Midterm Exam

1. For the events A, B, and C, we know that at least one of A or B always occurs, A and C are independent, and $\mathbb{P}(A) = \mathbb{P}(B) = 0.6$, $\mathbb{P}(C) = 0.1$. Answer (with justification!) the following questions.
   (a) What is the probability that both A and C occur?
   (b) What is the probability that both A and B occur?
   (c) What is $\mathbb{P}(B|A)$?
   (d) What is the probability that neither A nor C occurs?

2. At the reception of Building I there are 2 blue, 3 green, 1 black, and 2 red markers. Before the Probability class, we randomly (with equal probability) choose one of the markers to write on the board. We know that a blue marker runs out of ink by the end of the class with probability $\frac{2}{5}$, a green one with $\frac{1}{4}$, a black one with $\frac{1}{5}$, and a red one with $\frac{1}{10}$.
   (a) What is the probability that a green marker is chosen and it runs out of ink by the end of the class?
   (b) What is the probability that the chosen marker (regardless of its color) runs out of ink by the end of the class?
   (c) Given that the chosen marker ran out of ink by the end of the class, what is the probability that it was red?

3. On a tram line, a service failure occurs each day independently with probability $\frac{1}{10}$.
   (a) We observe the line for 20 consecutive days, recording each day whether a failure occurred. Let X denote the number of days (out of 20) when a failure occurred. Determine the expected value and standard deviation of X, and also compute $\mathbb{P}(X > 2)$.
   (b) Starting from a certain day (call it day 1), we observe daily whether a failure occurred until the first failure happens. What is the probability that the first failure occurs exactly on the 3rd day? And given that no failure occurred during the first three days, what is the probability that no failure occurs during the first five days?

4. We place two slips of paper into a hat, one labeled with the number 3 and one with 4. Into another hat, we place one slip labeled 2 and one labeled 3. We draw one number from each hat at random. Define the random variables X and Y as follows: let $X = 2$ if the number drawn from the first hat is 4, otherwise let $X = 1$; let Y be the number of hats from which a 3 was drawn. (For example, if we drew 3 from the first hat and 2 from the second, then $X = 1$ and $Y = 1$.)
   (a) Determine the joint distribution of X and Y (that is, the values of $\mathbb{P}(X = k, Y = \ell)$ for all k and ℓ where this probability is positive) in tabular form.
   (b) Determine the marginal distributions of X and Y.
   (c) Are X and Y independent?
   (d) Determine the expected value of the random variable X^3 (note: the third power!).

5. A continuous random variable X has the probability density function

$$f_X(x) = \begin{cases} ax, & \text{if } 1 < x < 2, \\ 0, & \text{otherwise,} \end{cases}$$

where a is a suitable real constant.

   (a) Determine the value of a.
   (b) Determine the distribution function of X.
   (c) Determine the expected value of X.

6. * On the open interval $(0, 2)$ we randomly (uniformly) choose a real number y. Consider the right triangle in $\mathbb{R}^2$ with vertices at $(0, 0)$, $(1, 0)$, and $(0, y)$. Let X denote the length of the hypotenuse of this triangle (that is, the distance between the points $(1, 0)$ and $(0, y)$).

   (a) What is the probability that X is less than 2?
   (b) Determine the distribution function of X.
   (c) Determine the density function of X.
   (d) Determine the expected value of the random variable $Z := \dfrac{2\sqrt{X^2 - 1}}{X}$. (You do not have to determine the distribution of Z.)

Distribution	Notation	Ran(X)	$F_X(t)$	$p_X(k)$, $f_X(t)$	$\mathbb{E}(X)$	$\mathbb{D}^2(X)$
indicator Bernoulli	$\mathbb{1}_A$ B(p)	$\{0, 1\}$		$p_X(0) = 1 - p$, $p_X(1) = p$ $(p = \mathbb{P}(A))$	p	$p(1 - p)$
binomial	Bin($n; p$)	$\{0, 1, \ldots, n\}$		$\binom{n}{k} p^k (1 - p)^{n-k}$	np	$np(1 - p)$
Poisson	Pois(λ)	$\{0, 1, 2, \ldots\}$		$\frac{\lambda^k}{k!} e^{-\lambda}$	λ	λ
geometric	Geo(p)	$\{1, 2, \ldots\}$		$(1 - p)^{k-1} p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
uniform	U($a; b$)	$(a; b)$	$\frac{t-a}{b-a}$ (if $t \in (a;b)$)	$\frac{1}{b-a}$ (if $t \in (a;b)$)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
exponential	Exp(λ)	$[0; \infty)$	$1 - e^{-\lambda t}$	$\lambda e^{-\lambda t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$