Probability Theory and Statistics

Exercise 8

10.27. – 10.31.

Covariance, Joint Density Function

1. Let $X \sim \mathrm{U}(0,3)$ and $Y \sim \mathrm{U}(-1,4)$ be independent random variables. Plot the level sets of the joint distribution function of (X,Y). Compute the following quantities:

   a) $\mathbb{P}(X < Y) =?$ b) $\mathbb{P}(X + Y = 1) =?$ c) $\mathbb{P}(XY < 1) =?$

2. Let $X, Y \sim \mathrm{U}(0,1)$ be independent, and define $Z = 2X + 1$, $V = 3Y$. Compute $\mathbb{P}(V < Z)$.
3. Let the joint density of X and Y be

$$f_{X,Y}(x,y) = \begin{cases} 2(x^3 + y^3), & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) $\mathbb{P}(X + Y < 1) = ?$
   (b) $\mathbb{P}(X^2 < Y) = ?$
   (c) Determine the marginal density functions of X and Y.
   (d) $\mathbb{E}(X) =?$
   (e) Are X and Y independent?

4. The continuous random vector (X,Y) has distribution function

$$F_{X,Y}(x,y) = \frac{xy^3 + x}{2}, \qquad 0 < x < 1,\ |y| < 1.$$

   The range of X is $[0,1]$, and the range of Y is $[-1,1]$. Find the probability that (X,Y) lies inside the triangle with vertices $A(0,0)$, $B\left(\frac{1}{2},0\right)$, and $C\left(\frac{1}{2},-\frac{1}{4}\right)$. (Hint: the joint density function may be useful.)

5. Let the joint density of X and Y be

$$f_{X,Y}(x,y) = \begin{cases} a(4x + y) + bxy + \frac{2}{5}, & \text{if } 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{otherwise,} \end{cases}$$

   where $a, b \in \mathbb{R}$. For which values of a and b are X and Y independent random variables?

6. The continuous random vector (X,Y) is uniformly distributed over the region bounded by the x–axis and the sine curve $y = \sin(x)$ between the points $(0,0)$ and $(\frac{\pi}{2},0)$. Determine the joint density of (X,Y) and the marginal density of X.

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7. A hat contains the numbers $1,2,3$ written on slips of paper. Two slips are drawn successively without replacement. Let X be the result of the first draw and Y that of the second.

   a) $\mathrm{cov}(X,Y) =?$ b) $\mathrm{cov}(X,X) =?$ c) $\mathrm{cov}(Y,Y) =?$ d) Are X and Y independent?

8. Let X be a random variable with $\mathbb{E}(X) = 2$, $\mathbb{E}(X^2) = 5$, and $\mathbb{E}(X^3) = 14$. Compute $\mathrm{cov}(X, X^2 - 4X + 4)$. Are $X - 2$ and $(X-2)^2$ independent?
9. Two dice are rolled. Let X denote the number of ones obtained and Y the outcome of the second die. Determine the covariance of X and Y.
10. Prove that if X and Y are random variables with the same (finite) variance, then the covariance of $X + Y$ and $X - Y$ is 0.