Probability Theory and Statistics

Sample Midterm

1. We know about events A, B, C that at least one of them always occurs, moreover A and B are independent, while B and C are mutually exclusive. Determine the probabilities of the above events if
$$\mathbb{P}(A \mid B) = \tfrac{1}{4}, \qquad \mathbb{P}(C \mid A \cup C) = \tfrac{3}{4}, \qquad \mathbb{P}(A \cap C) = \tfrac{1}{12}.$$

2. Aladár and Bea are playing cards. First Aladár draws two cards from a thoroughly shuffled 32-card *Hungarian (German-suited) deck*, then (without seeing which two cards were removed) Bea draws one card from Aladár's *hand of two.*[1]
   (a) What is the probability that Bea's card is of the Red (Piros) suit?
   (b) What is the probability that Aladár's *remaining single* card is also Red, given that Bea's card is Red?

3. Choose two numbers independently and uniformly at random, one from the interval $[0, 2]$ and one from $[0, 4]$. What is the probability that their sum lies between 3 and 5?

4. Sándor wants to send an n-bit string to Tamás, but on their channel each transmitted bit flips independently with probability 1% (i.e., a sent 1 may become 0 and a sent 0 may become 1). As a simple remedy, Sándor repeats each bit three times in a row (then moves on to the next bit), thus sending a total of $3n$ bits. Tamás decodes by majority in each consecutive triple (e.g., 101 is decoded as 1).

   How large can n be at most so that the whole message is correct with probability at least 99%?

5. Consider the function
$$f(x) = \begin{cases} \dfrac{(x-1)^2}{9}, & x \in (1, c), \\ 0, & \text{otherwise}, \end{cases}$$
   where $c > 1$ is a real parameter.
   (a) Determine the value of c that makes f a probability density function of some continuous random variable.
   (b) Let X be a random variable with density $f(x)$. Compute $\mathbb{E}[X]$.

6.* Let $X \sim \mathrm{U}[0, 2]$ and define $Y = \dfrac{1}{(1+X)^2}$.
   (a) Find the density of Y and compute $\mathbb{E}[Y]$.
   (b) Compare the probabilities $\mathbb{P}\left(Y < \dfrac{1}{\big(1+\mathbb{E}[X]\big)^2}\right)$ and $\mathbb{P}\big(Y < \mathbb{E}[Y]\big)$.

Information. The duration of the midterm is 90 minutes. You may use a calculator. Round all numerical answers to *4 significant digits*. To obtain full credit, you must show your work, including the properties and theorems used at each step. You may not leave the examination room during the first 30 minutes.

Common distributions (reference table).

[1] A Hungarian 32-card deck has four suits (Piros/Red, Zöld/Green, Makk/Acorns, Tök/Bells), 8 cards per suit; thus exactly 8 "Red" (Piros) cards.

Distribution	Notation	Ran(X)	$\mathbb{P}(X=k)$ or $F_X(t)$	$f_X(t)$	$\mathbb{E}(X)$	$\mathbb{D}^2(X)$
indicator	$\mathbf{1}(p)$	$\{0,1\}$	$p,\ 1-p$		p	$p(1-p)$
binomial	$\mathrm{B}(n;p)$	$\{0,1,...,n\}$	$\binom{n}{k}p^k(1-p)^{n-k}$		np	$np(1-p)$
Poisson	$\mathrm{Pois}(\lambda)$	$\{0,1,...\}$	$\frac{\lambda^k}{k!}e^{-\lambda}$		λ	λ
geometric	$\mathrm{Geo}(p)$	$\{1,2,...\}$	$(1-p)^{k-1}p$		$\frac{1}{p}$	$\frac{1-p}{p^2}$
uniform	$\mathrm{U}(a;b)$	$(a;b)$	$\frac{t-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
exponential	$\mathrm{Exp}(\lambda)$	$\mathbb{R}^+$	$1-e^{-\lambda t}$	$\lambda e^{-\lambda t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$