Probability Theory and Statistics

Exercise 4

09.29 – 10.03.

Expected value, Expected value of transformed random variables, deviation

1. Rolling two dice, what is the expected value of the maximum of the two outcomes?

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2. A store sells light bulbs. 1% of the bulbs are defective. If we buy 100 bulbs, how many defective ones do we expect to have bought? What is the variance of the number of defective bulbs?

3. We flip a coin until the first time that two consecutive flips have the same outcome. What is the expected number of flips and the variance?

4. On the square $[-1,1]\times[-1,1]$ we repeatedly (independently, uniformly) select random points. We stop when the first selected point falls inside the unit circle centered at the origin. What is the distribution of the number of selected points? What is its expected value?

5. At the university there are many telephones that break down independently with the same small probability. Out of 360 days in a year, on average there are 12 days when no phone breaks down. How many phones are expected to break down on a given day? How many days are expected (in a 360-day year) when 2 or more phones break down?

6. The engineers of a light bulb factory observed that the more bulbs they produce in a day, the smaller the probability that a single bulb is defective. If $n \geq 1$ bulbs are produced in a day, then each bulb is defective with probability $\frac{1}{2n}$, independently.

1. If $n = 10$ bulbs are produced in one day, what is the expected value and variance of the number of defective bulbs that day (and why)?
2. For $n = 10$, what is the probability that more than 4 but fewer than 8 bulbs are defective out of the 10? (Give the exact probability.)
3. Suppose that n bulbs are produced in a day, where n is very large. Approximate (with a suitable approximation) the probability that exactly 3 bulbs are defective on that day. (The exact probability does not need to be computed.)

7. Let X denote the outcome of a die roll. Compute $\mathbb{E}\left((X-3)^2\right)$.

8. Let X be the index of a randomly chosen month from April to December. The probability of choosing the i-th month is $\frac{i}{72}$ ($\forall i = 4, \ldots, 12$). Let $Y = (-1)^X$.

a) Determine the distribution of Y.

b) Compute $\mathbb{E}(Y)$ using the distribution of Y.

c) Derive $\mathbb{E}\Big((-1)^X\Big)$ using the distribution of X as well.

9. The first four prime numbers are written on the sides of a regular tetrahedron. We roll it three times. Let X denote the number of 7s obtained, and define $Y = X^2$, $Z = X^2 + X + 1$. Determine the expected values of Y and Z, and the variance of X.

10. We perform independent trials, each succeeding with probability p. We repeat the trials until we get 3 successes. Let X be the number of failures required. Compute $\mathbb{E}\left(\frac{1}{(X+2)(X+1)}\right)$ (as a function of p).

11. In a meadow, three deer are grazing unsuspectingly. Unaware of each other, three hunters sneak to the clearing and fire simultaneously. Each shot hits and is lethal. What are the expected value and variance of the number of deer running away after the shots? (Note: several hunters may shoot at the same deer, and each shot hits each deer with equal probability.)

12. Let X be a Poisson random variable with parameter $\lambda > 0$, and let $Y = 2X + 1$. Determine the variance of Y.

13. If we know that $\mathbb{E}X = 1$ and $\mathbb{D}^2X = 5$, compute a) $\mathbb{E}((2+X)^2)$ b) $\mathbb{D}^2(4+3X)$.

14. Let $X \sim \mathrm{Geo}\left(\frac{1}{3}\right)$. Compute $\mathbb{E}\Big((3-X)^2\Big)$ and $\mathbb{D}(5-2X)$.