and the problem is then equivalent to the construction of its interval model, which can be done in linear time. However, real data are always inconsistent and contaminated by a few but crucial errors, which have to be detected and fixed. In particular, on the detection of false-positive errors that correspond to false edges, Goldberg et al. [1995] proposed the INTERVAL EDGE DELETION problem (to make the graph an interval graph by the deletion of at most k edges) and showed its NP-hardness. This problem is equivalent to the maximum *spanning* interval subgraph, and is not known to be FPT or not. Moreover, false-negative errors are also possible, which significantly complicates the situation.

In this regard, we turn to the clones (vertices) involved in erroneous relations (edges) instead of the relations themselves, and try to identify them based on a similar assumption. More specifically, we study the INTERVAL (VERTEX) DELETION problem, which is equivalent to finding the maximum *induced* interval subgraph. Conceptually, this formulation is capable of dealing with both false-negatives and false-positives. Computationally, the number of clones involved in mis-observed relations is never larger, and believed to be significantly smaller, than the number of erroneous relations. It might thus provide better assistance to biologists by revealing more meaningful information in less time, as proclaimed by Karp [1993]:

> *Thus, optimization methods should be viewed not as vehicles for solving a problem, but for proposing a plausible hypothesis to be confirmed or disconfirmed by further experiments. The search for the correct solution of a reconstruction problem must inevitably be an iterative process involving a close interaction between experimentation and computation.*

In a seriation problem of archeology, overlap information of a collection of artifacts is given, and we are asked to put them in chronological order. Again we cannot expect the data to be consistent and have to deal with errors first. In particular, the famous *Berge mystery story* [Golumbic 2004] is essentially a seriation problem with false overlap information given by a *cheater*, and can be viewed as INTERVAL DELETION with $k = 1$.

2. OUTLINE

The purpose of this section is to describe the main steps of our algorithm at a high level. We say that a set $Q \subset V(G)$ is an *interval deletion set* to a graph G if $G - Q$ is an interval graph. An interval deletion set Q is *minimum* if there is no interval deletion set strictly smaller than $|Q|$, and it is *minimal* if no proper subset $Q' \subset Q$ is an interval deletion set. A set X of vertices is called a *minimal forbidden set* if X does not induce an interval graph but every proper subset $X' \subset X$ does; the subgraph $G[X]$ is called a *minimal forbidden induced subgraph*. Clearly, set Q is an interval deletion set if and only if it intersects every minimal forbidden set. Our goal is to find an interval deletion set of size at most k. For technical reasons, it will be convenient to define the problem as follows:

> INTERVAL DELETION: Given a graph G and an integer parameter k, return
> — if an interval deletion set of size $\leq k$ exists, a *minimum* interval deletion set $Q \subset V(G)$;
> — if no interval deletion set of size $\leq k$ exists, "NO."

PHASE 1: Preprocessing. The first phase of the algorithm applies two reduction rules exhaustively. They either simplify the instance or branch into a constant number of instances with strictly smaller parameter value. The first reduction rule is straightforward: we destroy every forbidden set of size at most 10.

REDUCTION 1. **[Small forbidden sets]** *Given an instance* (G, k) *and a minimal forbidden set* X *of no more than* 10 *vertices, we branch into* $|X|$ *instances,* $(G - v, k - 1)$ *for each* $v \in X$.

A graph on which Reduction 1 cannot be applied is called *prereduced*.

The second reduction rule is less obvious and more involved. Recall that a subset M of vertices forms a *module* if every vertex in M has the same neighbors outside M [Gallai 1967]. A module M of G is *nontrivial* if $1 < |M| < |V(G)|$. We observe (see Section 4.2) that a minimal forbidden set X of at least 5 vertices is either fully contained in a module M or contains at most one vertex of M. Moreover, if $X \cap M = \{x\}$, then replacing x by any other vertex $x' \in M \setminus \{x\}$ in X results in another minimal forbidden set. This permits us to branch on modules, as described in the following reduction rule.

REDUCTION 2. **[Main]** *Let* $I = (G, k)$ *be an instance where the graph* G *is prereduced, and a nontrivial module* M *that does not induce a clique.*

(1) *If every minimal forbidden set is contained in* M*, then return the instance* $(G[M], k)$.
(2) *If no minimal forbidden set is contained in* M*, then return the instance* (G_M, k)*, where* G_M *is obtained from* G *by inserting edges to make* $G[M]$ *a clique.*
(3) *Otherwise, we solve three instances:* $I_1 = (G - M, k - |M|)$*,* $I_2 = (G[M], k - 1)$*, and* $I_3 = (G', k - 1)$*, where* G' *is obtained from* G *by adding a clique* M' *of* $(k + 1)$ *vertices, connecting every pair of vertices* $u \in M'$ *and* $v \in N(M)$*, and deleting* M*; letting* Q_1*,* Q_2*, and* Q_3 *be the solutions of these instances respectively, we return either* $Q_1 \cup M$ *or* $Q_2 \cup Q_3$ *("NO" when* $|Q_2 \cup Q_3| > k$*), whichever is smaller.*

That is, in the third case we branch into two directions: the solution is obtained either as the union of M and the solution of I_1, or as the union of solutions of I_2 and I_3. The two branches correspond to the two cases where the solution fully contains M or only a minimum interval deletion set to $G[M]$ (i.e., Q_2), respectively. Note that in the second branch, it can be shown that Q_3 is disjoint from M'; hence $Q_2 \cup Q_3$ is indeed a subset of $V(G)$. Moreover, we have to clarify what the behavior of the reduction is if one or more of Q_1, Q_2, and Q_3 are "NO." If Q_2 or Q_3 is "NO," then we define $Q_2 \cup Q_3$ to be "NO" as well. If one of Q_1 and $Q_2 \cup Q_3$ is "NO," we return the other one; if both of them are "NO," we return "NO" as well.

A graph on which neither reduction rule applies is called *reduced*; in such a graph, every nontrivial module induces a clique. In Section 4, we prove the correctness of the reductions rules and that it can be checked in polynomial time if a reduction rule is applicable. Hence after exhaustive application of the reductions, we may assume that the graph is reduced.

The reductions are followed by a comprehensive study on reduced graphs that yields two crucial combinatorial statements. The first statement is on an AT $\{x, y, z\}$ that are witnessed by a minimal forbidden induced subgraph W different from a hole. We say that x is the *shallow terminal* if $W - N[x]$ is an induced path. We prove the shallow terminal x is simplicial, i.e., $N(x)$ induces a clique.

THEOREM 2.1. **[Shallow terminals]** *All shallow terminals in a reduced graph are simplicial.*

We say that two holes are *congenial* to each other if each vertex of one hole is a neighbor of the other hole. It turns out that the holes are pairwise congenial in a reduced graph.

THEOREM 2.2. **[Congenial holes]** *All holes in a reduced graph are congenial to each other.*

We point out that circular-arc graphs form an important example of graphs where the holes are pairwise congenial. Indeed, all holes of a reduced graph induce a circular-arc graph, but such a proof will not be given in this paper, as it is unnecessary for our purpose here. One may refer to [van 't Hof and Villanger 2013] for more intuition.

PHASE 2: Breaking holes. A consequence of Theorem 2.2 is that if a vertex v is in a hole, then $N[v]$ intersects every hole and thus makes a *hole cover*. Intuitively, this suggests that a minimal hole cover has to be very local in a certain sense. Indeed, by relating minimal hole covers in the reduced graph to minimal separators in the subgraph $G - N[v]$, we are able to establish a quadratic bound on the number of minimal hole covers, and more importantly, a cubic time algorithm to construct them.

THEOREM 2.3. **[Hole covers]** *Every reduced graph of n vertices contains at most n^2 minimal hole covers, and they can be enumerated in $O(n^3)$ time.*

Any interval deletion set must be a hole cover, and thus contains a minimal hole cover. This allows us to branch into at most n^2 instances, in each of which the input graph is chordal. Note that this branching step is applied only once; hence only a polynomial factor will be induced in the running time.

PHASE 3: Breaking ATs. As all the holes have been broken, the graph is already chordal at the onset of the third phase. It should be noted that, however, the graph might not be reduced, as new nontrivial non-clique modules can be introduced with the deletion of a hole cover in Phase 2. In principle, we could rerun the reductions of Phase 1 to obtain a reduced instance, but there is no need to do so at this point. The properties that we need in this phase are that graph is prereduced, chordal, and every shallow terminal is simplicial (Theorem 2.1). We give a name to such graphs and compare it with previously defined notions here.

— A graph is *prereduced* if Reduction 1 does not apply.
— A prereduced graph is *reduced* if Reduction 2 does not apply.
— A prereduced graph is *nice* if it is chordal and every shallow terminal in it is simplicial.

While both reduced graphs and nice graphs are prereduced, they are incomparable to each other. As only vertex deletions are applied after Phase 1, in the remainder of this algorithm the graph is an induced subgraph of that in a previous step. In other words, once a hereditary property is obtained after Phase 1, it remains true thereafter. It is easy to verify that the three defining properties of nice graphs are all hereditary. On the one hand, after the end of Phase 1, a reduced graph is prereduced by definition, and according to Theorem 2.1, every shallow terminal in it is simplicial. On the other hand, Phase 2 destroys all holes and the chordal property is obtained. Therefore, the graph becomes nice after Phase 2 and will remain nice till the end of our algorithm.

The removal of all simplicial vertices from a nice graph breaks all ATs (Theorem 2.1), thereby yielding an interval graph. This implies that a nice graph has a very special structure: It has a clique tree decomposition where the tree is a caterpillar, i.e., a path with degree-1 vertices attached to it. In other words, all vertices other than the shallow terminals can be arranged in a linear way, which greatly simplifies the examination of interactions between ATs. As a consequence, we can select an AT that is minimal in a certain sense, and single out 10 vertices such that there must exist a minimum interval deletion set destroying this AT with one of these 10 vertices. We can therefore safely branch on removing one of these 10 vertices.

THEOREM 2.4. **[Nice graphs]** *There is a $10^k \cdot n^{O(1)}$-time algorithm for* INTERVAL DELETION *on nice graphs.*

Algorithm Interval-Deletion(G, k)
INPUT: a non-interval graph G and a positive integer k.
OUTPUT: a minimum interval deletion set $Q \subset V(G)$ of size $\leq k$ or "NO."
1 Reduction 1: Let U be a minimal forbidden set of at most 10 vertices;
branch on deleting one vertex of U;
\\ *the graph will then be prereduced and remains so hereafter;*
2 Reduction 2: Let M be a nontrivial module of G not inducing a clique;
2.1 **if** all minimal forbidden sets of G are contained in M **then**
return Interval-Deletion($G[M], k$);
2.2 **else if** no minimal forbidden set is contained in M **then**
return Interval-Deletion(G_M, k), where edges are inserted to make $G[M]$ a clique;
2.3 **else branch** into three instances I_1, I_2, I_3;
\\ *now the graph is reduced;*
3 use the algorithm of Theorem 2.3 to enumerate the at most n^2 minimal hole covers of G;
\\ *the graph will then be nice and remains so hereafter;*
4 **for each** minimal hole cover HC **do**
use the algorithm of Theorem 2.4 to solve **(**$G - HC, k - |HC|$**)**;
5 **return** the smallest solution obtained, or "NO" if all solutions are "NO."

Fig. 1: Outline of algorithm for INTERVAL DELETION

Putting together these steps, the fixed-parameter tractability of INTERVAL DELETION follows (see Figure 1).

Theorem 1.1 (restated). There is a $10^k \cdot n^{O(1)}$ time algorithm for deciding whether or not there is a set of at most k vertices whose deletion makes an n-vertex graph G an interval graph.

PROOF. The algorithm described in Figure 1 solves the problem by making recursive calls to itself, or calling the algorithm of Theorem 2.4 $O(n^2)$ times. In the former case, at most 10 recursive calls are made, all with parameter value at most $k-1$. In the latter case, the running time is $10^k \cdot n^{O(1)}$. It follows that the total running time of the algorithm is $10^k \cdot n^{O(1)}$. $\square$

The paper is organized as follows. Section 3 sets the definitions and recalls some basic facts. Section 4 presents the details of the first phase. The next four sections are devoted to the proofs of Theorems 2.1–2.4. Sections 5 and 6 put shallow terminals and congenial holes under thorough examination, and prove Theorems 2.1 and 2.2, respectively. Section 7 fully characterizes minimal hole covers in reduced graphs and proves Theorem 2.3. Section 8 presents the algorithm that destroys ATs in nice graphs and proves Theorem 2.4. Section 9 closes this paper by some possible improvement and new directions.

3. PRELIMINARIES

All graphs discussed in this paper shall always be undirected and simple. A graph G is given by its vertex set $V(G)$ and edge set $E(G)$. If a pair of vertices v_1 and v_2 is connected by an edge, they are *adjacent* to each other, and denoted by $v_1 \sim v_2$, otherwise *nonadjacent* and denoted by $v_1 \not\sim v_2$. By $v \sim X$ we mean v is adjacent to at least one vertex of the set X. Two vertex sets X and Y are completely connected if $x \sim y$ for each pair of $x \in X$ and $y \in Y$. A graph is *complete* if each pair of vertices are adjacent. A *clique* in a graph is a subgraph that is complete, and a clique is maximal if it is not contained in another clique. A vertex is *simplicial* if its neighbors induce a clique. A neighbor of a vertex is another vertex that is adjacent to it, and the set of *neighborhood* of a vertex v is denoted by $N(v)$. The *closed neighborhood* of v is defined as $N[v] = N(v) \cup \{v\}$. This is generalized to a vertex set U, whose closed neighborhood and neighborhood are defined to be $N[U] = \bigcup_{v \in U} N[v]$ and $N(U) = N[U] \backslash U$. The notation $N_U(v)$ ($N_U[v]$) stands for the neighbors of v in the set

U, i.e., $N_U(v) = N(v) \cap U$ ($N_U[v] = N[v] \cap U$), regardless of whether $v \in U$ or not. The subgraph of a graph G induced by a subset of vertices U is denoted by $G[U]$, and $G - U$ is used as a shorthand for the subgraph induced by $V(G) \setminus U$.

A sequence of *distinct* vertices $(v_0 v_1 \dots v_\ell)$ such that $v_i \sim v_{i+1}$ for each $0 \leq i < \ell$ is called a v_0-v_ℓ *path*, whose *length* is defined to be ℓ. Vertices v_0 and v_ℓ are the *ends* of the path, while others, $\{v_1, \dots, v_{\ell-1}\}$, are called *inner vertices*. If the ends are distinct and adjacent, i.e., $\ell > 1$ and $v_0 \sim v_\ell$, then $(v_0 v_1 \dots v_\ell v_0)$ is called a *cycle*, whose *length* is defined to be $\ell + 1$. As an abuse of notation, by $u \in P$ (resp. $u \in C$) we mean that the vertex u appears in the path P (resp. cycle C), i.e., we use P or C as the set of vertices in the path (resp. cycle). A *chord* in a path or cycle is an edge between two non-consecutive vertices in the path or cycle. It is worth noting that the edge $v_0 v_\ell$, if exists, is a chord in the path $(v_0 v_1 \dots v_\ell)$, but not in the cycle $(v_0 v_1 \dots v_\ell v_0)$. It is easy to verify that no shortest path can contain a chord, so between each pair of vertices of a connected graph there is a chordless path. A chordless cycle of length ℓ, where $\ell \geq 4$, is called an *(ℓ-)hole*. A graph is *chordal* if it contains no hole, in other words, any cycle of length at least 4 contains a chord.

Chordal graphs admit several important and related characterizations. A set S of vertices *separates* x and y, and is called an *x-y separator* if there is no x-y path in the subgraph $G - S$, and *minimal x-y separator* if no proper subset of S separates x and y. For any pair of vertices x and y, a minimal x-y separator is also called a *minimal separator*. A graph is chordal if and only if each minimal separator in it induces a clique [Dirac 1961]. A nontrivial chordal graph contains at least two simplicial vertices, and there is at least one simplicial vertex in each component after the removal of any separator.

