Probability Theory and Statistics

Exercise 6

10.13. – 10.17.

Exponential and Uniform Distributions. Distribution Transformations and Expected Value in the Continuous Case

1. While stargazing on an early August night, we wait for a shooting star. We know that the probability of seeing a shooting star within the first 20 minutes is $1 - e^{-\frac{2}{3}}$. We may assume that the time remaining until the next shooting star does not depend on the time already elapsed (i.e. the distribution is continuous and memoryless). What is the probability of seeing a shooting star within the first hour? What is the variance of the waiting time until the first sighting?
2. Suppose that a certain washing machine model lasts on average 2 years until its first malfunction, and the time to first failure follows a continuous, memoryless distribution. What is the probability that it will not break down within the first 3 years, given that it worked perfectly during the first 2 years?
3. A faulty software has a memory leak, which occasionally causes the program to crash and restart. The running time between crashes (measured in hours) is exponentially distributed with parameter $1/10$. Dominik starts the program at 4 p.m., just before leaving work. Given that at 8 a.m. the next morning he finds the program still running (i.e. it has not crashed once), what is the probability that it will continue running until the lunch break at noon?
4. Let X and Y be exponentially distributed random variables. We know that the expected value of X is twice that of Y, and that $3\mathbb{P}(X > 1) = 2\mathbb{P}(Y < 1)$. What is the expected value of X?
5. In a certain bank branch, the waiting time follows a continuous, memoryless distribution. It is known that the probability that a client has to wait at least one hour is e^{-2}. Three friends, Adam, Betty, and Cecilia, visit this bank branch at different times. Assume that their waiting times are independent. What is the probability that at least two of them have to wait at least half an hour?

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6. Roll a fair die and let the result be X. Define $Y = |X - 3|$. Find the distribution function of Y, and compute $\int_0^\infty (1 - F_Y(y))\,\mathrm{d}y$ and $\mathbb{E}(Y)$.
7. Let X be a continuous random variable with density

$$f_X(x) = \begin{cases} 2x, & \text{if } 0 < x < a, \\ 0, & \text{otherwise,} \end{cases}$$

   where $a > 0$ is an appropriately chosen constant. Determine the value of a. What is the expected value of X?
8. A gas station's storage tank is refilled weekly. Let X denote the weekly consumption (measured in hundreds of thousands of liters), with density

$$f_X(x) = \begin{cases} 5(1-x)^4, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   How large should the tank's capacity be so that the probability of running out of fuel during the week is less than 0.05? What is the expected weekly consumption?

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9. Let X have density $f_X(x) = \frac{1}{2\sqrt{x}}$ for $0 < x < 1$ and 0 otherwise. Let $Y = X\sqrt{X}$.
   (a) Determine the distribution function of X.
   (b) Determine the distribution function of Y.
   (c) Determine the density function of Y.
   (d) Compute $\mathbb{E}(Y)$ using the density of Y.
   (e) Compute $\mathbb{E}(X\sqrt{X})$ directly using the density of X.
10. Let $X \sim \mathrm{Exp}(\lambda)$ and $Y = X^2$. Find the density and the expected value of Y.
11. Let $X \sim \mathrm{U}(0;1)$, and define $Y = \sqrt{2X}$, $V = \ln\frac{1}{X}$, and $Z = \arctan(X)$. Find the density function of Y, V, and Z.
12. In the United States, car fuel consumption is measured in miles per gallon (mpg), that is, how many miles a vehicle travels on one gallon of fuel. In Europe, consumption is expressed in liters per 100 km. Suppose that the mpg consumption X has density f_X. How should f_X be transformed when converting to the liters/100 km scale? (Assume 1 mile $= a$ km and 1 gallon $= b$ liters, where $a = 1.609$ and $b = 3.785$. Assume that F_X is differentiable wherever it is nonzero.)
13. Let

$$F_X(x) = \begin{cases} 0, & \text{if } x \leq 1, \\ \frac{1}{3}x^2 - \frac{1}{3}, & \text{if } 1 < x < 2, \\ 1, & \text{if } x \geq 2. \end{cases}$$

   (a) Determine the density function of X.
   (b) Find the expected value of X.
   (c) Let Y be uniformly distributed on $(1,2)$. Prove that for every $k \in \{1, 2, \ldots\}$,

$$\frac{3}{2}\mathbb{E}(X^k) = \mathbb{E}(Y^{k+1}).$$

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14. Let X be exponentially distributed, and suppose $\mathbb{P}(X > 3) = e^{-6}$.
   (a) What is the parameter λ of the distribution? (b) Compute $\mathbb{P}(X < 2)$. (c) Compute $\mathbb{E}(X)$.
15. A sleepy passenger falls asleep on the M3 metro line, which is 16.4 km long, and wakes up at a random point along the line. Let X be the distance (in km) from the terminus Kőbánya-Kispest to the point where he wakes up. Then X is uniformly distributed. What is the standard deviation of X? Would the answer change if the same experiment were conducted on another metro line of equal length?
16. On a unit square, two opposite sides are chosen. Select one point a uniformly at random on one side and one point b on the opposite side. Let X denote the square of the distance between a and b. What is the expected value of X?
17. On the interval $(0,1)$, three points are chosen uniformly at random. Let Y denote the middle (median) point. What is the expected value of Y?
18. Let X be a random variable with distribution function $x \mapsto F_X(x)$. Express the distribution functions of the following random variables in terms of F_X:
   (a) $Y = \max\{0, X\}$ (b) $Z = -X$ (c) $V = |X|$ (d) $W = \min\{0, -X\}$.