Retake Midterm Exam

1. A real number is chosen uniformly at random from the interval $(0, 2)$, and independently another real number is chosen uniformly from the interval $(1, 3)$. What is the probability that the number chosen from $(1, 3)$ is larger, but at most twice as large as the number chosen from $(0, 2)$?

2. A midterm exam of an elective course is written in two rooms: IB027 and IB028. In room IB027, there are 45 Hungarian-track and 15 German-track students; in room IB028, there are 80 Hungarian-track and 40 English-track students. A student writing the exam is chosen uniformly at random.
   (a) What is the probability that the selected student is in the Hungarian track?
   (b) Given that the selected student is in the Hungarian-track, what is the probability that they are writing the exam in IB028?
   (c) Are the following events independent (and why)?
   {the selected student is *not* in the Hungarian track} and
   {the selected student writes the exam in IB027}.

3. We examine the number of Colorado potato beetles found on a small patch of a potato field. This number is well approximated by a Poisson distribution, and the probability of finding exactly two beetles equals $2e^{-2}$.
   (a) What is the parameter (λ) of the distribution?
   (b) What is the probability that at least one beetle is found on the patch?
   (c) Let X denote the number of beetles on the patch. Determine the expected value of the random variable $2X - 3$ and the variance of the random variable $-4X - 7$.

4. Two fair coins are tossed, each having 1 written on one side and 0 on the other. Let U and V denote the numbers shown on the upper sides of the first and second coin, respectively. Define $X = U + V$ and $Y = U - V$.
   (a) Give the joint distribution of X and Y, that is, specify $\mathbb{P}(X = k, Y = \ell)$ for all (k, ℓ) for which it is positive, in tabular form.
   (b) Determine the marginal distributions of X and Y.
   (c) Is it true that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$?
   (d) Are X and Y independent?

5. The lifetime of a battery (in months) follows a continuous, memoryless distribution. It is known that the probability that the battery runs out within two months is $1 - e^{-4}$. Let X denote the battery lifetime.
   (a) Determine the value of the density function of X at $x = 3$.
   (b) Determine the expected value and the standard deviation of X.
   (c) Given that the battery lasts longer than eight months, what is the probability that its lifetime (including those eight months) exceeds thirteen months?

6. * Let X be uniformly distributed on $(-1, 1)$ and define $Y = X^4$. Determine the distribution function, the density function, and the expected value of Y.

Distribution	Notation	Ran(X)	$F_X(t)$	$p_X(k)$, $f_X(t)$	$\mathbb{E}(X)$	$\mathbb{D}^2(X)$
Indicator Bernoulli	$\mathbb{1}_A$ B(p)	$\{0, 1\}$		$p_X(0) = 1 - p$, $p_X(1) = p$ $(p = \mathbb{P}(A))$	p	$p(1 - p)$
Binomial	Bin($n; p$)	$\{0, 1, \ldots, n\}$		$\binom{n}{k} p^k (1 - p)^{n-k}$	np	$np(1 - p)$
Poisson	Pois(λ)	$\{0, 1, 2, \ldots\}$		$\frac{\lambda^k}{k!} e^{-\lambda}$	λ	λ
Geometric	Geo(p)	$\{1, 2, \ldots\}$		$(1 - p)^{k-1} p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Uniform	U($a; b$)	$(a; b)$	$\frac{t-a}{b-a}$ (if $t \in (a;b)$)	$\frac{1}{b-a}$ (if $t \in (a;b)$)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	Exp(λ)	$[0; \infty)$	$1 - e^{-\lambda t}$	$\lambda e^{-\lambda t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$