

Probability Theory and Statistics

Lecture 16

Bence Csonka

Budapest University of Technology and Economics
csonkab@edu.bme.hu

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Theorem (Law of Total Expectation)

Let X and Y be random variables such that $\mathbb{E}(|X|)$ and $\mathbb{E}(|Y|)$ are finite. Then

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)).$$

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Theorem (Law of Total Expectation with a Partition)

Let $A_1, \dots, A_n$ be a partition of Ω such that $\mathbb{P}(A_i) > 0$ for all i . If $\mathbb{E}(|Y|) < \infty$, then

$$\mathbb{E}(Y) = \sum_{i=1}^n \mathbb{E}(Y|A_i)\mathbb{P}(A_i).$$

Theorem (Law of Total Expectation, Discrete Case)

Let X be a discrete random variable with possible values $\{x_1, \dots, x_n, \dots\}$ such that $\mathbb{P}(X = x_i) > 0$. If $\mathbb{E}(|Y|) < \infty$, then

$$\mathbb{E}(Y) = \sum_i \mathbb{E}(Y|X = x_i)\mathbb{P}(X = x_i).$$

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Theorem (Law of Total Expectation, Continuous Case)

Let X be a continuous random variable with $\mathbb{E}(|Y|) < \infty$, and let f_X denote its density function. Then

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} \mathbb{E}(Y|X = x)f_X(x)dx.$$

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Exercise 2: Toss a fair coin. If it is head, then we take a number from $Y \sim \text{Exp}(1/2)$, otherwise we take a number from $Z \sim N(2, 1)$. What is the expectation of the final number?

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Exercise 3: We can choose two different boxes, A and B . The probability of choosing A is $\frac{1}{3}$ and the probability of choosing B is $\frac{2}{3}$. In the A box, there are 3 red and 5 blue balls. In the B box, there are 4 red and 3 blue balls. From the chosen box we take out 3 balls. (Without replacement.) What is the expectation of the number of blue balls that are discarded?

Reminder

Theorem (Law of Total Probability for the Discrete Case)

Let X be a discrete random variable and A an event. Then

$$\mathbb{P}(A) = \sum_{k \in \text{Ran} X} \mathbb{P}(A|X = x) \mathbb{P}(X = k).$$

Reminder

Theorem (Law of Total Probability for the Discrete Case)

Let X be a discrete random variable and A an event. Then

$$\mathbb{P}(A) = \sum_{k \in \text{Ran} X} \mathbb{P}(A|X = x) \mathbb{P}(A|X = k).$$

Theorem (Law of Total Probability for the Continuous Case, or “Continuous LTP”)

Let X be a continuous random variable and A an event. Then

$$\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A|X = x) f_X(x) dx,$$

where f_X is the density of X .

Exercises 2

Exercise 1: We choose an x number uniformly from 2 to 5. After the choose a y number from x and x^2 . What is the probability that $y < 20$?

Exercises 2

Exercise 2: A wireless communication channel's daily average noise level is modeled by a random variable Y , where $Y \sim U(0, 2)$. On a given day, if the noise level is $Y = y$, then the probability that a given 0-1 bit is received in error is

$$p(y) = \frac{y}{4}, \quad 0 \leq y \leq 2.$$

Let N denote the number of errors in a 10-bit message. What is the probability that the 10-bit message is received without error?

Convolution of random variables

Convolution: $X + Y$

We have seen:

Theorem

If $X \sim \text{Poi}(\lambda)$ and $Y \sim \text{Poi}(\mu)$ are independent random variables, then $X + Y \sim \text{Poi}(\lambda + \mu)$.

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Theorem

If $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$ are independent random variables, then $X + Y \sim \text{Bin}(n + m, p)$.

Convolution of Normal Distributions

We haven't seen the case of two independent normal distributions.

Theorem

If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent random variables, then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. Moreover $cX \sim N(c\mu_1, c^2\sigma_1^2)$

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Sketch of proof:

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y) = \mu_1 + \mu_2$$

$$\mathbb{D}^2(X + Y) = \mathbb{D}^2(X) + \mathbb{D}^2(Y) = \sigma_1^2 + \sigma_2^2$$

$$\mathbb{E}(cX) = c\mathbb{E}(X)$$

$$\mathbb{D}^2(cX) = c^2\mathbb{D}^2(X).$$

Exercise for the convolution of normal distributions

Exercise: Let $X \sim N(2, 1^2)$ and $Y \sim N(3, 2^2)$ be independent random variables. If $V := X + Y$ and $W := X - Y$, then

- what is $\text{cov}(V, W)$?
- what is the linear regression of V on W ?

