

Midterm Exam in Probability and Statistics

October 24, 2023

1. Two numbers are independently and uniformly chosen at random from the interval $[0, 1]$. What is the probability that the sum of the two numbers is greater than twice the difference between the larger and the smaller number?
2. In a forest, there are two species of felines in very large numbers: domestic cats and wildcats. Wildcats make up ten percent of the total feline population. All wildcats are striped, and on average one out of five domestic cats is striped.
 - (a) If we randomly select a feline from the forest, what is the probability that it is striped?
 - (b) If we spot a striped feline in the forest but cannot tell from a distance whether it is domestic or wild, what is the probability that it is a domestic cat?
 - (c) Suppose we randomly select a feline from the forest. Are the following events independent?

$A = \{\text{the selected feline is not striped}\},$

$B = \{\text{the selected feline is a domestic cat}\}.$

3. Anna, Bea and Csilla play cards with a Hungarian deck. The deck has 32 cards, of which exactly 8 are red. After shuffling, each player receives 8 cards.
 - (a) What is the probability that Anna has no red cards?
 - (b) What is the probability that neither Bea nor Csilla has any red cards?
 - (c) What is the probability that each player has at least one red card?
4. A nine-member student committee must decide between two program options, called A and B. Voting is mandatory, abstention is not allowed. Three members will definitely vote for A, two members will definitely vote for B, and the remaining four are uncertain. Each of these four tosses a fair coin independently; if it lands heads, they vote for A, otherwise for B. Let X denote the number of members voting for A.
 - (a) What is the probability that one of the two options wins by more than two votes?
 - (b) What is the distribution of the random variable $X - 3$ and what are its parameters?
 - (c) Let Y denote the number of members voting for option B. Determine the joint distribution of X and Y , as well as their marginal distributions (preferably in tabular form, but it is sufficient to give $P(X = k, Y = l)$, $P(X = k)$ and $P(Y = l)$ for all k, l with positive probability).

5. A continuous random variable X has density function

$$f_X(x) = \begin{cases} \frac{c}{x^2}, & x > 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $c > 0$ is an appropriately chosen constant.

- (a) Determine the value of c .
- (b) Define a discrete random variable Y as follows:

$$Y = \begin{cases} 3, & \text{if } X < 3, \\ 0, & \text{if } 3 \leq X < 4, \\ 4, & \text{otherwise.} \end{cases}$$

Determine $P(Y = k)$ for all $k \in \mathbb{R}$ for which this value is positive.

- (c) Compute the expectation and the standard deviation of Y .

6. * Let X be a random variable with distribution function $F_X : \mathbb{R} \rightarrow [0, 1]$ defined by

$$F_X(x) = e^{-e^{-x}} \quad (e \approx 2.71).$$

(You may use that this is indeed a valid distribution function, see Exercise Sheet 5.)

- (a) Prove that there exists an $\varepsilon > 0$ such that $P(X \leq \varepsilon) < 1/2$.
- (b) Let $Y = e^{-X}$. Determine the density function and the standard deviation of Y .