A tree $\mathcal{T}$ whose nodes are the maximal cliques of a graph G is a *(maximal) clique tree* of G if it satisfies the following conditions: any pair of adjacent nodes K_i and K_j defines a minimal separator that is $K_i \cap K_j$; for any vertex $x \in V$, the maximal cliques containing x correspond to a subtree of $\mathcal{T}$. A graph is chordal if and only if it has such a clique tree. A clique tree of a graph G will be denoted by $\mathcal{T}(G)$, or $\mathcal{T}$ when the graph G is clear from the context. Without distinguishing the node in a clique tree and the maximal clique in the graph G corresponding to it, we use K to denote both. A set of vertices is a minimal separator of G if and only if it is the intersection of K_i and K_j for some edge $K_i K_j$ in $\mathcal{T}$ [Buneman 1974]. This separator, $K_i \cap K_j$, is a minimal x-y separator for any pair of vertices $x \in K_i \setminus K_j$ and $y \in K_j \setminus K_i$.

As interval graphs are chordal, all aforementioned properties also apply to interval graphs. Moreover, by the following characterization of Fulkerson and Gross, each interval graph has a clique tree that is a path.

THEOREM 3.1 ([FULKERSON AND GROSS 1965]). *A graph G is an interval graph if and only if the maximal cliques of G can be linearly ordered such that, for each vertex v, the maximal cliques containing v occur consecutively.*

For a comprehensive treatment and for references to the extensive literature on chordal graphs and interval graphs, one may refer to the monograph of Golumbic [2004] and the survey of Brandstädt et al. [1999].

4. REDUCTION RULES AND BRANCHING

This section discusses the reduction rules described in Section 2 in more details.

4.1. Forbidden induced subgraphs

Three vertices form an *asteroidal triple*, AT for short, if each pair of them is connected by a path that avoids the neighborhood of the third one. We use *asteroidal witness* (AW) to refer to a minimal induced subgraph that is not a hole and contains an AT but none of its proper induced subgraphs does. It should be easy to check that an AW contains precisely one AT, and its vertices are the union of these three defining paths for this triple; the three

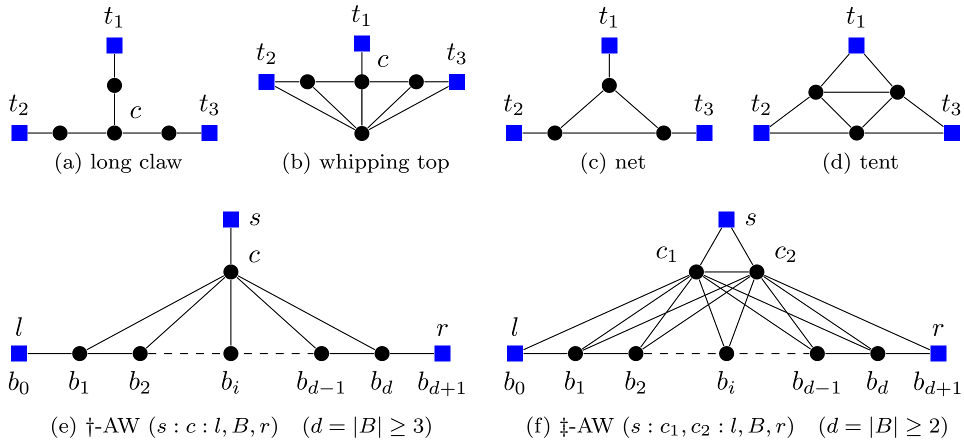


Fig. 2: Minimal chordal asteroidal witnesses (terminals are marked as squares).

defining vertices will be called *terminals* of this AW. It can be observed from Figure 2 that the three terminals are the only simplicial vertices of this AW and they are nonadjacent to each other. Lekkerkerker and Boland [1962] observed that a graph is an interval graph if and only if it is chordal and contains no AW. Not stopping here, they rolled up their sleeves and got their hands dirty by checking each possible forbidden induced subgraph. Their effort brought the following less beautiful but more useful characterization, here a minimal non-interval graph refers to a graph whose every proper induced subgraph is an interval graph but itself is not.

THEOREM 4.1 ([LEKKERKERKER AND BOLAND 1962]). *A minimal non-interval graph is either a hole or an AW depicted in Figure 2.*

Some remarks are in order. First, it is easy to verify that a hole of six or more vertices witnesses an AT (specifically, any three nonadjacent vertices from it) and is minimal, but following convention, we only refer to it as a hole, while reserve the term AW for graphs listed in Figure 2. Second, the set of AWs depicted in Figure 2 is not a literal copy of the original list in [Lekkerkerker and Boland 1962], which contains neither net nor tent; they are viewed as †-AW with $d = 2$ and ‡-AW with $d = 1$, respectively. We single them out for the convenience of later presentation. To avoid ambiguities, in this paper we explicitly require a †-AW (resp., ‡-AW) to contain at least 7 (resp., 8) vertices. Third, each of the four subgraphs in the first row of Figure 2 consists of a constant number, 6 or 7, of vertices, and thus can be easily located and disposed of by standard enumeration. For the purpose of the current paper, we are mainly concerned with the two kinds of AWs in the second row, whose sizes are unbounded. A †- or ‡-AW W contains a unique terminal s, called the *shallow terminal*, such that $W - N[s]$ is an induced path. The neighbor(s) of the shallow terminal are the *center(s)*. The other two terminals are called *base terminals*, and other vertices are called *base vertices*. The whole set of base vertices is called the *base*. We use $(s : c : l, B, r)$ (resp., $(s : c_1, c_2 : l, B, r)$) to denote the †-AW (resp., ‡-AW) with shallow terminal s, center c (resp., centers c_1 and c_2), base terminals l and r, and base $B = \{b_1, \ldots, b_d\}$. For the sake of notational convenience, we will also use b_0 and b_{d+1} to refer to the base terminals l and r, respectively, even though they are not part of the base B. The center(s) and base vertices are called *non-terminal vertices*.

Clearly, Reduction 1 can be applied in polynomial time: we can find a minimal forbidden set of size at most 10 in polynomial time, e.g., by complete enumeration. There are ways to improve this, but optimizing the exponent is not the focus of this paper. After the exhaustive application of Reduction 1, the graph is *prereduced*. By definition, any AW in a prereduced graph contains at least 11 vertices, which rules out long claws, whipping tops, nets, and tents. Furthermore, the base of a †-AW (resp., ‡-AW) in a prereduced graph contains at least 7 (resp., 6) vertices.

The purpose of the following proposition and a detailed proof is twofold. These special structures arise frequently in this paper, and we do not want to repeat the same argument again and again. The proof is exemplary in the sense that, by and large, most proofs of this paper exploit a similar contradictory arguments: They explicitly construct a forbidden induced subgraph, either a small AW or a short hole, assuming the property under discussion does not hold; because all graphs discussed henceforth are prereduced, such a contradiction will suffice to prove the desired property.

PROPOSITION 4.2. *Let $P = (v_0 \dots v_p)$ be a chordless path of length p in a prereduced graph, and u be adjacent to every inner vertex of P.*

(1) If $p \geq 4$ and u is also adjacent to v_0 and v_p, then $N[v_\ell] \subseteq N[u]$ for every $2 \leq \ell \leq p-2$.
(2) If $p \geq 3$ and u is also adjacent to v_0 and v_p, then $N[v_\ell] \cap N[v_{\ell+1}] \subseteq N[u]$ for every $1 \leq \ell \leq p-2$.
(3) If $p \geq 4$, then $N[v_\ell] \setminus (N(v_1) \cup N(v_{p-1})) \subseteq N[u]$ for every $2 \leq \ell \leq p-2$.

PROOF. Suppose to the contrary of statement (1), there is a vertex $x \in N[v_\ell] \setminus N[u]$, then we show the existence of a short hole or small AW in G, thus contradicting the assumption that G is prereduced. Note that $x \not\sim v_i$ for any $i \leq \ell-2$ or $i \geq \ell+2$, as otherwise, there is a 4-hole $(uv_ixv_\ell u)$ (here $v_i \not\sim v_\ell$ because P is chordless). There is • a 4-hole $(uv_{\ell-1}xv_{\ell+1}u)$ when x is also adjacent to both $v_{\ell-1}$ and $v_{\ell+1}$; • a tent $\{u, v_{\ell-1}, v_\ell, x, v_{\ell+1}, v_{\ell+2}\}$ when x is adjacent to $v_{\ell+1}$ but not $v_{\ell-1}$; • a tent $\{u, v_{\ell-2}, v_{\ell-1}, x, v_\ell, v_{\ell+1}\}$ when x is adjacent to $v_{\ell-1}$ but not $v_{\ell+1}$; or • a whipping top $\{x, u, v_{\ell-2}, v_{\ell-1}, v_\ell, v_{\ell+1}, v_{\ell+2}\}$ otherwise (x is only adjacent to v_ℓ in the path).

Suppose, for contradiction to statement (2), $x \in N[v_\ell] \cap N[v_{\ell+1}] \setminus N[u]$. If x is adjacent to $v_{\ell-1}$ or $v_{\ell+2}$, then there is a 4-hole; otherwise, there is a tent $\{u, v_{\ell-1}, v_\ell, x, v_{\ell+1}, v_{\ell+2}\}$.

Statement (3) will follow from statement (1) if u is also adjacent to v_0 and v_p; hence we assume otherwise, and without loss of generality, $u \not\sim v_0$. Suppose to the contrary of statement (3), there is a vertex $x \in N[v_\ell] \setminus (N(v_1) \cup N(v_{p-1}) \cup N[u])$. If v_2 is the only inner vertex of P that is adjacent to x, then there is • a 4-hole $(xv_0v_1v_2x)$ when $x \sim v_0$; • a 4-hole $(xv_4v_3v_2x)$ when $x \sim v_4$; • a net $\{v_0, v_1, x, v_2, u, v_4\}$ when $x \not\sim v_0, v_4$ and $u \sim v_4$; or • a †-AW $(x : v_2 : v_0, v_1uv_3, v_4)$ when $x \not\sim v_0, v_4$ and $u \not\sim v_4$.

A symmetric argument proves the case when $u \not\sim v_p$ and v_{p-2} is the only inner vertex of P that is adjacent to x. Other cases follow from statements (1) and (2). □

Let X be a nonempty set of vertices. A vertex v is a *common neighbor* of X if it is adjacent to every vertex $x \in X$. We denote by $\widehat{N}(X)$ the set of all common neighbors of X. It is easy to verify that in a prereduced graph, at least one of X and $\widehat{N}(X)$ induces a clique, as otherwise two nonadjacent vertices in $\widehat{N}(X)$, together with two nonadjacent vertices in X, will induce a 4-hole. In particular, we have the following proposition.

PROPOSITION 4.3. *Let X be a set of vertices of a prereduced graph that induces either a hole, an AW, or a path of length at least 2. Then $\widehat{N}(X)$ induces a clique.*

4.2. Modular decomposition

A subset M of vertices forms a *module* of G if all vertices in M have the same neighborhood outside M. In other words, for any pair of vertices $u, v \in M$ and vertex $x \notin M$, $u \sim x$ if and only if $v \sim x$. The set $V(G)$ and all singleton vertex sets are modules, called *trivial*.

A brief inspection shows that no graph in Figure 2 has any nontrivial modules and this is true also for holes of length greater than 4:

PROPOSITION 4.4. *Let M be a module, and X be a minimal forbidden set. If $|X| > 4$, then either $X \subseteq M$, or $|M \cap X| \leq 1$.*

Indeed, the only minimal forbidden induced subgraph of no more than 4 vertices is a 4-hole, of which the pair of nonadjacent vertices might belong to a module. This observation allows us to prove the following statement, which is the main combinatorial reason behind the correctness of the branching in Reduction 2.

THEOREM 4.5. *Let G be a graph that contains no 4-hole and M be a module of G. A minimum interval deletion set to G contains either all vertices of M, or only a minimum interval deletion set to $G[M]$.*

PROOF. Let Q be a minimum interval deletion set to G such that $M \not\subseteq Q$; otherwise we are already done. To show that $Q_M = Q \cap M$ is precisely a minimum interval deletion set to $G[M]$, it suffices to show that for any minimum interval deletion set Q'_M to $G[M]$, the set $Q' = (Q \setminus Q_M) \cup Q'_M$ is an interval deletion set to G: Trivially Q_M is an interval deletion set to $G[M]$; if it is not minimum, then $|Q_M| > |Q'_M|$, and $|Q| > |Q'|$, which contradicts the fact that Q is minimum.

Suppose the contrary and X is a minimal forbidden set in $G - Q'$. By construction, Q'_M intersects every minimal forbidden set in $G[M]$, while $Q \setminus Q_M$ intersects every minimal forbidden set in $G - M$. Thus X intersects both M and $V(G) \setminus M$. On the other hand, $|X| > 4$ as the graph is 4-hole free. According to Proposition 4.4, $X \cap M$ contains exactly one vertex; let it be x. Let x' be a vertex in $M \setminus Q$, which is nonempty by the assumption $M \not\subseteq Q$, and let $X' = X \setminus \{x\} \cup \{x'\}$; it is immaterial whether $x' = x$ or not. The set X' is disjoint from Q, and by definition of modules, $G[X']$ and $G[X]$ are isomorphic. In other words, X' is a minimal forbidden set in $G - Q$, which is impossible. Therefore, Q' is a interval deletion set to G and this finishes this proof. □

To apply Reduction 2, we have to first find a nontrivial module that is not a clique. For this purpose, we do not need to compute a modular decomposition tree of the graph. The simple algorithm described in Figure 3 is sufficient.

LEMMA 4.6. *We can find in polynomial time a nontrivial module M such that $G[M]$ is not a clique, or report no such a module exists.*

PROOF. The algorithm described in Figure 3 finds such a module in a greedy manner. It starts from a pair of nonadjacent vertices u and v, and generates the module by adding vertices. Note that each vertex in the set X defined at step 2.1 is a witness for the fact that M is not a module, in other words, M is a module only if $X = \emptyset$. When a nonempty vertex set M is returned at step 2.2, from the algorithm we can derive that $X = \emptyset$ and $M \neq V(G)$; hence M must be a nontrivial module. Now it remains to show that as long as there is a nontrivial non-clique module U in the graph, the algorithm is guaranteed to return a nonempty set (not necessarily U itself). As U does not induce a clique, it contains a pair of nonadjacent vertices u and v, which shall be considered in some iteration of the for-loop. In this iteration, initially $M \subseteq U$, and by induction we are able to show that no vertex of $V(G) \setminus U$ can be included in X during this iteration; hence $M \subseteq U$ will remain an invariant. As a consequence, a subset M that satisfies $\{u, v\} \subseteq M \subseteq U$ is returned. □

Indeed, one can easily verify that the module found as above is the inclusive-wise minimal one containing both u and v. We are now ready to explain the application of Reduction 2 and prove its correctness.

```
for each pair of nonadjacent vertices u and v do
1       M = {u, v};
2       while M ≠ V(G) do
2.1         X = {x ∉ M : 0 < |N_M(x)| < |M|};
2.2         if X = ∅ then return M;
2.3         else M = M ∪ X;
return ∅.       \\ there is no such a module
```

Fig. 3: Algorithm Find-Module

LEMMA 4.7. *Reduction 2 is correct and it can be checked in polynomial time whether Reduction 2 (and which case of it) is applicable.*

PROOF. The correctness of the reduction is clear in case 1: removing the vertices of $V(G) \setminus M$ does not make the problem any easier, as these vertices do not participate in any minimal forbidden set.

In case 2, the correctness of the reduction follows from the fact that G and G_M have the same set of minimal forbidden sets. Note that a clique is an interval graph, and more importantly, the insertion of edges to make M a clique neither breaks the modularity of M nor introduces any new 4-hole; thus Proposition 4.4 is applicable to G_M. As M induces an interval graph in both G and G_M, if X is a minimal forbidden set of G or G_M, then Proposition 4.4 implies that X contains at most one vertex of M. In other words, the insertion of edges has no effect on any minimal forbidden set, which means that Q is an interval deletion set to G if and only if it is an interval deletion set to G_M.

The correctness of case 3 can be argued using Theorem 4.5, which states the two possibilities of any interval deletion set to G with respect to M. In particular, the two branches of case 3 correspond to these two cases. The first branch is straightforward: we simply remove all vertices of M from the graph and solve the instance $I_1 = (G - M, k - |M|)$. It is the second branch (where we assume $M \not\subseteq Q$) that needs more explanation. Recall that by construction of I_3, the set M' is a module of G' and induces an interval graph. It is clear that either solution Q_2 or Q_3 being "NO" will rule out the existence of an interval deletion set of G that does not fully contain M. Hence we may assume Q_2 and Q_3 are minimum interval deletion sets of I_2 and I_3, respectively; and $Q = Q_2 \cup Q_3$. Note that both $|Q_2|$ and $|Q_3|$ are upper bounded by $k - 1$.

