Probability Theory and Statistics

Exercise 5

10.13. – 10.17.

Discrete Joint Distributions, Independence of Random Variables, Introduction to Continuous Distributions

1. We roll a fair die twice. Let X be the number of sixes, and let Y be the number of even outcomes.
   (a) Are X and Y independent?
   (b) Give the table of the joint distribution of X and Y, as well as their marginal distributions.
2. The joint distribution of random variables X and Y is given by the table below.

Y \ X	-1	0	1
-1	p	3p	6p
1	5p	15p	30p

   a) $p = ?$ b) $\mathbb{P}(X \leq 0,\, Y = 1) = ?$ c) Are X and Y independent? d) $\mathbb{E}(XY) = ?$
3. A box contains 6 balls: 2 white, 2 green, and 2 red. We draw balls one by one *without replacement* until the first red appears. Let X be the number of draws, and let Y be the number of white balls drawn. Give the table of their joint distribution. Are X and Y independent?
4. Let $X, Y \sim \mathrm{Geo}(\frac{2}{3})$ be independent. Determine the following quantities:
   a) $\mathbb{E}(XY)$ b) $\mathbb{P}(X = 2 \mid Y = 5)$ c*) $\mathbb{P}(X = Y)$.
5. The joint distribution of X and Y is given by the following table, but two entries are missing. Determine these entries assuming that the events $\{X = 2\}$ and $\{Y = 0\}$ are independent. Are X and Y themselves independent? Compute the probability $\mathbb{P}(XY > 0 \mid X < 2)$ and the expectations $\mathbb{E}(4X + Y)$ and $\mathbb{E}(XY)$.

Y \ X	0	1	2
0	1/10	1/5	
2		1/4	1/5

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6. Are the following mappings $F : \mathbb{R} \to \mathbb{R}$ distribution functions?
   a) $F(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$ b) $F(x) = e^{-e^{-x}}$
   c) $F(x) = 1 - e^{-x^2}$ d) $F(x) = \frac{1}{\pi}\mathrm{arctg}(x) + \frac{1}{2}$
7. Determine the value of α so that f is a probability density function. Also give the corresponding distribution function.
   a) $f : x \mapsto \begin{cases} \alpha\,(2x - x^2) & \text{if } 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$ b) $f : x \mapsto \begin{cases} \alpha\sqrt{x-2} & \text{if } 2 < x < 3, \\ 0 & \text{otherwise.} \end{cases}$
   c) $f : x \mapsto \begin{cases} \alpha\sqrt{x-2} & \text{if } 3 < x < 4, \\ 0 & \text{otherwise.} \end{cases}$ d) $f : x \mapsto \begin{cases} \alpha\cos\dfrac{x}{2} & \text{if } 0 < x < \pi, \\ 0 & \text{otherwise.} \end{cases}$

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   (a) Are X and Y independent?
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2. The joint distribution of random variables X and Y is given by the table below.

$Y \setminus X$	-1	0	1
-1	p	3p	6p
1	5p	15p	30p

   a) $p = ?$ b) $\mathbb{P}(X \leq 0,\, Y = 1) = ?$ c) Are X and Y independent?   d) $\mathbb{E}(XY) = ?$
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8. The distribution of crater diameters (in km) on the dwarf planet Pluto can be approximated by a random variable S with density

$$f_S : x \mapsto \begin{cases} c\,x^{-\frac{5}{2}} & \text{if } x > d, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $\mathbb{P}(S > 9) = 0.2689$. a) Determine c. b) Determine d.

9. Let $X \sim \mathrm{Pois}(3)$ and $Y = 3X - 1$. Determine the value of the distribution function of Y at the point π.

10. We roll a (fair, six-sided) die. Let X be the outcome, and let Y be the number of positive divisors of the outcome. Plot the distribution functions of X and Y. Are these distribution functions continuous?

The following problems will (partly) be discussed during the lecture at a previously announced time. Except for Problems 11(a), 12(a), and 14(a), problems of these types can only appear in the midterm as part of Problem 6.

11. On the interval $[0, 1]$, we choose a point P uniformly at random. Let X be the distance between P and the point 0.3.
    (a) Let $\Omega = [0, 1]$ be our sample space. Write X explicitly as a function $\Omega \to \mathbb{R}$.
    (b) Determine the distribution function of X.
    (c) Determine the density function of X.

12. On the interval $[0, 1]$, we choose a point P uniformly at random. Let X be the square of the distance from P to the endpoint of the interval that is closer to P.
    (a) Let $\Omega = [0, 1]$ be our sample space. Write X explicitly as a function $\Omega \to \mathbb{R}$.
    (b) Determine the distribution function and the density function of X.

13. In the plane, within the right isosceles triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$, we choose a point (X, Y) uniformly at random.
    (a) Determine the distribution function and density of X.
    (b) Determine the distribution function and density of Y.

14. In $\mathbb{R}^2$, within the triangle with vertices $(0, 0)$, $(1, 0)$, and $(\frac{1}{2}, \frac{1}{2})$, we choose a point P uniformly at random. Let X be the distance from P to the x-axis.
    a) Let the triangle (as a subset of $\mathbb{R}^2$) be our sample space, denoted by Ω. Write X explicitly as a function $\Omega \to \mathbb{R}$.
    b) What are the distribution function and the density function of X?

15. On a unit square, pick one point a uniformly at random on one side, and another point b uniformly at random on the opposite side (independently). Let X be the square of the distance between the two points.
    a) Determine the distribution function of X.
    b) Determine the density function of X.
    c) Where does the density of X attain its maximum value?

16. On the interval $(0, 1)$ we select three points independently and uniformly at random. Let Y be the middle point (the median). Determine the distribution function and the density function of Y.

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$$f_S : x \mapsto \begin{cases} c\,x^{-\frac{5}{2}} & \text{if } x > d, \\ 0 & \text{otherwise.} \end{cases}$$

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14. In $\mathbb{R}^2$, within the triangle with vertices $(0,0)$, $(1,0)$, and $(\frac{1}{2}, \frac{1}{2})$, we choose a point P uniformly at random. Let X be the distance from P to the x-axis.
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