# Probability Theory and Statistics

Exercise 9

11.03. – 11.07.

Covariance and correlation of continuous random variables, linear regression

1. Let $X$ and $Y$ be independent random variables such that $\mathbb{E}(X) = 4$, $\mathbb{E}(Y) = 0$, $\mathbb{D}^2(X) = 1$, and $\mathbb{D}^2(Y) = 2$. Determine $\mathbb{E}(5X - 6Y)$, $\mathbb{E}(XY)$, $\mathbb{D}^2(5X - 6Y + 8)$, and $\operatorname{cov}(5X + 2Y + 2, X + 6Y - 3)$.
2. Let $X, Y \sim \mathrm{U}(0, 1)$ be independent, and define $Z_1 = XY$ and $Z_2 = X + Y$.
   a) Find the expected values of $Z_1$ and $Z_2$.
   b) Is it true that $\mathbb{D}^2(XY) = \mathbb{D}^2(X) \cdot \mathbb{D}^2(Y)$?
   c) Compute $\mathbb{D}^2(Z_1 + 2Z_2)$.
3. Let the joint density function of $X$ and $Y$ be
$$f_{X,Y} : (x, y) \mapsto \begin{cases} 2(x^3 + y^3), & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$
   Determine $\operatorname{cov}(X, Y)$.
4. Let $X \sim \mathrm{N}(m, \sigma^2)$, $Y = 3X + 8$, and $Z = 5 - 2X$. Compute the correlation coefficient $\operatorname{corr}(Y, Z)$.
5. Let $X \sim \mathrm{U}(0, 2\pi)$, $Y = \cos(X)$, and $Z = \sin(X)$. Determine $\operatorname{cov}(Y, Z)$. Are $Y$ and $Z$ independent?
6. Let the joint density function of $U$ and $V$ be
$$f_{U,V} : (u, v) \mapsto \begin{cases} \dfrac{1}{\sqrt{v}}, & \text{if } 0 < u < 1 \text{ and } 0 < v < u^2, \\ 0, & \text{otherwise.} \end{cases}$$
   Determine $\operatorname{cov}(U, V^2)$.

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7. Let $X_1$, $X_2$, and $X_3$ be uncorrelated random variables, each with mean 0 and standard deviation 1.
   a) Compute $\operatorname{cov}(X_1 + X_2, X_2 + X_3)$, $\mathbb{D}^2(X_1 + X_2)$, and $\mathbb{D}^2(X_2 + X_3)$.
   b) What is the correlation between $X_1 + X_2$ and $X_2 + X_3$?
   c) Find the regression line of $X_2 + X_3$ on $X_1 + X_2$.
   d) Determine the variance of the approximation error using the above regression line.
8. Find the regression line of $X^2 + X + 1$ on $X$, where the distribution of $X$ is
   a) $\mathrm{Exp}(5)$ b) $\mathrm{N}(0, 1)$ c*) $\mathrm{Bin}(10, 0.3)$
9. Let the joint density of $X$ and $Y$ be the same as in Problem 3. Knowing that $\mathbb{D}^2(X) = 0.0775$, find the regression line of $Y$ on $X$ and the regression line of $X$ on $Y$. Determine the mean squared error (i.e., the variance of the error) for each regression.
10. Suppose that the regression line of $Y$ on $X$ is given by $\{(x, y) \in \mathbb{R}^2 \mid y = x - 2\}$, while the regression line of $X$ on $Y$ is $\{(y, x) \in \mathbb{R}^2 \mid x = \frac{1}{4}y - \frac{11}{2}\}$. Find $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.
11. Two dice are rolled. Let $X$ denote the number of sixes, and $Y$ the number of even results. Find the regression line of $Y$ on $X$ and that of $X$ on $Y$.