$$f_{X,Y}(x,y) := \begin{cases} axy^2, \text{ if } 1 < x < 2 \text{ and } 1 < y < 3 \\ 0 \text{ otherwise} \end{cases}$$

1. $a =?$
2. $F_{X,Y} =?$
3. Calculate F_X, F_Y (marginal distributions)
4. Calculate f_X, f_Y (marginal densities)
5. Calculate $\mathbb{E}(X)$ and $\mathbb{E}(Y)$!

Solution: 1.) We know $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y = 1$. Solve the following double integral equation!

$$\int_1^3\int_1^2 axy^2\mathrm{d}x\mathrm{d}y = 13a = 1 \Rightarrow a = \frac{1}{13}.$$

2.) To avoid confusion between the variables of $f_{X,Y}$ and the bounds of the integral, we will calculate $F_{X,Y}(t_1,t_2)$ instead of using $F_{X,Y}(x,y)$. There are 4 different cases. If $2 \geq t_1$ and $t_2 \geq 3$, then $F_{X,Y}(t_1,t_2) = 1$. Now we take $1 \leq t_1 < 2$ and $t_2 \geq 3$.

$$\int_1^3\int_1^{t_1} \frac{1}{13}xy^2\,\mathrm{d}x\,\mathrm{d}y = \int_1^3 \frac{y^2}{13}\left[\frac{x^2}{2}\right]_{x=1}^{x=t_1}\mathrm{d}y = \int_1^3 \frac{y^2}{26}\left(t_1^2-1\right)\mathrm{d}y =$$

$$= \frac{t_1^2-1}{26}\left[\frac{y^3}{3}\right]_{y=1}^{y=3} = \frac{t_1^2-1}{26}\cdot\frac{26}{3} = \frac{t_1^2-1}{3}.$$

If $t_1 \geq 2$ and $1 \leq t_2 < 3$, then

$$\int_1^{t_2}\int_1^2 \frac{1}{13}xy^2\,\mathrm{d}x\,\mathrm{d}y = \int_1^{t_2} \frac{y^2}{13}\left[\frac{x^2}{2}\right]_{x=1}^{x=2}\mathrm{d}y =$$

$$= \int_1^{t_2}\frac{y^2}{26}(4-1)\,\mathrm{d}y = \int_1^{t_2}\frac{3y^2}{26}\,\mathrm{d}y = \frac{3}{26}\left[\frac{y^3}{3}\right]_{y=1}^{y=t_2} = \frac{t_2^3-1}{26}.$$

Assume that $1 \leq t_1 < 2$ and $1 \leq t_2 < 3$ so

$$\int_1^{t_2}\int_1^{t_1} \frac{1}{13}xy^2\,\mathrm{d}x\,\mathrm{d}y = \int_1^{t_2} \frac{y^2}{13}\left[\frac{x^2}{2}\right]_{x=1}^{x=t_1}\mathrm{d}y = \int_1^{t_2} \frac{y^2}{26}\left(t_1^2-1\right)\mathrm{d}y =$$

$$= \frac{t_1^2-1}{26}\int_1^{t_2} y^2\,\mathrm{d}y = \frac{t_1^2-1}{26}\left[\frac{y^3}{3}\right]_{y=1}^{y=t_2} = \frac{(t_1^2-1)(t_2^3-1)}{78}.$$

The final case when $t_1 < 1$ or $t_2 < 0$, hence $F_{X,Y}(t_1, t_2) = 0$. (Since the joint density function 0.)

Summarize the results:

$$F_{X,Y}(t_1, t_2) = \begin{cases} 1 \text{ , if } t_1 \geq 2 \text{ and } t_2 \geq 3 \\ \frac{t_1^2 - 1}{3} \text{ , if } 1 \leq t_1 < 2 \text{ and } t_2 \geq 3 \\ \frac{t_2^3 - 1}{26} \text{ , if } t_1 \geq 2 \text{ and } 1 \leq t_2 < 3 \\ \frac{(t_1^2 - 1)(t_2^3 - 1)}{78} \text{ , if } 1 \leq t_1 < 2 \text{ and } 1 \leq t_2 < 3 \\ 0 \text{ , otherwise} \end{cases}$$

3.) Let us compute $F_X(t_1) = \lim_{t_2 \to \infty} F_{X,Y}(t_1, t_2)$ and $F_Y(t_2) = \lim_{t_1 \to \infty} F_{X,Y}(t_1, t_2)$. We can easily see that

$$F_X(t_1) = \begin{cases} 1 \text{ , if } 2 \geq t_1 \\ \frac{t_1^2 - 1}{3} \text{ , if } 1 \leq t_1 < 2 \\ 0 \text{ , otherwise} \end{cases}$$

$$F_X(t_w) = \begin{cases} 1 \text{ , if } 2 \geq t_2 \\ \frac{t_2^3 - 1}{26} \text{ , if } 1 \leq t_2 < 3 \\ 0 \text{ , otherwise} \end{cases}$$

4.) The marginal density functions can be computed in two different ways:

$$f_X(x) = \partial_x F_X(x)$$

$$f_Y(y) = \partial_y F_Y(y).$$

or

$$f_X(x) = \int_1^3 \frac{1}{13} x y^2 \mathrm{d}y$$

$$f_Y(y) = \int_1^2 \frac{1}{13} x y^2 \mathrm{d}x.$$

Both case give us

$$f_X(x) = \begin{cases} \frac{2x}{3} \text{ , if } 1 < x < 2 \\ 0 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{3y^2}{26} \text{ if } 1 < y < 3 \\ 0. \end{cases}$$

5.) At first, $\mathbb{E}(X)$ will be calculated by using the marginal density function f_X.

$$\mathbb{E}(X) = \int_1^2 x \frac{2x}{3} \mathrm{d}x = \frac{2}{3} \int_1^2 x^2 \mathrm{d}x = \frac{14}{9}.$$

In addition,

$$\mathbb{E}(Y) = \int_1^3 y \frac{3y^2}{26} \mathrm{d}y = \frac{3}{26} \int_1^3 y^3 \mathrm{d}y = \frac{30}{13}.$$