CLAIM 1. *Set Q is an interval deletion set of G.*

PROOF. According to Theorem 4.5, if Q_3 intersects M', which is a module of G', then it must contain all $(k + 1)$ vertices in M',[1] i.e., $|Q_3| > k$; a contradiction. Therefore, $Q_3 \cap M' = \emptyset$, which means $Q \subset V(G)$. Suppose that there is a minimal forbidden set X of G disjoint from Q. It cannot be fully contained in M, as $Q_2 \subseteq Q$ is an interval deletion set of $G[M]$. Then by Proposition 4.4, X contains exactly one vertex x of M and $X' = X \setminus \{x\} \cup \{x'\}$ is also a minimal forbidden set of G' for any $x' \in M'$. Since Q_3 is an interval deletion set of G' disjoint from M', it has to contain a vertex of $X' \setminus \{x'\} = X \setminus \{x\}$; a contradiction. ⌟

CLAIM 2. *Set Q is not larger than the smallest interval deletion set Q' satisfying $M \not\subseteq Q'$.*

PROOF. Suppose that Q' is an interval deletion set of G of size at most k with $M \not\subseteq Q'$; let $Q'_2 = Q' \cap M$ and $Q'_3 = Q' \setminus M$. We claim that Q'_2 and Q'_3 are interval deletion sets of I_2 and I_3, respectively. First, we argue that Q'_2 and Q'_3 are not empty; hence both of them

[1]Indeed, $\min(k + 1, |N(M)|)$ vertices will suffice for our bookkeeping purpose, and an alternative way to this is to add only one vertex but mark it as "forbidden."

have sizes at most $k-1$. The assumption that $G[M]$ is not an interval graph implies $Q'_2 \neq \emptyset$. By assumption, $M \not\subseteq Q'$, thus there is a vertex $x \in M \setminus Q'$. Now $Q'_3 = \emptyset$ would imply that $G-(M\setminus\{x\})$ is an interval graph, that is, there is no minimal forbidden set containing only one vertex of M, and it follows that we should have been in Case 1. Since $|Q'_2| \leq k-1$, it is clear that Q'_2 is a solution of instance $I_2 = (G[M], k-1)$. The only way Q'_3 is not a solution of I_3 is that there is a minimal forbidden set X containing a vertex of the $(k+1)$-clique introduced to replace M. As this $(k+1)$-clique is a module, Proposition 4.4 implies that X contains exactly one vertex y of this clique. But in this case $X' = X \setminus \{y\} \cup \{x\}$ (where x is a vertex of $M \setminus Q'$) is a minimal forbidden set disjoint from Q', a contradiction. Thus $|Q| \leq |Q'|$ follows from the fact that both Q_2 and Q_3 are minimum. ⌟

As a consequence of Claim 2, if $|Q| > k$, then there cannot be an interval deletion set of size no more than k that does not fully include M. This finishes the proof of the correctness of Reduction 2.

On the applicability of Reduction 2, we first use Lemma 4.6 to find a nontrivial module that does not induce a clique. If such a module M is found, then Reduction 2 is applicable, and it remains to figure out which case should apply by checking the conditions in order. To check whether case 1 holds, we need to check if there is a minimal forbidden set X not contained in M. By Proposition 4.4, such an X, if exists, contains at most one vertex x from M; and x can be replaced by any other vertex of M. Therefore, it suffices to pick any vertex $x \in M$, and test in linear time whether $G - (M \setminus \{x\})$ is an interval graph. If it is not an interval graph, then there is a minimal forbidden set X not contained in M (as it contains at most one vertex of M). Otherwise, $G - (M \setminus \{x\})$ is an interval graph for every $x \in M$, and there is no such X; hence case 1 holds. To check whether case 2 holds, observe that the condition "there is no minimal forbidden set contained in M" is equivalent to saying that $G[M]$ is an interval graph, which can be checked in linear time. In all remaining cases, we are in case 3. □

5. SHALLOW TERMINALS

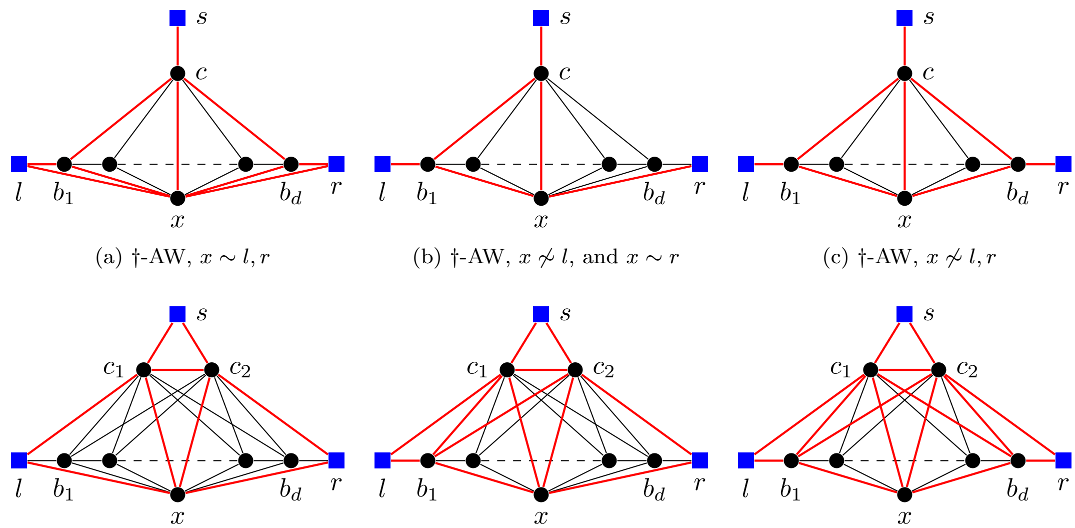
This section proves Theorem 2.1 by showing that each shallow terminal is contained in a module whose neighborhood induces a clique. This module either is trivial (consisting of only this shallow terminal), or induces a clique (after the application of Reduction 2). Therefore, this shallow terminal is always simplicial. Recall that an AW in a prereduced graph G has to be a †- or ‡-AW. Let us start from a thorough scrutiny of neighbors of its shallow terminal, which, by definition, is disjoint from the base and base terminals.

LEMMA 5.1. *Let W be an AW in a prereduced graph. Every common neighbor x of the base B is adjacent to the shallow terminal s.*

PROOF. The center(s) of W are also common neighbors of B, and hence according to Proposition 4.3, they are adjacent to x. Suppose, for contradiction, $x \in \widehat{N}(B) \setminus N(s)$. If W is a †-AW, then there is (see the first row of Figure 4) • a whipping top $\{s, c, l, b_1, x, b_d, r\}$ centered at c when $x \sim l, r$; • a net $\{s, c, l, b_1, r, x\}$ when $x \sim r$ but $x \not\sim l$ (similarly for $x \sim l$ but $x \not\sim r$); or • a †-AW $(s : c : l, b_1 x b_d, r)$ when $x \not\sim l, r$. If W is a ‡-AW, then there is (see the second row of Figure 4) • a tent $\{x, c_1, b_1, s, b_d, c_2\}$ when $x \sim l, r$; • a ‡-AW $(s : c_1, c_2 : l, b_1 x, r)$ when $x \sim r$ but $x \not\sim l$ (similarly for $x \sim l$ but $x \not\sim r$); or • a ‡-AW $(s : c_1, c_2 : l, b_1 x b_d, r)$ when $x \not\sim l, r$. As none of these structures can exist in a prereduced graph, this lemma is proved. □

LEMMA 5.2. *Let W be an AW in a prereduced graph, and x be adjacent to the shallow terminal s.*

(1) Then x is also adjacent to the center(s) of W (different from x).



(a) †-AW, $x \sim l, r$ (b) †-AW, $x \not\sim l$, and $x \sim r$ (c) †-AW, $x \not\sim l, r$

(d) ‡-AW, $x \sim l, r$ (e) ‡-AW, $x \not\sim l$, and $x \sim r$ (f) ‡-AW, $x \not\sim l, r$

Fig. 4: Adjacency between a common neighbor x of B and s [Lemma 5.1].

(2) Classifying x with respect to its adjacency to the base B, we have the following categories:

(full) x is adjacent to every base vertex.
Then x is also adjacent to every vertex in $N(s) \setminus \{x\}$.

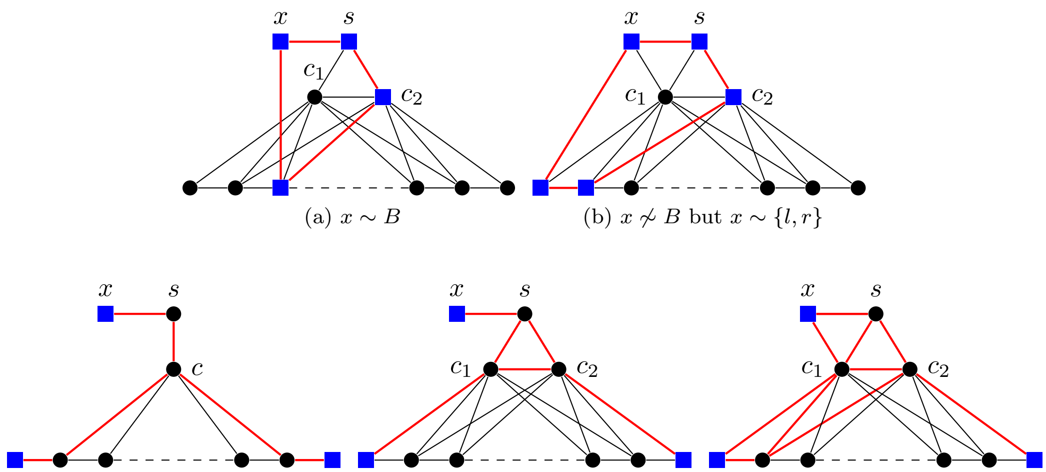
(partial) x is adjacent to some, but not all base vertices.
Then there is an AW whose shallow terminal is s, one center is x, and base is a proper sub-path of B.

(none) x is adjacent to no base vertex.
Then x is adjacent to neither base terminals, and thus replacing the shallow terminal of W by x makes another AW.

Proof. Suppose to the contrary of statement (1), $x \not\sim c$ if W is a †-AW or (without loss of generality) $x \not\sim c_2$ if W is a ‡-AW. If $x \sim b_i$ for some $1 \leq i \leq d$ then there is a 4-hole $(xscb_ix)$ or (xsc_2b_ix) (See Figure 5(a)). Hence we may assume $x \not\sim B$. (See Figure 5(b,c,d,e).) There is • a 5-hole $(xscb_1lx)$ or $(xscb_drx)$ if W is a †-AW, and $x \sim l$ or $x \sim r$, respectively; • a 5-hole (xsc_2b_1lx) or 4-hole (xsc_2rx) if W is a ‡-AW, and $x \sim l$ or $x \sim r$, respectively; • a long claw $\{x, s, c, b_1, l, b_d, r\}$ if W is a †-AW and $x \not\sim l, r$; • a net $\{x, s, l, c_1, r, c_2\}$ if W is a ‡-AW and $x \not\sim c_1, l, r$; or • a whipping top $\{r, c_2, s, x, c_1, l, b_1\}$ centered at c_2 if W is a ‡-AW and $x \not\sim l, r$, but $x \sim c_1$. Neither of these cases is possible, and thus statement (1) is proved.

For statement (2), let us handle category "none" first. Note that x, nonadjacent to B, cannot be a center of W. If $x \sim l$, then there is a 4-hole (xcb_1lx) or (xc_2b_1lx) when W is a †-AW or ‡-AW, respectively. A symmetrical argument will rule out $x \sim r$. Now that x is adjacent to the center(s) but neither base terminals nor base vertices of W, then $(x : c : l, B, r)$ (resp., $(x : c_1, c_2 : l, B, r)$) makes another †-AW (resp., ‡-AW).

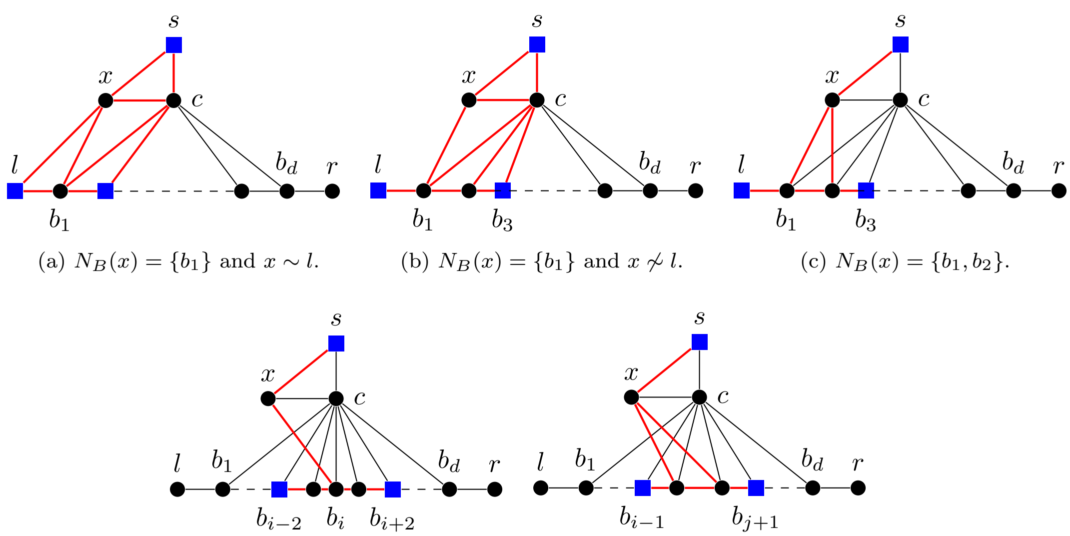
Assume now that x is in category "full." Suppose for contradiction that $x \not\sim v$ for some $v \in N(s) \setminus \{x\}$. We have already proved in statement (1) that v and x are adjacent to the center(s) of W (different from them). In particular, if one of v and x is a center, then they



(a) $x \sim B$ (b) $x \not\sim B$ but $x \sim \{l, r\}$

(c) $x \sim s$ but $x \not\sim B$ (d) $x \sim s$ but $x \not\sim B$ (e) $x \sim c_1$ but $x \not\sim c_2, l$

Fig. 5: Adjacency between a neighbor x of s and centers [Lemma 5.2].



(a) $N_B(x) = \{b_1\}$ and $x \sim l$. (b) $N_B(x) = \{b_1\}$ and $x \not\sim l$. (c) $N_B(x) = \{b_1, b_2\}$.

(d) $N_B(x) = \{b_i\}$ $(1 < i < d)$. (e) $N_B(x) = \{b_i, b_{i+1}\}$ $(1 < i < d - 1)$.

Fig. 6: Vertex x in category "partial" w.r.t. W [Lemma 5.2].

are adjacent. Therefore, we can assume that v and x are not centers. If $v \sim b_i$ for some $1 \leq i \leq d$, then there is a 4-hole $(xsvb_ix)$. Otherwise, $v \not\sim B$, and it is in category "none." Let W' be the AW obtained by replacing s in W by v; then $x \sim v$ follows from Lemma 5.1.

