Probability Theory and Statistics

Exercise 7

10.20. – 10.24.

Normal Distribution, Central Limit Theorem

1. Suppose a battery's lifetime is normally distributed with mean 6.3 years and standard deviation 2 years.
   a) If the warranty is 8 years, what is the probability that a failure occurs before the warranty expires?
   b) How many years of warranty should we offer so that the probability the device fails only *after* the warranty period is 0.95?
2. A machine is set to fill 2 kg of flour into bags, but due to the technology, the amount of flour in a bag follows $N(m; 0.002^2)$. From prior observations we know that the probability the amount is less than 2 kg is 0.01. What is m?
3. A normally distributed random variable takes a value less than 10 with probability 0.2, and greater than 14 with probability 0.3. What are the parameters of its distribution?
4. In Texas, temperatures are measured in degrees Fahrenheit. It was determined that summer temperatures there follow $N(86; 16)$. How does the distribution change if we switch to the Celsius scale? $\left(\frac{5}{9}(X-32)\ [°\mathrm{F}] = Y\ [°\mathrm{C}]\right)$
5. Let X be a normally distributed random variable such that $Y = 2X + 3$ is standard normal. Determine the parameters of the distribution of X.
6. Let $X \sim \mathrm{N}(m; \sigma^2)$ and $Z = \left(\frac{X-m}{\sigma}\right)^2$. Compute the density of Z.
7. Let $X \sim \mathrm{N}(m; \sigma^2)$. Compute $\mathbb{E}(e^{\alpha X})$ for $\alpha \in \mathbb{R}$.

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8. Béla is picky about movies: on average he likes 1 out of 10 films. What is the probability that among 300 films he watches, he finds more than 42 that he likes?
9. In a BME-VIK cohort, 500 students take a course. A consultation is organized before the exam. Based on preliminary surveys, students attend independently with probability 0.25. How large should the room be so that, with 90% confidence, all attendees fit?
10. Using a computer, we simulated $X_1, X_2, \ldots, X_{12} \sim \mathrm{U}(0; 1)$ (jointly) independent random numbers. Using these, generate an *approximately* $N(5; 4)$-distributed random number.
11. The time required to complete a subtask is $\mathrm{U}(1; 3)$-distributed, and the times for different subtasks are (jointly) independent. Subtasks are performed sequentially. For 1000 subtasks, what is the probability that we finish in less than 1984 time units?
12. Let $X_1, \ldots, X_n \sim \mathrm{Exp}(\lambda)$ be (jointly) independent random variables, where $\lambda > 0$ is fixed. Give an approximation for λ if we know that
$$\mathbb{P}\left(\sum_{i=1}^{n} X_i > \frac{n}{\lambda} + \frac{\sqrt{n}}{3}\right) = 0.1587.$$
13. A live TV program broadcasts short interviews one after another. Guests are asked to plan their message for m minutes. Empirically, an interview then has expectation m and standard deviation 10% of its expectation. What m should organizers tell the 10 guests that day if they want the chance of overrunning the 100-minute slot to be approximately 3%?

14. The number of worms in a given cherry is Poisson with parameter $\lambda = 0.1$, and the worm counts in any distinct cherries are jointly independent.
    (a) What is the probability that a given cherry contains at least one worm?
    (b) We take 100 cherries; let X be the total number of worms in them. What are the expectation and standard deviation of X?
    (c) Using a suitable approximation, estimate the probability that in the 100 cherries we find at least 13 worms in total. *You do not need the exact probability; no points are awarded for that.* *(Exam course, Problem 3, 2025-06-12.)*
15. We roll ten fair dice; let X be the sum of the outcomes. Estimate, using the CLT, the probabilities $\mathbb{P}(24 < X < 46)$ and $\mathbb{P}(31 \le X \le 39)$.

Table of the Standard Normal Distribution Function

z	0,00	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767
2,0	0,9772	0,9778	0,9783	0,9788	0,9793	0,9798	0,9803	0,9808	0,9812	0,9817
2,1	0,9821	0,9826	0,9830	0,9834	0,9838	0,9842	0,9846	0,9850	0,9854	0,9857
2,2	0,9861	0,9864	0,9868	0,9871	0,9875	0,9878	0,9881	0,9884	0,9887	0,9890
2,3	0,9893	0,9896	0,9898	0,9901	0,9904	0,9906	0,9909	0,9911	0,9913	0,9916
2,4	0,9918	0,9920	0,9922	0,9925	0,9927	0,9929	0,9931	0,9932	0,9934	0,9936
2,5	0,9938	0,9940	0,9941	0,9943	0,9945	0,9946	0,9948	0,9949	0,9951	0,9952
2,6	0,9953	0,9955	0,9956	0,9957	0,9959	0,9960	0,9961	0,9962	0,9963	0,9964
2,7	0,9965	0,9966	0,9967	0,9968	0,9969	0,9970	0,9971	0,9972	0,9973	0,9974
2,8	0,9974	0,9975	0,9976	0,9977	0,9977	0,9978	0,9979	0,9979	0,9980	0,9981
2,9	0,9981	0,9982	0,9982	0,9983	0,9984	0,9984	0,9985	0,9985	0,9986	0,9986
3,0	0,9987	0,9987	0,9987	0,9988	0,9988	0,9989	0,9989	0,9989	0,9990	0,9990
3,1	0,9990	0,9991	0,9991	0,9991	0,9992	0,9992	0,9992	0,9992	0,9993	0,9993
3,2	0,9993	0,9993	0,9994	0,9994	0,9994	0,9994	0,9994	0,9995	0,9995	0,9995
3,3	0,9995	0,9995	0,9995	0,9996	0,9996	0,9996	0,9996	0,9996	0,9996	0,9997
3,4	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9998