Finally, assume that x is in category "partial," that is, $x \sim B$, but $x \not\sim b_i$ for some $1 \leq i \leq d$. In this case, we construct the claimed AW as follows. As the case $x \not\sim l$ but $x \sim r$ is symmetric to $x \sim l$ but $x \not\sim r$, it is ignored in the following, i.e., we assume that $x \sim r$ only if $x \sim l$. Let p be the smallest index such that $x \sim b_p$, and q be the smallest index such

		$q = p + 1$	$q = p + 2$	$q > p + 2$
†	$p = 0$ Fig. 6(a)	4-hole $(xcb_1lx)^a$	tent $\{l, x, s, c, b_2, b_1\}$	‡-AW $(s : x, c : l, b_1 \dots b_{q-1}, b_q)^b$
	$p = 1$ Fig. 6(b,c)	whipping top $\{l, b_1, x, s, c, b_3, b_2\}^c$	net $\{l, b_1, s, x, b_3, b_2\}$	†-AW $(s : x : l, b_1 \dots b_{q-1}, b_q)^b$
	$p > 1$ Fig. 6(d,e)	long claw[1] $\{b_{p-2}, b_{p-1}, b_p, s, x, b_{p+2}, b_{p+1}\}$	net $\{b_{p-1}, b_p, s, x, b_q, b_{q-1}\}$	†-AW $(s : x : b_{p-1}, b_p \dots b_{q-1}, b_q)$
‡	$p = 0$	4-hole $(xc_2b_1lx)^a$	tent $\{l, x, s, c_2, b_2, b_1\}$	‡-AW $(s : x, c_2 : l, b_1 \dots b_{q-1}, b_q)^b$
	$p = 1$	whipping top $\{l, b_1, x, s, c_2, b_3, b_2\}^c$	net $\{l, b_1, s, x, b_3, b_2\}$	†-AW $(s : x : l, b_1 \dots b_{q-1}, b_q)^b$
	$p > 1$	long claw $\{b_{p-2}, b_{p-1}, b_p, s, x, b_{p+2}, b_{p+1}\}$	net $\{b_{p-1}, b_p, s, x, b_q, b_{q-1}\}$	†-AW $(s : x : b_{p-1}, b_p \dots b_{q-1}, b_q)$

[a]The vertex x is in category "none."
[b]The vertex x would be in category "full" if $q = d + 1$.
[c]A 4-hole $(xb_pb_{p+1}b_{p+2}x)$ would be introduced if $x \sim b_{p+2}$;

Table I: Structures used in the proof of Lemma 5.2 (category "partial")

that $p < q \leq d + 1$ and $x \not\sim b_q$ (q exists by assumptions). See Table I for the structures for †-AW and ‡-AW respectively (see also Figure 6).[2]

As the graph is prereduced and contains no small forbidden induced subgraph, it is immediate from Table I that the case $q > p + 2$ holds; otherwise there always exists a small forbidden induced subgraph. This completes the categorization of vertices in $N(s) \setminus T$ and the proof. □

The proof of our main result of this section is an inductive application of Lemma 5.2. To avoid the repetition of the essentially same argument for †-AWs and ‡-AWs, especially for the interaction between AWs, we use a generalized notation to denote both. We will uniformly use c_1, c_2 to denote center(s) of an AW, and while the AW under discussion is a †-AW, both c_1 and c_2 refer to the only center c. As long as we do not use the adjacency of c_1 and l, c_2 and r, or c_1 and c_2 in any of the arguments, this unified (abused) notation will not introduce inconsistencies.

THEOREM 5.3. *Let W be a †- or ‡-AW in a prereduced graph G with shallow terminal s and base B. Let $C = N(s) \cap N(B)$ and let M be the vertex set of the component of $G - C$ containing s. Then M is completely connected to C, and $G[C]$ is a clique.*

PROOF. Denote by $W = (s : c_1, c_2 : l, B, r)$, where $c_1 = c_2$ when W is a †-AW. Let x and y be any pair of vertices such that $x \in C$ and $y \in M$. By definition, $G[M]$ is connected, and there is a chordless path $P = (v_0 \dots v_p)$ from $v_0 = s$ to $v_p = y$ in $G[M]$. We claim that no vertex in P is adjacent to B. It holds vacuously if $p = 1$ and then $y \sim s$; hence we assume $p > 1$. Suppose the contrary and let q be the smallest index such that $v_q \sim B$. This means that every v_i with $i < q$ is in category "none" of Lemma 5.2(2). Therefore, applying Lemma 5.2(1,2) on v_i and AW $(v_{i-1} : c_1, c_2 : l, B, r)$ inductively for $i = 1, \dots, q - 1$, we conclude that there is an AW $W_i = (v_i : c_1, c_2 : l, B, r)$ for each $i < q$. One more application of Lemma 5.2(1) shows that v_q is adjacent to the center(s) of W_{q-1} as well. If v_q is adjacent to all vertices of B, i.e., in the category "full" with respect to every W_i, then Lemma 5.2(2)

[2]We omit the figure for ‡-AWs: For a ‡-AW $(s : c_1, c_2 : l, B, r)$, we are only concerned with the relation between center c_2 and $B \cup \{l\}$, which is the same as the relation between c and $B \cup \{l\}$ in a †-AW.

on v_q and W_{q-1} implies that v_q is adjacent to $v_{q-2} \in N(v_{q-1})$, contradicting the assumption that P is chordless. Otherwise (the category "partial"), according to Lemma 5.2(2), there is another AW $W' = (v_{q-1} : c'_1, c'_2 : l', B', r')$, where $B' \subset B$, and $v_q \in \{c'_1, c'_2\}$. Now an application of Lemma 5.2(1) on v_q and W' shows that v_q is adjacent to $v_{q-2} \in N(v_{q-1})$, again a contradiction. From these contradictions we can conclude P is disjoint from $N(B)$. Applying Lemma 5.2 inductively on v_{i+1} and $W_i = (v_i : c_1, c_2 : l, B, r)$, we get an AW with the same centers for every $0 \leq i \leq p$.

As x is adjacent to both s and B, it cannot be in category "none" with respect to W. We now separate the discussion based on whether x is in the category "full" or "partial." Suppose first that x is in the category "full"; as $x \in N(s)$, Lemma 5.2(1) implies that $x \sim c_1, c_2$. Then applying Lemma 5.2(2) inductively, where $i = 1, \ldots, p$, on vertex x and W_{i-1} we get that $x \sim v_i$ for every $i \leq p$; in particular, $x \sim v_p$ $(= y)$. Suppose now that x is in in category "partial." Then by Lemma 5.2(2), there is an AW $W'_0 = (v_0 : c'_1, c'_2 : l', B', r')$, where $B' \subset B$, and $x \in \{c'_1, c'_2\}$. For any $0 \leq i \leq p$, as $v_i \not\sim B$, we have $v_i \not\sim B'$ as well, i.e., v_i is in category "none" with respect to W'_0. Therefore, by an inductive application of Lemma 5.2(2) on the vertex v_i and AW $W'_{i-1} = (v_{i-1} : c'_1, c'_2 : l', B', r')$ for $i = 1, \ldots, p$, we conclude that there is an AW $W'_p = (v_p : c'_1, c'_2 : l', B', r')$, from which $x \sim y$ follows immediately.

Now we show the second assertion. For any pair of vertices x and y in C, we apply Lemma 5.2 on x and W; by definition, $x \sim B$ and thus cannot be in category "none." If x is in category "full" with respect to W, then Lemma 5.2(2) and the fact $y \in N(s)$ imply that $x \sim y$. Otherwise, if x is in category "partial" with respect to W, then Lemma 5.2(2) implies that there is an AW $W' = (s : c'_1, c'_2 : l', B', r')$ where $B' \subset B$ and $x \in \{c'_1, c'_2\}$. Therefore, by Lemma 5.2(1) on the vertex $y \in N(s)$ and W', we get that $y \sim c'_1, c'_2$ and hence $x \sim y$. □

Now Theorem 2.1 follows from Theorem 5.3: the set M containing s is in a module whose neighborhood is a clique, hence every vertex in M is simplicial. We point out that the set C is the minimal M-B separator.

6. LONG HOLES

This section proves Theorem 2.2 by showing that the holes in a reduced graph are pairwise congenial. During the study of vertices of a hole, their indices become very subtle. To simplify the presentation, we will frequently apply a common technique, that is, to number the vertices of a hole starting from a vertex of special interest for the property at hand. Needless to say, indexing two adjacent vertices in a hole will determine the indices of all the vertices in the hole, as well as the ordering used to traverse the hole. All indices of vertices in a hole H should be understood as modulo $|H|$, e.g., $h_0 = h_{|H|}$.

We start from two simple facts on the relations between vertices and holes, from which we derive the relations between two holes, and finally generalize them to multiple holes.

PROPOSITION 6.1. *For any vertex v and hole H of a prereduced graph, $N_H[v]$ are consecutive in H. Moreover, either $N_H[v] = H$ or $|N_H[v]| < |H| - 7$.*

PROOF. Both assertions are trivially true when $v \not\sim H$, $N_H[v] = H$, or $v \in H$ (then $|N_H[v]| = 3$); it is hence assumed that none of them holds true. We number the vertices of H in a way that $h_0 \sim v$ but $h_1 \not\sim v$. Suppose the first assertion is not true, then we can find the following three vertices of H, whose existence is clear from assumptions: (a) p_1 is the smallest index such that $p_1 > 1$ and $h_{p_1} \sim v$; (b) p_2 is the smallest index such that $p_2 > p_1$ and $h_{p_2} \not\sim v$; and (c) p_3 is the smallest index such that $p_2 < p_3 < |H|$ and $h_{p_3} \sim v$, or $p_3 = |H|$ if $h_i \not\sim v$ for each $i > p_2$ (then $h_{p_3} = h_0$). By the selection of p_1, p_2, and p_3, we have $v \sim h_i$ for each $p_1 \leq i < p_2$ and $i = 0, p_3$, but $v \not\sim h_j$ for $0 < j < p_1$ or $p_2 \leq j < p_3$ (relations between v and h_i where $p_3 < i < |H|$ are immaterial).

Now we examine the distances between the three indices. If $p_1 < 4$ or $p_3 < p_2 + 4$, then there is a small hole, $(vh_0h_1 \dots h_{p_1}v)$ or $(vh_{p_2-1}h_{p_2} \dots h_{p_3}v)$, respectively, of length at most 6. Thus $4 \leq p_1 < p_2 < p_2 + 4 \leq p_3$. Nonetheless, there is • a long claw $\{h_0, v, h_{p_1}, h_{p_1-2}, h_{p_1-1}, h_{p_2}, h_{p_2+1}\}$ if $p_2 = p_1 + 1$; • a net $\{h_0, v, h_{p_1-1}, h_{p_1}, h_{p_2}, h_{p_2-1}\}$ if $p_2 = p_1 + 2$; or • a long claw $\{h_1, h_0, v, h_{p_1-1}, h_{p_1}, h_{p_2}, h_{p_2-1}\}$ if $p_2 > p_1 + 2$. None of these forbidden induced subgraphs involves both h_0 and h_{p_3} and thus they exist regardless of $h_{p_3} = h_0$ or not. This contradiction ensures that $N_H[v]$ are consecutive, so are the vertices of $H \setminus N_H[v]$. On the second assertion, note that there is a hole of length at most 10 if $1 \leq |H \setminus N_H[v]| \leq 7$. □

Recall that $\widehat{N}(H)$ is the set of all common neighbors of the hole H. If a vertex $v \notin \widehat{N}(H)$ is adjacent to more than three vertices of H, then we can use v as a shortcut for the inner vertices of the path induced by $N_H[v]$ to obtain another hole that is strictly shorter than H.

COROLLARY 6.2. *Let H be a shortest hole. If $v \notin \widehat{N}(H)$, then $N_H[v] \leq 3$.*

Note that each hole H in a prereduced graph contains at least 11 vertices. If $v \in \widehat{N}(H)$, then on any five consecutive vertices of the hole H and v, Proposition 4.2(1) applies, which implies that v is dominating in the closed neighborhood of H.

COROLLARY 6.3. *Let H be a hole in a prereduced graph. If $v \in \widehat{N}(H)$, then v is adjacent to all vertices in $N[H] \setminus \{v\}$.*

So far we characterized neighbors of holes in a prereduced graph: Any vertex v is adjacent to a (possibly empty) set of consecutive vertices of a hole H; if v is adjacent to all vertices of H, then it is also adjacent to every neighbor of H. From these facts we now derive the relations between holes. Following is the most crucial concept of the section:

Definition 6.4. Two holes H_1 and H_2 are called *congenial* (to each other) if each vertex of one hole is a neighbor of the other hole, that is, $H_1 \subseteq N[H_2]$ and $H_2 \subseteq N[H_1]$.

We remark that every hole is congenial to itself by definition. The definition is partially motivated by:

PROPOSITION 6.5. *Let $\mathcal{H}$ be a set of holes all congenial to H. For each $v \in H$, every hole in $\mathcal{H}$ intersects $N[v]$.*

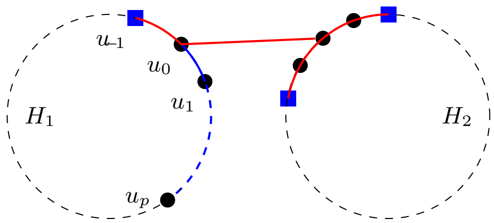
Since a vertex in a hole cannot be a common neighbor of it, Corollary 6.3 and the definition of congenial holes immediately imply:

COROLLARY 6.6. *For any pair of congenial holes H_1 and H_2 in a prereduced graph, $\widehat{N}(H_1) = \widehat{N}(H_2)$. Moreover, no vertex of H_1 (resp., H_2) is a common neighbor of H_2 (resp., H_1).*

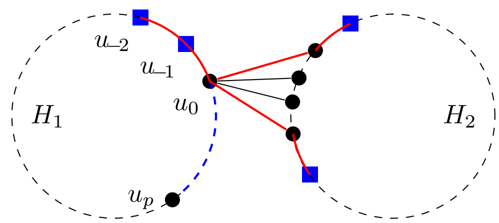
We analyze next the relation between two non-congenial holes. It turns out that if not all vertices of a hole H_1 are adjacent to another hole H_2, then, as shown in the following lemma, every vertex of H_1 is adjacent to either all or none of the vertices of H_2.

LEMMA 6.7. *Let H_1 and H_2 be two adjacent holes in a prereduced graph such that $H_1 \not\subseteq N[H_2]$. Each neighbor of H_2 in H_1 is a common neighbor of H_2, i.e., $N_{H_1}[H_2] \subseteq \widehat{N}(H_2)$. In particular, H_1 and H_2 are disjoint.*

PROOF. Let u be any vertex in $N_{H_1}[H_2]$, which is nonempty by assumption, and let P be the maximal path in H_1 with the property that $u \in P \subseteq N_{H_1}[H_2]$; denote by p the number of vertices of P. Note that some vertices of P can belong to H_2 (in particular, u can be in H_2). Observe that $p < |H_1|$, as by assumption, H_1 is not contained in $N[H_2]$. Numbering



(a) $|N_{H_2}(u_0)| = 1$



(b) $|N_{H_2}(u_0)| > 1$

Fig. 7: Adjacencies between two non-congenial holes ($u_{-1} \not\sim H_2$)

the vertices in H_1 such that $P = u_0 \dots u_{p-1}$ (the ordering of H_1 is immaterial when $p = 1$ and then u_1 can be either neighbor of u_0 in H_1), the selection of P means $u_i \sim H_2$ for each $0 \le i < p$, and $u_{-1}, u_p \not\sim H_2$ (it is immaterial whether $u_{-1} = u_p$ or not). In the following, we show that both ends of P belong to $\widehat{N}(H_2)$, which induces a clique (Proposition 4.3). Thus either $u_0 = u_{p-1}$ (i.e., $p = 1$) or u_0 and u_{p-1} are adjacent (i.e., $p = 2$); in either case, we have $u \in \{u_0, u_{p-1}\} \subseteq \widehat{N}(H_2)$. This proves the first assertion, and the second assertion ensues, as otherwise their common vertices will be common neighbors of H_2, which is not possible.

Note that $u_0 \notin H_2$, as otherwise u_{-1} is also adjacent to H_2, contradicting the maximality of P. Similarly, $u_{-1}, u_{-2} \notin H_2$. If u_0 has a unique neighbor v in H_2, then the subgraph induced by u_{-1}, u_0 and five consecutive H_2 vertices centered at v is a long claw (see Figure 7a). Now we consider the case $2 \le |N_{H_2}[u_0]| \le |H_2| - 7$ (Proposition 6.1), and number the vertices of H_2 such that $N_{H_2}[u_0] = \{v_1, v_2 \dots, v_q\}$. Note that $|N_{H_2}[u_0]| \le |H_2| - 7$ implies that $v_0 \ne v_{q+1}$. If u_{-2} is adjacent to v_0, v_1, v_q, or v_{q+1}, then there is a hole $(u_{-2}u_{-1}u_0v_1v_0u_{-2})$, $(u_{-2}u_{-1}u_0v_1u_{-2})$, $(u_{-2}u_{-1}u_0v_qu_{-2})$, or $(u_{-2}u_{-1}u_0v_qv_{q+1}u_{-2})$, respectively. Otherwise, $u_{-2} \not\sim \{v_0, v_1, v_q, v_{q+1}\}$, then there is a net $\{u_{-1}, u_0, v_0, v_1, v_{q+1}, v_q\}$ when $|N_{H_2}(u_0)| = 2$, or long claw $\{u_{-2}, u_{-1}, u_0, v_0, v_1, v_{q+1}, v_q\}$ when $|N_{H_2}(u_0)| > 2$ (see Figure 7b). This proves $u_0 \in \widehat{N}(H_2)$, and with a symmetrical argument we can prove $u_{p-1} \in \widehat{N}(H_2)$. □

We are now ready to establish the transitivity of the congenial relation. The reflexivity and symmetry of this relation are clear from definition; therefore congenial holes form an equivalence class.

Lemma 6.8. *Let H, H_1, and H_2 be three holes in a prereduced graph G. If both H_1 and H_2 are congenial to H, then H_1 and H_2 are congenial.*

Proof. According to Corollary 6.6, $\widehat{N}(H_1) = \widehat{N}(H) = \widehat{N}(H_2)$. If H_1 and H_2 are adjacent, then they have to be congenial, as otherwise Lemma 6.7 implies that one of them contains a common neighbor of the other, hence a common neighbor of all three holes, which is impossible. Assume hence that there is no edge between H_1 and H_2; in particular, they are disjoint. Let h be any vertex in H, and we number the vertices of H_1 and H_2 such that $N_{H_1}[h] = \{u_1, \dots, u_p\}$ and $N_{H_2}[h] = \{v_1, \dots, v_q\}$. Proposition 6.1 implies that $u_0 \ne u_{p+1}$ and $v_0 \ne v_{p+1}$. Note that h is adjacent to some but not all vertices of both H_1 and H_2. There is • a long claw $\{v_1, h, u_{-1}, u_0, u_1, u_2, u_3\}$ when $p = 1$; • a net $\{v_1, h, u_0, u_1, u_3, u_2\}$ when $p = 2$; or • a long claw $\{v_0, v_1, u_0, u_1, h, u_p, u_{p+1}\}$ when $p \ge 3$. □

To prove Theorem 2.2, we show that if there are two holes that are not congenial, then one of them is contained in a nontrivial module. This is impossible in a reduced graph, where every nontrivial module induces a clique. We construct this nontrivial module with the help of the following lemma, which shows that the common neighbors form a separator.

LEMMA 6.9. *Let H be a hole that is the shortest among all the holes congenial to it in a prereduced graph G. Then the set $\widehat{N}(H)$ of common neighbors of H separates $N[H] \setminus \widehat{N}(H)$ from $V(G) \setminus N[H]$.*

PROOF. Suppose to the contrary, $N[H] \setminus \widehat{N}(H)$ and $V(G) \setminus N[H]$ are still connected in $G - \widehat{N}(H)$, then there is a pair of adjacent vertices $u \in N[H] \setminus \widehat{N}(H)$ and $v \in V(G) \setminus N[H]$. Note that $u \notin H$, and we have two adjacent vertices only one of which is adjacent to part of the hole H. Depending on the number of neighbors of u in H, we have either a long claw (when $|N_H(u)| = 1$), a net (when $|N_H(u)| = 2$), or a †-AW of size 7 (when $|N_H(u)| = 3$), none of which can exist in a prereduced graph. On the other hand, if $|N_H(u)| > 3$ then we can use u to find another hole H' that is strictly shorter than H; it is surely congenial to H, which contradicts the assumption. □

We are now ready to prove Theorem 2.2:

Theorem 2.2 (restated). All holes in a reduced graph are congenial to each other.

PROOF. Suppose, for contradiction, that not all holes are congenial to each other. By Lemma 6.8, being congenial is an equivalence relation. Hence there are two equivalence classes of holes, from each of which we pick a shortest one; let them be H_1 and H_2. Assume without loss of generality that H_2 has a vertex v not in $N[H_1]$. Lemma 6.9 implies that $\widehat{N}(H_1)$ separates $N[H_1] \setminus \widehat{N}(H_1)$ and $V(G) \setminus N[H_1]$. Either $\widehat{N}(H_1) = \emptyset$ and then G is disconnected where $N[H_1]$ induces a connected component ($v \notin N[H_1]$); or $\widehat{N}(H_1)$ is the neighbor of $N[H_1]$ and they are completely connected (Corollary 6.3). In either case, the set $N[H_1] \setminus \widehat{N}(H_1)$ is a nontrivial module that does not induce a clique. Thus Reduction 2 is applicable and the graph is not reduced. □

7. HOLE COVERS

A set of vertices is called a *hole cover* of a graph G if it intersects every hole in G, and the removal of any hole cover makes the graph chordal. A hole cover is minimal if any proper subset of it is not a hole cover. Any interval deletion set makes a hole cover of the input graph, and thus contains a minimal hole cover. The goal of this section is to prove Theorem 2.3, that is, to provide a polynomial bound on the number of minimal hole covers in a reduced graph and give a polynomial time algorithm to find all of them.

To simplify the task, observe that no minimal hole cover contains a vertex that is not in any hole.

PROPOSITION 7.1. *Let $\mathcal{H}$ be the set of all holes in a reduced graph G, and G_0 be the subgraph induced by $\bigcup_{H \in \mathcal{H}} H$. A set HC of vertices is a minimal hole cover of G if and only if it is a minimal hole cover of G_0.*

In this section we will focus on the subgraph G_0 induced by the union of all holes in the reduced graph G. The subgraph G_0 has the same set of holes as G, and they remain pairwise congenial. Moreover, each vertex of G_0 is in the closed neighborhood of each hole H of G_0, which means G_0 is connected. As we have said earlier, circular-arc graphs form an important example of graphs of which all holes are pairwise congenial. Thinking of G_0 as a circular arc graph gives the intuition behind most statements to follow. But since this fact is not directly used in this paper, we are not giving a proof for it.

PROPOSITION 7.2. *The subgraph $G_0 - HC$ is an interval graph for each hole cover HC of G_0.*

PROOF. Each vertex of G_0 belongs to some hole, and thus cannot be simplicial. Therefore, by Theorem 2.1, G_0 contains no AW. By definition, $G_0 - HC$ contains no hole; thus $G_0 - HC$ is an interval graph. □

In what follows we prove a series of claims on how the neighborhood of a vertex v of a hole H_1 looks like in another hole H_2. The first statement is a paraphrase of Corollary 6.6:

COROLLARY 7.3. *No vertex v of G_0 can be a common neighbor of any hole in G_0.*

Therefore, by definition of congenial holes and Proposition 6.1, we can assume that for every $v \in V(G_0)$ and hole H, we have that $N_H[v]$ is a proper nonempty subset of H and its vertices induce a path in H. Fixing any ordering of the vertices in H, we can denote two ends of the path as $\texttt{begin}_H(v)$ and $\texttt{end}_H(v)$ respectively; when $N_H[v]$ contains both h_0 and $h_{|H|-1}$, we number vertices of $N_H[v]$ as $\{h_{-p}, \dots, h_0, \dots, h_q\}$ where both p and q are nonnegative, and then $\texttt{begin}_H(v) = -p$, $\texttt{end}_H(v) = q$.

PROPOSITION 7.4. *Let H be a hole of G_0. For any pair of adjacent vertices u, v of G_0, their closed neighborhoods in H satisfy the following properties.*

(1) $N_H[u] \cap N_H[v] \neq \emptyset$ and $N_H[u] \cup N_H[v] \neq H$.
(2) If v is adjacent to neither $h_{\texttt{begin}_H(u)}$ nor $h_{\texttt{end}_H(u)}$, then $N[v] \subset N[u]$.

PROOF. We number vertices of H such that $N_H[u] = \{h_0, \dots, h_\ell\}$; the order can be either way if $|N_H[u]| = 1$, i.e., $\ell = 0$.

Statement (1) holds trivially if either or both of u and v belong to H (Proposition 6.1). Hence we assume $u, v \notin H$. Suppose first, for contradiction, $N_H[u] \cap N_H[v] = \emptyset$; we may assume $\{h_{\ell_1}, \dots, h_{\ell_2}\} = N_H[v]$ where $\ell < \ell_1 \leq \ell_2 < |H|$. If $\ell_2 \geq |H|-3$, then $(uvh_{\ell_2} \dots h_{|H|}u)$ is a hole of length at most 6. Otherwise, $(uvh_{\ell_1} \dots h_\ell u)$ is a hole not congenial to H: in particular, the vertex $h_{|H|-2}$ in H is nonadjacent to it. In either case, we end with a contradiction; hence $N_H[u]$ and $N_H[v]$ must intersect. Suppose now, for contradiction, $N_H[u] \cup N_H[v] = H$. Then v is adjacent to every vertex in $(h_{\ell+1}h_{\ell+2} \cdots h_{|H|-1})$. Proposition 6.1 and Corollary 7.3 imply $6 < \ell < |H|-6$. If $v \not\sim h_\ell$, then $(uh_\ell h_{\ell+1}vu)$ is a 4-hole. A symmetric argument applies when $v \not\sim h_0$. Now suppose v is adjacent to both h_0 and h_ℓ, then $(uh_\ell h_{\ell+1} \cdots h_{|H|-1}h_0u)$ is a hole and v is a common neighbor of it (contradicting Corollary 7.3). None of the cases is possible, and hence $N_H[u] \cup N_H[v] \neq H$.

The condition of statement (2) means that $v \not\sim h_0$ and $v \not\sim h_\ell$. According to statement (1), and since $N_H[v]$ is consecutive in H, we must have $N_H[v] \subseteq \{h_1, \dots, h_{\ell-1}\}$. Note that $N_H[v]$ is nonempty and thus $\ell \geq 2$. If $u \in H$, then $N_H[v] = \{u\} = \{h_1\}$, and statement (2) follows from statement (1); here we use the fact that every $x \in N[v]$ is in the neighborhood of H and that u is the only neighbor of v in H. Assume now $u \notin H$; the argument below holds regardless of whether $v \in H$ or not. Let x be any vertex in $N[v]$ different from u, and we argue $x \sim u$. By statement (1), $N_H[x]$ must intersect $\{h_1, \cdots, h_{\ell-1}\}$. If x is nonadjacent to $\{h_0, h_\ell\}$, then $N_H[x]$ is also a subset of $\{h_1, \cdots, h_{\ell-1}\}$, and $x \sim u$ follows from Proposition 4.2(3) (taking $(h_{-1}h_0 \cdots h_\ell h_{\ell+1})$ as the path). Otherwise, x is adjacent to at least one of $\{h_0, h_\ell\}$, then $x \not\sim u$ will imply a 4-hole (h_0uvxh_0) or $(h_{\ell+1}uvxh_0)$, which is impossible. The proof is now completed. □

Noting that the closed neighborhoods of two consecutive vertices in a hole are incomparable, Proposition 7.4(2) has the following corollary.

COROLLARY 7.5. *Let H and H_1 be two holes in G_0. For each pair of consecutive vertex $u_i, u_{i+1} \in H_1$, at least one end of $N_H[u_i]$ is in $N_H[u_{i+1}]$.*

The following lemmas characterize minimal hole covers of G_0.

LEMMA 7.6. *Any minimal hole cover of G_0 induces a clique.*

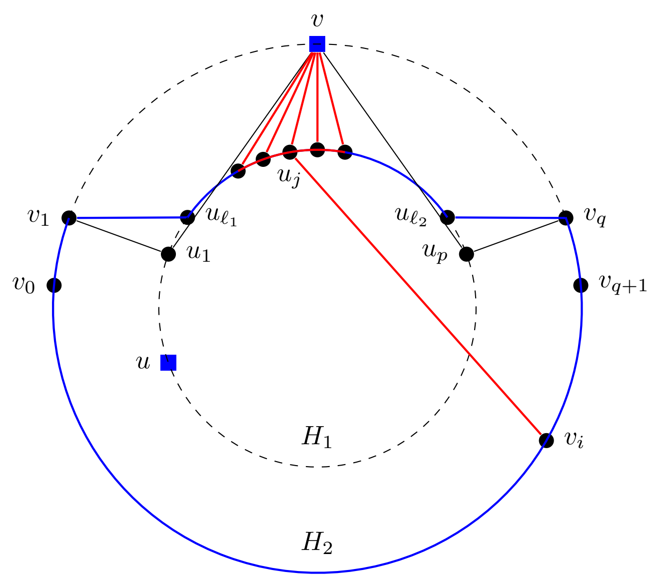


Fig. 8: Illustration for the proof of Lemma 7.6.

Proof. Suppose to the contrary, there is a minimal hole cover HC that contains two nonadjacent vertices u and v. By the minimality of HC, there are two (unnecessarily disjoint) holes H_1 and H_2 such that $HC \cap H_1 = \{u\}$ and $HC \cap H_2 = \{v\}$. In particular, $u \notin H_2$ and $v \notin H_1$. We number the vertices of H_1 such that $N_{H_1}[v] = \{u_1, u_2, \cdots, u_p\}$. The union of $N_{H_2}[u_1]$ and $N_{H_2}[u_p]$ is a consecutive set of vertices in H_2: they both contain v, and, by Proposition 6.1, are consecutive in H_2. We number the vertices of H_2 such that $u_1 \sim v_1$ and $N_{H_2}[u_1] \cup N_{H_2}[u_p] = \{v_1, \ldots, v_q\}$.

Claim 3. *At least one vertex of H_2 is adjacent to neither u_1 nor u_p.*

Proof. The claim follows from Proposition 7.4(1) when $p = 2$; hence we may assume $p > 2$, which means $u_1 \not\sim u_p$ (note that $u_0 \neq u_{p+1}$). Suppose $N_{H_2}[u_1] \cup N_{H_2}[u_p] = H_2$, then by Proposition 6.1, we have $7 < |N_{H_2}[u_1]| < |H_2| - 7$, which means at least one end of the path induced by $N_{H_2}[u_1]$ is not adjacent to v. Without loss of generality, let it be v_i where $i = \mathtt{end}_{H_2}[u_1]$; noting that by assumption $v_{i+1} \sim u_p$, there is either a 4-hole $(vu_1v_iu_pv)$ (if $v_i \sim u_p$) or a 5-hole $(vu_1v_iv_{i+1}u_pv)$ (if $v_i \not\sim u_p$). ⌟

In what follows we show the existence of a hole in $G - HC$, which contradicts the assumption that HC is a hole cover and thus proves this lemma. Denote by $P_1 = (u_1u_2 \ldots u_p)$ and $P_2 = (v_qv_{q+1} \ldots v_0v_1)$. By definition $u \notin P_1$; to show $v \notin P_2$ it suffices to rule out the possibility that $v \in \{v_1, v_q\}$, as by the numbering of H_2, v is in $\{v_1, v_1, \ldots, v_q\}$. According to Corollary 7.5, the two neighbors of v in H_2 are adjacent to either u_1 or u_p; however, Claim 3 implies that v_0 and v_{q+1} are inner vertices of P_2 and hence are not adjacent to u_1 or u_p. We now argue that each inner vertex v_i of P_2 is not adjacent to P_1 (see the thick edges in Figure 8). Suppose to the contrary, v_i is adjacent to P_1. Noting that $v_i \not\sim u_1$, $v_i \not\sim u_p$, and $u_1 \neq u_{p+1}$, Proposition 4.2(3) applies, and we can conclude $v_i \sim v$, which is impossible. (It is immaterial whether $v_i \in H_1$ or not.) Now we construct the hole in $G - HC$ as follows. Claim 3 implies that the length of P_2 is at least 2. If $u_1 \sim v_q$, then $(u_1P_2u_1)$ is such a hole. Otherwise by assumption we have $u_p \sim v_q$. Let $\ell_1 = \max\{i : u_i \sim v_1 \text{ and } 0 \leq i \leq p\}$, and

$\ell_2 = \min\{i : u_i \sim v_q \text{ and } \ell_1 \leq i \leq p\}$. Then $(u_{\ell_1} u_{\ell_2} P_2 u_{\ell_1})$ will be such a hole (see the solid hole in Figure 8). $\square$

LEMMA 7.7. *For any minimal hole cover HC of G_0 and any shortest hole H, there is a vertex $v \in H$ such that $N_{G_0}[v] \not\sim HC$.*

PROOF. We show this by construction. By Corollaries 6.2 and 7.3, each vertex in G_0 has at most 3 neighbors in H. By Lemma 7.6, HC is a clique and hence $|H \cap HC| \leq 2$. We number the vertices of H in a way that $h_0 \in HC$ and $h_1 \notin HC$, and claim that $v = h_5$ is the asserted vertex. Suppose to the contrary, $N_{G_0}[h_5]$ and HC are adjacent, then there is an h_0-h_5 path P of length at most 3 and all its inner vertices belong to G_0. The case $P = (h_0 v h_5)$ is impossible, as by Proposition 6.1 and Corollary 6.2, v is adjacent to at most 3 consecutive vertices in H. Now we may assume $P = (h_0 v_1 v_2 h_5)$, and examine the neighbors of v_1 and v_2 in H. By Corollary 6.2, we have $\texttt{end}_H(v_1) \leq 2$ and $\texttt{begin}_H(v_2) \geq 3$. This means that there is a hole $(v_1 h_i h_{i+1} \dots h_j v_2 v_1)$, where $i = \texttt{end}_H(v_1)$ and $j = \texttt{begin}_H(v_2)$, of length at least 4 and at most 8. $\square$

We now relate minimal hole covers of G_0 to minimal separators in some interval subgraphs. In one direction of the proof, we need the following claim. Observe that in an interval representation of a connected interval graph, the union of all the intervals also forms an interval. Similarly, if there is a point p in the real line such that there are intervals not containing p both to the left and to the right of p, then the set of intervals containing p is a clique separator.

PROPOSITION 7.8. *Let v be a vertex in an interval graph G. If v is not adjacent to any simplicial vertex, then $N[v]$ is a separator of G.*

PROOF. We consider an interval representation of G. Without loss of generality, we assume that no two intervals have the same ends. Denote by x the interval with the smallest right end, and by y the interval with the largest left end. It is easy to see that x and y are simplicial. If $x \sim y$, then the graph is a complete graph (every interval contains the interval between the left end of y and the right end of x); thus every vertex is adjacent to a simplicial vertex, and the assertion is vacuously true. Therefore, we can assume $x \not\sim y$, and let p be an arbitrary point in interval v. By assumption v is not adjacent to x or y, which means that x is to the left of p and y is to the right of p. As every interval that contains p is in $N[v]$, in the subgraph $G - N[v]$ that contains x and y, no interval contains p; hence x and y are disconnected. In other words, $N[v]$ is an x-y separator. $\square$

According to Lemma 7.7, every minimal hole cover satisfies the condition in the following lemma; hence the lemma applies to all of them. Note that $G_0 - N_{G_0}[v]$ is the same as $G_0 - N[v]$.

LEMMA 7.9. *Let v be a vertex in a shortest hole H of G_0, and X induce a clique nonadjacent to $N_{G_0}[v]$. Set X forms a minimal hole cover of G_0 if and only if X is a minimal separator of $G_0 - N[v]$.*

PROOF. It suffices to show that X is a hole cover of G_0 if and only if it is a separator of $G_0 - N[v]$.

$\Rightarrow$ Clearly, each component of $G_0 - N[v]$ contains a neighbor of $N[v]$. As X is not adjacent to $N[v]$, the set X cannot fully contain a component of $G_0 - N[v]$, which implies that the number of components of $G_0 - N[v] - X$ is no less than that of $G_0 - N[v]$. Therefore, if $G_0 - N[v]$ is not connected, then neither is $G_0 - N[v] - X$, and X makes a trivial separator for $G_0 - N[v]$. In the following argument of this direction we may assume $G_0 - N[v]$ is connected, and it suffices to show that $G_0 - N[v] - X$ is not connected. By Proposition 7.2, $G_0 - X$ is an interval subgraph; as G_0 itself contains no simplicial vertex, any vertex x that is simplicial in $G_0 - X$ must be a neighbor of X: otherwise $N_{G_0 - X}(x) = N_{G_0}(x)$ and cannot be a clique. As $N[v]$ is not adjacent to X by assumption, v is not adjacent to any simplicial vertex of the interval graph $G_0 - X$. Therefore, according to Proposition 7.8, the removal of $N[v]$ disconnects $G_0 - X$. This finishes the proof of the "only if" direction.

$\Leftarrow$ Let us start from a close scrutiny of $G_0 - N[v]$. According to Proposition 6.1, the removal of $N[v]$ transforms each hole into a path of length at least 7; in particular, let P be the path induced by $H \setminus N_H[v]$. In the argument to follow, we show that ends of each such path are connected to the ends of P respectively; the further removal of X separates each path into at most two sub-paths; hence if there is a hole disjoint from X, then the path left by it is able to connect every sub-path and thereby every vertex, which is impossible.

We number the vertices of H such that $v = v_0$. Then $N_H[v] = \{v_{-1}, v_0, v_1\}$ and the ends of P are v_{-2} and v_2. Let H' be another hole of G_0, and P' be the path induced by $H' \setminus N_{H'}[v]$. We may number the vertices of H' such that $N_{H'}[v] = \{h_1, \dots, h_p\}$. As a result, the ends of P' are h_0 and h_{p+1}.

CLAIM 4. *The ends h_0 and h_{p+1} of P' are adjacent to $\{v_1, v_2\}$ and $\{v_{-1}, v_{-2}\}$, respectively.*

PROOF. By Corollary 6.2, $N_H[h_1] \subset \{v_{-2}, v_{-1}, v_0, v_1, v_2\}$, and according to Proposition 7.4(1), h_0 is adjacent to either $\{v_1, v_2\}$ or $\{v_{-1}, v_{-2}\}$; a symmetric argument works for h_{p+1}. Since the length of H is at least 11, the sets $\{v_1, v_2\}$ and $\{v_{-1}, v_{-2}\}$ are disjoint. It remains to show that the ends of P' cannot be adjacent to both $\{v_1, v_2\}$ or both $\{v_{-1}, v_{-2}\}$. Suppose, for contradiction, both h_0 and h_{p+1} are adjacent to $\{v_1, v_2\}$. We consider three cases.

Suppose first that both h_0 and h_{p+1} are adjacent to v_1. According to Corollary 7.5 (applied on the adjacent vertices v_0, v_1 and the hole H'), at least one end of $N_{H'}[v_1]$ is in $N_{H'}[v_0]$, i.e., $\{h_1, \dots, h_p\}$. As a result, $N_{H'}[v_1]$ must contain all of $\{h_{p+1}, h_{p+2}, \dots, h_0\}$, and then $N_{H'}[v_0] \cup N_{H'}[v_1] = H'$. This contradicts Proposition 7.4(1) and is impossible.

Suppose now that h_0 and h_{p+1} are both adjacent to v_2. Note that $v_2 \notin H'$; otherwise, since $v_0 \not\sim v_2$, we must have $v_2 = h_{-1}$, and then $(v_0 h_1 h_0 v_2 h_{p+1} h_p v_0)$ is a 6-hole, which is impossible. We can apply Proposition 4.2(1) on v_2, v_0, and path $(h_{-1} h_0 h_1 \dots h_p h_{p+1} h_{p+2})$ to conclude $v_0 \sim v_2$, which is impossible.

In the remaining cases, h_0 and h_{p+1} are adjacent to v_1 and v_2, respectively. Without loss of generality, we consider $h_0 \sim v_1$ and $h_{p+1} \sim v_2$. Clearly $h_p \neq v_2$ as they have different adjacencies to v_0; likewise, $h_{p+1} \neq v_1$ and $h_{p+1} \neq v_2$. We exclude $h_p = v_1$: then $p = 1$ by $h_0 \sim v_1$, and $v_0 \sim h_0$ by Corollary 7.5, which contradicts the numbering of H'. Then $h_p \sim v_2$, as otherwise there is a hole $(v_2 v_1 v_0 h_p h_{p+1} v_2)$ or $(v_2 v_1 h_p h_{p+1} v_2)$; likewise, $h_p \sim v_1$. It follows that, by Corollary 6.2, $h_p \not\sim v_{-1}$, and $p > 1$. On the other hand, $h_0 \not\sim v_{-1}$, as otherwise there is a hole $(h_0 v_{-1} v_0 v_1 h_0)$. We can apply Proposition 4.2(3) on v_1, v_{-1}, and path $(h_{-1} h_0 h_1 \dots h_p v_{p+1})$ to conclude $v_{-1} \sim v_1$, which is impossible.

A symmetric argument applies to $\{v_{-1}, v_{-2}\}$. $\lrcorner$

We may assume without loss of generality, $h_{p+1} \sim \{v_1, v_2\}$, and then $h_0 \sim \{v_{-1}, v_{-2}\}$. Let ℓ_1 be the smallest index such that $\ell_1 > p$ and $h_{\ell_1} \in N[v_2]$; for its existence, observe that $\ell_1 = p + 1$ if $h_{p+1} \in N[v_2]$, otherwise by Corollary 7.5, $h_{\texttt{end}_{H'}(v_1)}$ must be in $N[v_2]$. For $p + 1 \leq i \leq \ell_1$, it holds that $h_i \sim v_1$, and since $v_1 \in N[v]$, by assumption ($X \not\sim N[v]$), we have $h_i \notin X$. Symmetrically, we can find the largest index ℓ_2 such that $\ell_2 \leq |H|$ and $h_{\ell_2} \in N[v_{-2}]$. For $\ell_2 \leq i \leq |H|$, it holds that $h_i \sim v_{-1}$ and $h_i \notin X$. On the other hand, as X is a clique, it contains at most two vertices of P', which are in $\{h_{\ell_1+1}, \dots, h_{\ell_2-1}\}$. Therefore, the removal of X either leaves P' intact (when X is disjoint from P'), or separates P' into two sub-paths. In the later case, the two sub-paths, containing h_{ℓ_1} and h_{ℓ_2} respectively, are connected to v_2 and v_{-2} respectively.

To prove the "if" direction, we need to show that X intersects every hole. Suppose, for contradiction, that X is disjoint from some hole H_1. Then the path P_1 induced by $H_1 \setminus N_{H_1}[v]$

remains a path of $G_0 - N[v] - X$. Since P_1 is adjacent to both v_2 and v_{-2}, we conclude that v_2 and v_{-2} are connected in $G_0 - N[v] - X$. We have seen that for each hole H', the vertices left from H' after the removal of $N[v]$ and X are connected to at least one of v_2 and v_{-2}. Therefore, the subgraph $G_0 - N[v] - X$ is connected, contradicting the assumption that X is a separator of $G_0 - N[v]$. This finishes the proof of the "if" direction. $\square$

We are now ready to prove Theorem 2.3. We remark that the quadratic bound can be improved to linear with more careful analysis.

Theorem 2.3 (restated). Every reduced graph of n vertices contains at most n^2 minimal hole covers, and they can be enumerated in $O(n^3)$ time.

PROOF. Let G_0 be induced by the union of the holes of G. On the one hand, according to Lemmas 7.7 and 7.9, each minimal hole cover of G corresponds to a minimal separator of the interval subgraph $G_0 - N[v]$ for some vertex v of a shortest hole H. On the other hand, there are at most n minimal separators in $G_0 - N[v]$ for each vertex $v \in H$, which implies a quadratic bound for the total number of minimal hole covers of G. To enumerate them, we try every vertex $v \in H$ and enumerate all minimal separators of $G_0 - N[v]$. $\square$

8. CATERPILLAR DECOMPOSITIONS

This section proves Theorem 2.4 by providing the claimed algorithm for INTERVAL DELETION on nice graphs. Recall that a nice graph is chordal and contains no small AW, and every shallow terminal in a nice graph is simplicial; nice graphs are hereditary. Our algorithm finds an AW satisfying a certain minimality condition, from which we can construct a set of ten vertices that intersects some minimum interval deletion set. Hence it branches on deleting one of these ten vertices. The set of all shallow terminals, denoted by $ST(G)$, can be found in polynomial time as follows. For each triple of vertices, we check whether or not they forms the terminals for an AW. If yes, then one of them is necessarily shallow. The following lemma ensures that all shallow terminals can be found as such.

PROPOSITION 8.1. *In a nice graph, all AWs with the same set of terminals have the same shallow terminal.*

PROOF. Of any AT $\{x, y, z\}$, there must be a vertex, say, x, such that the shortest y-z path in $G - N[x]$ has length at least 4, as otherwise there is an AW of size at most 9, which contradicts the definition of nice graphs. Therefore, neither y nor z can be the shallow terminal in an AW with terminals $\{x, y, z\}$. $\square$

It should be noted that this does not rule out the possibility of a vertex being a base terminal of an AW and the shallow terminal of another AW. If this happens, these AWs necessarily have at least one different terminal. Recall that by Theorem 2.1, every vertex in $ST(G)$ is simplicial in G. For each †- or ‡-AW, its shallow terminal is in $ST(G)$ by definition, its base terminals might or might not be in $ST(G)$, and none of the non-terminal vertices can be in $ST(G)$ (as they are not simplicial). From Lemma 5.2 we can derive

PROPOSITION 8.2. *Let s be a shallow terminal in a nice graph. There is an AW of which every base vertex is adjacent to all vertices of $N(s) \setminus ST(G)$.*

PROOF. Let W be an AW with shallow terminal s and shortest possible base. Applying Lemma 5.2 on any vertex $x \in N(s) \setminus ST(G)$ and W, it cannot be in category "partial" by the minimality of W. Vertex x cannot be in category "none" either, otherwise x is a shallow terminal, contradicting $x \in N(s) \setminus ST(G)$. Thus every vertex in $N(s) \setminus ST(G)$ is in category "full." $\square$

Now that the graph is chordal, it makes sense to discuss its clique tree, which shall be the main structure of this section. No generality will be lost by assuming G is connected.

Since no inner vertex of a shortest path can be simplicial, the removal of simplicial vertices will not disconnect a connected graph; hence $G - ST(G)$ is a connected interval graph. This observation suggests a clique tree of G with a very nice structure. A *caterpillar (tree)* is a tree that consists of a central path and all other vertices are leaves connected to it.

PROPOSITION 8.3. *In polynomial time we can construct a clique tree $\mathcal{T}$ for a connected nice graph G such that*

— *$\mathcal{T}$ is a caterpillar;*
— *every shallow terminal of G appears only in one leaf node of $\mathcal{T}$; and*
— *every other vertex in G appears in some nodes of the central path of $\mathcal{T}$.*

PROOF. Let us inspect every maximal clique K of G. If K contains some shallow terminal s, then K must be $N[s]$: Being a clique, $K \subseteq N[s]$; this containment cannot be proper as K is maximal and $N[s]$ induces a clique. Otherwise, $K \cap ST(G) = \emptyset$, then K is also a maximal clique of $G - ST(G)$. On the other hand, a maximal clique K' of $G - ST(G)$ has to be a maximal clique of G as well; otherwise it is a proper subset of some maximal clique K of G that must contain a shallow terminal s, hence $K = N[s]$ and $K' \subseteq N[s] \setminus ST(G)$, which, however, according to Proposition 8.2, cannot be maximal in $G - ST(G)$. Therefore, a maximal clique of G is either $N[s]$ for some vertex $s \in ST(G)$ or a maximal clique of $G - ST(G)$. We construct the claimed clique tree as follows. First use Theorem 3.1 to make a clique path $\mathcal{T}'$ for the interval subgraph $G - ST(G)$, then for each $s \in ST(G)$, attach $N[s]$ as a leave to $\mathcal{T}'$ at some maximal clique that properly contains $N[s] \setminus ST(G)$ (arbitrarily pick from multiple choices). □

Within a caterpillar decomposition, we number the nodes in the central path as $K_0, K_1, \ldots$. By Proposition 8.3 and the definition of clique trees, each vertex not in $ST(G)$ is contained in some consecutive nodes of the central path. For each vertex $v \notin ST(G)$, we denote by $\texttt{first}(v)$ and $\texttt{last}(v)$ the smallest and, respectively, largest indices of nodes that contain v. In any †- or ‡-AW, every vertex of the base is non-simplicial, hence belongs to the central path of the caterpillar decomposition. By assumption, $d = |B| \geq 3$ and $b_1 \not\sim b_d$; as a result, the nodes that contain b_1 and b_d are disjoint. When numbering the vertices of the base, we follow the convention that $\texttt{last}(b_1) < \texttt{first}(b_d)$, i.e., base B goes "from left to right." Given a numbering of the base, the base terminals l and r can be distinguished from each other based on their adjacency with b_1 and b_d. Similarly, in the case of a ‡-AW, the centers c_1 and c_2 can be distinguished from each other, as they have different adjacency relations with l and r.[3]

By observing the adjacencies and nonadjacencies between vertices of an AW and their possible positions in an interval representation of $G - ST(G)$, the following is straightforward and hence stated here without proof. In order to avoid pointless repetition, we are again using the same generalized notation for both †- and ‡-AW as stipulated in Section 4.

PROPOSITION 8.4. *Let $(s : c_1, c_2 : l, B, r)$ be a †- or ‡-AW in a nice graph G. In a caterpillar decomposition of G,*

$$\begin{aligned}\big(\texttt{first}(b_1) \leq & \texttt{last}(l) <\big)\ \texttt{first}(c_2), \texttt{first}(b_2) \leq \texttt{last}(b_1) < \ldots \\ & \leq \texttt{first}(b_i) \leq \texttt{last}(b_{i-1}) < \texttt{first}(b_{i+1}) \leq \texttt{last}(b_i) < \texttt{first}(b_{i+2}) \\ \leq \cdots < & \texttt{first}(b_d) \leq \texttt{last}(b_{d-1}), \texttt{last}(c_1)\ \big(< \texttt{first}(r) \leq \texttt{last}(b_d)\big), \end{aligned} \tag{1}$$

where relations in parentheses only hold when $l \notin ST(G)$ and $r \notin ST(G)$, respectively.

[3]Note that we are not relying on the relation between $\texttt{first}(c_1)$ and $\texttt{first}(c_2)$ or that between $\texttt{last}(c_1)$ and $\texttt{last}(c_2)$, and they will not matter in the proofs to follow.

Nodes that contain non-terminal vertices of an AW appear consecutively in the central path of $\mathcal{T}(G)$. We would like to identify a minimum set of consecutive nodes whose union contains all non-terminal vertices of the AW.

Definition 8.5. Let $\mathcal{T}$ be a caterpillar decomposition of a nice graph G. We define $\mathbb{I}[p,q] = \bigcup_{p \le i \le q} K_i$ for a pair of indices $p \le q$, and $\mathbb{I}(W) = \mathbb{I}[\texttt{last}(b_1), \texttt{first}(b_d)]$ for an AW W. Set $\mathbb{I}(W)$ will be referred to as the *container* of W, and we say it is *minimal* if there exists no AW W' such that $\mathbb{I}(W') \subset \mathbb{I}(W)$.

Let us observe that every base vertex of W appears in $\mathbb{I}(W)$ and no shorter subsequence of nodes contains every base vertex. Moreover, the following proposition shows that the centers also appear in $\mathbb{I}(W)$ (recall that $\widehat{N}(B)$ is the set of common neighbors of B and every center is in $\widehat{N}(B)$).

PROPOSITION 8.6. *$K_{\texttt{last}(b_1)} \cap K_{\texttt{first}(b_d)} = \widehat{N}(B)$.*

PROOF. By definition, a vertex of the left side is in K_i for every $\texttt{last}(b_1) \le i \le \texttt{first}(b_d)$, and thus belongs to $\widehat{N}(B)$. On the other hand, if a vertex v does not belong to the left side, then either $\texttt{first}(v) > \texttt{last}(b_1)$ or $\texttt{last}(v) < \texttt{last}(b_d)$, which implies $v \not\sim b_1$ or $v \not\sim b_d$ respectively. In either case, we have $v \notin \widehat{N}(B)$. $\square$

In Section 6, we considered holes of the shortest length and observed that a vertex sees either all or at most 3 vertices in such a hole. Here for an AW whose container is minimal and base consists of the inner vertices of a shortest l-r path specified below, we can observe an analogous statement about the number of base vertices a vertex can see.

Definition 8.7. Let $W = (s : c_1, c_2 : l, B, r)$ be an AW in a nice graph such that $\mathbb{I}(W)$ is minimal. We say B is a *short* base if (lBr) is a shortest l-r path in the subgraph induced by $\big(\mathbb{I}(W) \setminus \widehat{N}(B)\big) \cup \{l, r\}$.

The following lemma shows that if the base is not short, then we can get an AW with a shorter base. In particular, this implies that a vertex of $\mathbb{I}(W) \setminus \widehat{N}(B)$ can see at most three consecutive vertices of the base.

LEMMA 8.8. *Let $W = (s; c_1, c_2; l, B, r)$ be an AW such that $\mathbb{I}(W)$ is minimal. Then there is an W' such that $\mathbb{I}(W') = \mathbb{I}(W)$ and W' has a short base.*

PROOF. We show that if (lPr) is a chordless l-r path in the subgraph induced by $\big(\mathbb{I}(W) \setminus \widehat{N}(B)\big) \cup \{l, r\}$, then we can replace the base B of W by P to obtain another AW $W_P = (s : c_1, c_2 : l, P, r)$. Clearly the center(s) of W belong to $\widehat{N}(B)$, thereby adjacent to every other vertex in $\mathbb{I}(W)$, and hence to P. It is also easy to verify that no vertex in $\mathbb{I}(W) \setminus \widehat{N}(B)$ is adjacent to s: if such a vertex exists, then Lemma 5.2 classifies it as "partial" with respect to W, hence there is another AW W' such that $B' \subset B$ and $\mathbb{I}(W') \subset \mathbb{I}(W)$, which contradicts the minimality of $\mathbb{I}(W)$. Therefore, W_p is indeed an AW. Letting b'_1 and $b'_{d'}$ be the first and, respectively, last vertices of P, the selection of P implies $\texttt{last}(b'_1) \ge \texttt{last}(b_1)$ and $\texttt{first}(b'_{d'}) \le \texttt{first}(b_d)$, hence $\mathbb{I}(W_P) \subseteq \mathbb{I}(W)$; as the latter is already minimal, they must be equal. Therefore, if the base of W is not short, then we can find another AW with the same container and shorter base. Applying this argument repeatedly will eventually procure an AW with the same container and having a short base. $\square$

With all pertinent definitions and observations, we are now ready to present the main lemma of this section which justifies our branching rule. Without an upper bound on the number of vertices in an AW—in particular, the length of its base can be arbitrarily long—trying each vertex in it cannot be done in FPT time. Thus we have to avoid most but a (small) constant number of base vertices—those are close to the base terminals—to procure the claimed algorithm. To further decrease the number of vertices we need to consider, observing that the central path of the caterpillar decomposition has a linear structure, we start from the *leftmost* minimal container. By definition, minimal containers cannot properly contain each other, and thus the one with smallest begin-index also has the smallest end-index. In particular, the leftmost minimal container is unique, though it might be observed by more than one AWs, and can be identified in polynomial time. With this additional condition, if another AW intersects $\mathbb{I}(W)$, it has to come "from the right."

Let W be an AW of leftmost minimal container and having a short base. We claim that there is a minimum interval deletion set that breaks W in a canonical way: it contains either one of a constant number of specific vertices of W, or a specific minimum separator (details are given below) breaking the base of W. Therefore, by branching into ten directions, we can guess one vertex of this interval deletion set.[4]

For each $\texttt{last}(b_1) \leq i < \texttt{first}(b_{d-1})$, let us define $S_i = K_i \cap K_{i+1}$ to be the *ith separator.* Note that S_i contains $\widehat{N}(B)$ as a proper subset.

LEMMA 8.9. *Let $\mathcal{T}$ be a caterpillar decomposition of a nice graph G, and $W = (s : c_1, c_2 : l, B, r)$ be an AW in G such that*

— $\texttt{first}(b_d)$ *is the smallest among all AWs;*
— $\mathbb{I}(W)$ *is minimal; and*
— B *is a short base.*

Let ℓ be the minimum index such that $\texttt{last}(b_1) \leq \ell < \texttt{first}(b_{d-2})$ and the cardinality of S_ℓ is minimum among $\{S_i : \texttt{last}(b_1) \leq i < \texttt{first}(b_{d-2})\}$. There is a minimum interval deletion set to G that either contains one of the 9 vertices

$$V_B = \{s, c_1, c_2, l, b_1, b_{d-2}, b_{d-1}, b_d, r\},$$

or the whole set $X = S_\ell \setminus N$, where $N = \widehat{N}(B)$.

PROOF. We prove by construction. Let Q be any minimum interval deletion set; we may assume $Q \cap V_B = \emptyset$, and $X \not\subseteq Q$, as otherwise Q satisfies the asserted condition and we are finished. We claim $Q' = (Q \setminus V_I) \cup X$, where $V_I = \mathbb{I}[\texttt{last}(b_2), \texttt{first}(b_{d-3})] \setminus N$, is the desired interval deletion set, which fully contains X in particular.

As G is chordal, all minimal forbidden induced subgraphs in G are AWs. To show that Q' makes an interval deletion set to G, it suffices to argue that if there exists an AW W' avoiding Q' then we can also find an AW W'', not necessarily the same as W', avoiding Q. Suppose $W' = (s' : c'_1, c'_2 : l', B', r')$ is the AW in $G - Q'$. By the construction of Q', this AW must intersect $V_I \setminus X$; let $u \in W' \cap (V_I \setminus X)$. Clearly, u can neither be s', as $u \notin ST(G)$, nor r', as otherwise according to Proposition 8.4, $\texttt{first}(b'_{d'}) < \texttt{first}(u) < \texttt{first}(b_d)$, contradicting the selection of W. The following claim rules out the possibility that $u \in \{c'_1, c'_2\}$.

CLAIM 5. *For each vertex $v \in \mathbb{I}[0, \texttt{first}(b_{d-2})] \setminus N$, we have $\texttt{last}(v) < \texttt{first}(b_d)$, and $v \not\sim ST(G)$.*

PROOF. By definition, if v is adjacent to B, then $v \sim b_i$ for some $i \leq d-3$. If $v \sim b_d$, then B is not a short base, as there would be a a shorter (not necessarily chordless) l-r path $(l, \ldots, b_i, v, b_d, r)$. Therefore, $v \not\sim b_d$ and it follows that $\texttt{last}(v) < \texttt{first}(b_d)$. Suppose to the contrary of the second assertion, v is adjacent to the shallow terminal x of some AW W_1. We apply Lemma 5.2(2) on v and W_1. As $v \notin ST(G)$, it has to be in categories

[4]A slightly weaker version of Lem 8.9 is given in the appendix. The proof of Lem 8.9, trying to minimizing the number of branching directions, has to consider many cases and is ponderous. In contrast, the proof of the weaker version uses only the fact that a vertex that is not a common neighbor of B sees at most three vertices in it; hence the underlying ideas are easier to understand.

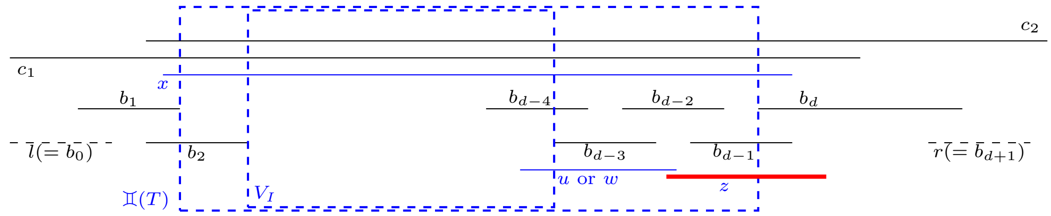


Fig. 9: Interval representation of non-terminal vertices of a leftmost minimal AW.

"full" or "partial." In either case, there exists an AW whose base is fully contained in $\mathbb{I}[\texttt{first}(v), \texttt{last}(v)]$, contradicting the selection of W. ⌟

Therefore, either $u = l'$ or $u \in B'$. Now we focus on the chordless path $l'B'r'$, which we shall refer to by P', and how it reaches u when going from r' to l'. Recall that every vertex of B' appears in the central path of the caterpillar decomposition. Figure 9 depicts non-terminal vertices of W in an interval representation of the interval subgraph $G - ST(G)$, where base terminals l and r are illustrated with dashed lines as they might belong to $ST(G)$. The main observation here is: for any vertex u in V_I, if another vertex $z \in N(u) \setminus N$ (the thick segment) reaches outside of $\mathbb{I}(W)$, then $z \sim b_d$, and u and z will make a short cut between b_{d-4} and b_d, which is impossible.

Claim 6. $B' \cap N = \emptyset$.

Proof. Suppose the contrary and let x be a vertex in $B' \cap N$ (see Figure 9). Then $s \sim x$ follows from Lemma 5.1. We claim that every neighbor z of u is adjacent to x. Note that $z \not\sim x \in N$ implies either $\texttt{last}(z) < \texttt{last}(b_1)$ or $\texttt{first}(z) > \texttt{first}(b_d)$ and the latter is ruled out by the definition of u and Claim 5. Let B_1 be the subset of the inner vertices of the z-r path $(z, u, \ldots, b_d, r)$. Then one of the following AW contradicts the minimality of the choice of W: $(s : c_1, x : z, B_1, r)$ (when $c_1 \sim z$ and $x \sim r$), $(s : c_1 : z, B_1, r)$ (when $c_1 \not\sim z$), or $(s : x : z, B_1, r)$.

As x and u are both in the chordless path P' and x is adjacent to every neighbor of u, vertex u has to be one end of P'. More specifically, $u = l'$ and $x = b'_1$. A further consequence is that u is the only vertex in $W' \cap V_I$: the argument above applies to any vertex $u' \in W' \cap V_I$, and thus $u' = l' = u$. Now we show, for any vertex w in $X \setminus Q$, which is nonempty by assumption, it has the same neighbors as u in W', and hence $(s' : c'_1, c'_2 : w, B', r')$ is an AW in $G - Q$, contradicting the assumption that Q is an interval deletion set to G.

First, if a vertex is in N, then it is adjacent to both w and u. Vertex b'_1 $(= x)$ is in N. We claim that c'_1 is also in N when W' is a ‡-AW. Otherwise, observe that $\texttt{first}(c'_1) \leq \texttt{last}(u) < \texttt{first}(b_d)$ as $c'_1 \sim u$ and u satisfies the condition of Claim 5. Let B_1 be the path $(b_1, \ldots, b_i, u, c'_1)$, where b_i is the first base vertex of W adjacent to u. Now one of the following AW contradicts the minimality of W: $(s : x, c_2 : l, B_1, b'_{d'})$ (when $x \sim l$ $c_2 \sim b'_{d'}$), $(s : x : l, B_1, b'_{d'})$ (when $x \not\sim l$), or $(s : c_2 : l, B_1, b'_{d'})$ (when $c_2 \not\sim b'_{d'}$).

Second, $c'_2 \not\sim u$ $(= l')$ implies $c'_2 \notin N$. We claim that $c'_2 \not\sim w$. Otherwise, observe that $\texttt{first}(c'_2) \leq \texttt{last}(w) < \texttt{first}(b_d)$ as $c'_2 \sim w$ and w satisfies the condition of Claim 5. Let B_1 be the path $(b_1, \ldots, b_i, w, c'_2)$, where b_i is the first base vertex of W adjacent to w. Now one of the following AWs contradicts the minimality of W: $(s : x, c_2 : l, B_1, b'_{d'})$ (when $x \sim l$ and $c_2 \sim b'_{d'}$), $(s : x : l, B_1, b'_{d'})$ (when $x \not\sim l$), or $(s : c_2 : l, B_1, b'_{d'})$ (when $c_2 \not\sim b'_{d'}$).

Finally, we claim that $w \not\sim b'_2$. Otherwise, observe that $\texttt{first}(b'_2) \leq \texttt{last}(w) < \texttt{first}(b_d) < \texttt{first}(b'_3)$ as $b'_2 \sim w$, w satisfies the condition of Claim 5, and $b'_3 \not\sim x$ $(= b'_1 \in N)$. Let B_1 be the path $(b_1, \ldots, w, b'_2)$. Now one of the following AWs contradicts the minimality of W: $(s : x, c_2 : l, B_1, b'_3)$ (when $x \sim l$ and $c_2 \sim b'_3$), $(s : c_2 : l, B_1, b'_3)$ (when $c_2 \not\sim b'_3$), or $(s : x : l, B_1, b'_3)$ (when $x \not\sim l$). Moreover, from $\texttt{last}(w) < \texttt{first}(b'_2) < \texttt{first}(b'_i)$, it can be easily inferred that $w \not\sim b'_i$ for any $3 \le i \le d' + 1$. ⌟

Now that P' reaches u not through N, next we show that the center c'_2 has to be in N as it is adjacent to all base vertices b_i of W for $d - 3 \le i \le d$.

CLAIM 7. $c'_2 \in N$.

PROOF. Suppose to the contrary, $c'_2 \notin N$, then c'_2 cannot be adjacent to b_1. From $c'_2 \sim b'_1$ and $c'_2 \not\sim b_1$ we can derive $\texttt{last}(b'_1) \ge \texttt{first}(c'_2) > \texttt{last}(b_1)$. On the other hand, as $\amalg(W)$ is minimal, it does not properly contain $\amalg(W')$, which implies $\texttt{first}(b'_d) > \texttt{first}(b_d)$. Therefore, $\texttt{last}(c'_2) \ge \texttt{first}(b'_d) > \texttt{first}(b_d)$. (See Figure 9.) By the selection of W and B, if a vertex z is adjacent to both u and b_d, then $z \in N$, as otherwise there exists a path $(l, b_1, \dots, b_p, u, z, b_d, r)$, where $p \le d - 4$, shorter than lBr. In particular, the vertex next to u in the path $l'B'r'$ is not adjacent to b_d; letting $u = b'_i$ where $i < d'$, it means $last(b'_{i+1}) < first(b_d)$. From $c'_2 \sim b'_{i+1}$ we can conclude $\texttt{first}(c'_2) < \texttt{first}(b_d)$, and then c'_2 and b_d are adjacent, which further implies that u is not adjacent to c'_2. In other words, u has to be l'. Let p be the index such that $b_p \not\sim c'_2$ and $b_{p+1} \sim c'_2$, which exists by assumption. We now show $b_{d-2} \not\sim c'_2$, and $p \ge d-2$, by contradiction. • If s' is adjacent to every vertex in N, then $(s' : c_1, c'_2 : b_p, b_{p+1} \dots b_d, r)$ or $(s' : c'_2 : b_p, b_{p+1} \dots b_d, r)$ would be an AW that contradicts the selection of W. • If s' is not adjacent to $x \in N$, then $(s : x, c_2 : l, b_1 \dots b_{p+1}, c'_2, s')$, $(s : x : l, b_1 \dots b_{p+1}, c'_2, s')$ or $(s : c_2 : l, b_1 \dots b_{p+1}, c'_2, s')$ would be an AW that contradicts the selection of W: noting that $\texttt{first}(c'_2) \le \texttt{last}(b_{d-2}) < \texttt{first}(b_d)$ as $c'_2 \sim b_{d-2}$. However, $(s' : c'_1, c'_2 : b_p, b'_q, \dots, b'_{d'}, r')$, $(s' : c'_1 : b_p, b'_q, \dots, b'_{d'}, r')$, or $(s' : c'_2 : b_p, b'_q, \dots, b'_{d'}, r')$, where q is the largest index such that $b'_q \sim b_p$, will be an AW in $G - Q$, which is impossible as Q is an interval deletion set to G. ⌟

An immediate consequence of Claim 7 is $c'_2 \sim u$, hence $u \in B'$. By Proposition 8.4, $\texttt{first}(b'_1) \le \texttt{first}(u) < \texttt{first}(b_{d-2})$. Then from Claim 5 and the fact $l' \sim b'_1$, it can be inferred that $l' \notin ST(G)$. Now $\texttt{last}(l')$ is defined, and $\texttt{last}(l') \le \texttt{first}(c'_2) < \texttt{last}(b_1)$; the selection of W implies $\texttt{first}(b'_{d'}) \ge \texttt{first}(b_d)$. Therefore, the l'-$b'_{d'}$ path $l'B'$ has to go through X, and we end with a contradiction.

This verifies that Q' is an interval deletion set to G, and it remains to show that Q' is minimum, from which the lemma follows.

CLAIM 8. $|Q'| \le |Q|$.

PROOF. It will suffice to show that $Q \cap V_I$ makes a b_1-b_{d-2} separator in $G - N$, and then the claim ensues as

$$|Q'| = |Q \setminus V_I| + |X| \le |Q \setminus V_I| + |Q \cap V_I| = |Q|.$$

Suppose to the contrary, there is a chordless b_1-b_{d-2} path P. We can extend P into an l-r path $P^+ = (lPb_{d-1}b_dr)$, which is disjoint from Q and N. Within P^+ there is a chordless l-r path (lB_1r). By assumption, $\{s, c_1, c_2\} \cap Q = \emptyset$; every vertex in B_1 satisfies the condition of Claim 9, and hence nonadjacent to s. Moreover, $c_1, c_2 \in N$, and therefore both c_1 and c_2 are adjacent to every vertex of B_1. Thus, $(s : c_1, c_2 : l, B_1, r)$ is an AW in $G - Q$, which is impossible. ⌟

This completes the proof of the lemma. □

To prove Theorem 2.4, we need one last piece of the jigsaw, i.e., to find the AW required by Lemma 8.9.

Theorem 2.4 (restated). There is a $10^k \cdot n^{O(1)}$-time algorithm for INTERVAL DELETION on nice graphs.

Proof. Based on Lemma 8.9, it suffices to show how to find such an AW, and then the standard branching will deliver the claimed algorithm. For any triple of vertices $\{x, y, z\}$ and pair of indices $\{p, q\}$ for the nodes in the central path of the caterpillar decomposition, we can check whether or not there is an AW W whose terminals are $\{x, y, z\}$ and non-terminal vertices are fully contained in $\mathbb{I}[p, q]$. Therefore, in $O(n^6)$ time we are able to find the correct terminals and indices, from which the short base B can also be easily constructed. This finishes the construction of the AW required by Lemma 8.9. □

9. CONCLUDING REMARKS

We have classified INTERVAL DELETION to be FPT by presenting a $c^k \cdot n^{O(1)}$-time algorithm with $c = 10$. The constant c might be improvable, and let us have a brief discussion on how to achieve this. The current constant 10 comes from Reduction 1 and Theorem 2.4. The constant in Reduction 1 is not tight, and it can be replaced by 8. We choose the current number for the convenience for later argument; for example, if we do not break AWs of size 9 in preprocessing, then we have to use a far more complicated proof for Proposition 8.1. In other words, the real dominating step is to break ATs in nice graphs, where we need to branch into 10 cases. As a nice graph exhibits a linear structure, it might help to apply dynamic programming here. To further lower the constant c, we need to break small forbidden induced subgraphs in a better way than the brute-force in our algorithm. So a natural question is: Can it be $c = 2$?

It is known that CHORDAL COMPLETION can be solved in polynomial time if the input graph is a circular-arc graph [Kloks et al. 1998] while INTERVAL COMPLETION remains NP-hard on chordal graphs [Peng and Chen 2006]. It would be interesting to inquire the complexity of INTERVAL DELETION on chordal graphs and other graph classes. At least, can it be solved in polynomial time if the input graph is nice, which, if positively answered, would suggest that all the troubles are small forbidden subgraphs. We leave open the parameterized complexity of INTERVAL EDGE DELETION, which instead asks for a set of k edges whose removal makes an interval graph [Goldberg et al. 1995; Bodlaender et al. 1995]. To adapt our approach to this problem, one needs a reasonable bound for the number of edge hole covers for congenial holes.

As having been explored by Narayanaswamy and Subashini [2013], we would also like to ask which other problems can be formulated as or reduced to INTERVAL DELETION and then solved with our algorithm. Both practical and theoretical consequences are worth further investigation.

Appendix. A simpler and weaker version of Lemma 8.9

Lemma a. Let $\mathcal{T}$ be a caterpillar decomposition of a nice graph G, and $W = (s : c_1, c_2 : l, B, r)$ be an AW in G such that

— $\texttt{first}(b_d)$ is the smallest among all AWs;
— $\mathbb{I}(W)$ is minimal; and
— B is a short base.

Let ℓ be the minimum index such that $\texttt{last}(b_2) \leq \ell < \texttt{first}(b_{d-5})$ and the cardinality of S_ℓ is minimum among $\{S_i : \texttt{last}(b_2) \leq i < \texttt{first}(b_{d-5})\}$. There is a minimum interval deletion set to G that either contains one of the 13 vertices

$$V_B = \{s, c_1, c_2, l, b_1, b_2, b_{d-5}, b_{d-4}, b_{d-3}, b_{d-2}, b_{d-1}, b_d, r\},$$

or the whole set $X = S_\ell \setminus N$, where $N = \widehat{N}(B)$.

Proof. We prove by construction. Let Q be any minimum interval deletion set; we may assume $Q \cap V_B = \emptyset$, and $X \not\subseteq Q$, as otherwise Q satisfies the asserted condition and we are finished. We claim $Q' = (Q \setminus V_I) \cup X$, where $V_I = \mathbb{I}[\texttt{last}(b_3), \texttt{first}(b_{d-6})] \setminus N$, is the desired interval deletion set, which fully contains X in particular. By definition of V_I, any vertex $z \in V_I$ is adjacent to some vertex b_i for $4 \leq i \leq d-7$, then as B is short and $z \notin N$, we have

$$\texttt{first}(b_2) \leq \texttt{last}(b_1) < \texttt{first}(z) \leq \texttt{last}(z) < \texttt{first}(b_{d-4}) \leq \texttt{last}(b_{d-5}). \qquad (2)$$

As G is chordal, all minimal forbidden induced subgraphs in G are AWs. To show that Q' makes an interval deletion set to G, it suffices to argue that if there exists an AW W' avoiding Q' then we can also find an AW, not necessarily the same as W', avoiding Q. Suppose $W' = (s' : c'_1, c'_2 : l', B', r')$ is an AW in $G - Q'$. By the construction of Q', this AW must intersect $V_I \setminus X$; let $u \in W' \cap (V_I \setminus X)$. Clearly, u can neither be s', as $u \notin ST(G)$, nor r', as otherwise according to Proposition 8.4, $\texttt{first}(b'_{d'}) < \texttt{first}(u) < \texttt{first}(b_d)$, contradicting the selection of W. The following claim further rules out the possibility that $u \in \{c'_1, c'_2\}$.

Claim 9. *For each vertex $v \in \mathbb{I}[0, \texttt{first}(b_{d-2})] \setminus N$, we have $\texttt{last}(v) < \texttt{first}(b_d)$, and $v \not\sim ST(G)$.*

Proof. By definition, if v is adjacent to B, then $v \sim b_i$ for some $i \leq d-3$. If $v \sim b_d$, then B is not a short base, as there would be a shorter (not necessarily chordless) l-r path $(l, \ldots, b_i, v, b_d, r)$. Therefore, $v \not\sim b_d$ and it follows that $\texttt{last}(v) < \texttt{first}(b_d)$. Suppose to the contrary of the second assertion, v is adjacent to the shallow terminal x of some AW W_1. We apply Lemma 5.2(2) on v and W_1. As $v \notin ST(G)$, it has to be in categories "full" or "partial." In either case, there exists an AW whose base is fully contained in $\mathbb{I}[\texttt{first}(v), \texttt{last}(v)]$, contradicting the selection of W. ⌟

Therefore, either $u = l'$ or $u \in B'$. Now we focus on the chordless path $l'B'r'$, which we shall refer to by P', and how it reaches u when going from r' to l'. Recall that every vertex of B' appears in the central path of the caterpillar decomposition.

Claim 10. $B' \cap N = \emptyset$.

Proof. Suppose the contrary and let x be a vertex in $B' \cap N$. By definition of N and (2), we have $\texttt{first}(x) < \texttt{first}(u) \leq \texttt{last}(u) < \texttt{last}(x)$. Then every neighbor of u, which is not in $ST(G)$ according to Claim 9, is thus adjacent to x. As x and u are both in the chordless path P', vertex u has to be one end of it. More specifically, $u = l'$ and $x = b'_1$. A further consequence is that u is the only vertex in $W' \cap V_I$: the argument above applies to any vertex $u' \in W' \cap V_I$, and thus $u' = l' = u$.

Now we show, for any vertex w in $X \setminus Q$, which is nonempty by assumption, it has the same neighbors as u in W', and hence $(s' : c'_1, c'_2 : w, B', r')$ is an AW in $G - Q$, contradicting the assumption that Q is an interval deletion set to G. Observe that any vertex in N is adjacent to both u and w.

— The assumption $w \notin N$ implies $w \not\sim s$: otherwise, w is adjacent to both s and B but not in N, and we can apply Lemma 5.2 to W and w, which is in category "partial," to obtain an AW with strictly smaller container.
— By the selection of W, we have $\texttt{last}(c'_i) \geq \texttt{first}(b'_d) \geq \texttt{first}(b_d)$ for both $i = 1, 2$. If c'_i, where $i = 1$ or 2, is adjacent to one of u and w, then (2) implies $\texttt{first}(c'_i) < \texttt{last}(b_{d-5})$; as B is short, c'_i must be in N, and then adjacent to both u and w.
— Vertex b'_1 $(= x)$ is in N, hence adjacent to w.
— By definition, $b'_3 \sim b'_2$ and $b'_3 \not\sim b'_1 (\in N)$ imply $\texttt{last}(b'_2) \geq \texttt{first}(b'_3) > \texttt{last}(b'_1) \geq \texttt{first}(b_d)$. On the other hand, $b'_2 \not\sim u$ implies $b'_2 \notin N$. Then as B is short, $\texttt{first}(b'_2) > \texttt{last}(b_{d-5})$. Therefore, from (2) we can conclude for $2 \leq i \leq d'+1$, it holds that $\texttt{first}(b'_i) > \texttt{last}(w)$ and thus $w \not\sim b'_i$.

⌟

Claim 11. $c'_2 \in N$.

Proof. As $u = b'_i$ for some $0 \leq i \leq d'$, Proposition 8.4 and (2) imply $\texttt{first}(b'_1)\,(\leq \texttt{last}(l')) \leq \texttt{last}(u) < \texttt{last}(b_{d-5})$. By Claim 10, b'_1 is not in N and adjacent to at most 3 vertices of B; thus $\texttt{last}(b'_1) < \texttt{first}(b_{d-2}) \leq \texttt{last}(b_{d-3})$. On the other hand, by the selection of W, we have $\texttt{last}(c'_2) \geq \texttt{first}(b'_d) \geq \texttt{first}(b_d)$. Therefore, c'_2 is adjacent to at least 4 vertices of B and is in N. ⌟

From Claim 11 we can conclude $c'_2 \sim u$ and then $u \in B'$. By Proposition 8.4, $\texttt{first}(b'_1) \leq \texttt{first}(u) < \texttt{first}(b_{d-4})$. Then from Claim 9 and the fact $l' \sim b'_1$, it can be inferred that $l' \notin ST(G)$. Now $\texttt{last}(l')$ is defined, and $\texttt{last}(l') < \texttt{first}(c'_2) \leq \texttt{last}(b_1)$; the selection of W implies $\texttt{first}(b'_{d'}) \geq \texttt{first}(b_d)$. Therefore, the l'-$b'_{d'}$ path $l'B'$ has to go through X, and we end with a contradiction. This verifies that Q' is an interval deletion set to G, and it remains to show that Q' is minimum, from which the lemma follows.

Claim 12. $|Q'| \leq |Q|$.

Proof. It will suffice to show that $Q \cap V_I$ makes a b_2-b_{d-5} separator in $G - N$, and then the claim ensues as

$$|Q'| = |Q \setminus V_I| + |X| \leq |Q \setminus V_I| + |Q \cap V_I| = |Q|.$$

Suppose to the contrary, there is a chordless b_2-b_{d-5} path P. We can extend P into an l-r path $P^+ = (lb_1Pb_{d-4}b_{d-3}b_{d-2}b_{d-1}b_dr)$, which is disjoint from Q and N. Within P^+ there is a chordless l-r path (lB_1r). By assumption, $\{s, c_1, c_2\} \cap Q = \emptyset$; every vertex in B_1 satisfies the condition of Claim 9, and hence nonadjacent to s. Moreover, $c_1, c_2 \in N$, and therefore both c_1 and c_2 are adjacent to every vertex of B_1. Thus, $(s : c_1, c_2 : l, B_1, r)$ is an AW in $G - Q$, which is impossible. ⌟

This completes the proof of the lemma. □

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