Probability Theory and Statistics

Exercise 3

09.22 – 09.26.

Combinatorics, Discrete Random Variables, Binomial-, Poisson-, Geometric distribution

1. A department has 12 students: 4 seniors and 8 juniors.

1. How many 5-person committees can be formed?
2. How many 5-person committees contain *at least two* seniors?
3. Two particular students, Alice and Bob, refuse to serve together. How many 5-person committees can be formed under this restriction?

2. An ice-cream shop sells 7 flavors. You order a cup with 10 scoops; scoops are indistinguishable except for flavor, order does not matter, and you may choose multiple scoops of the same flavor.

1. How many different cups can you order?
2. How many different cups can you order if the cup must contain *at least one* vanilla scoop?
3. How many different cups can you order if no single flavor is used more than 3 times?

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3. Flip a fair coin three times. Let the sample space Ω be the set of length-3 heads–tails sequences, and label its elements in the obvious way: $FFF, FIF, \ldots$. Define the function $X : \Omega \to \mathbb{R}$ by $X(FFF) = 0$, and for any other outcome let X be the index of the first occurrence of "tails" (e.g., $X(FIF) = 2$).

1. What is the probability that X is odd?
2. Define Y in the same way as X, except that $Y(FFF)$ is randomly either 0 or 1. Is Y a random variable on the sample space Ω?

4. Roll a fair die twice. Define the random variable X as the number of sixes rolled (e.g., if both rolls are six, then $X = 2$). What is the probability that X is even?

5. Let A, B, and C be three events with the following probabilities and intersection probabilities:

$$\mathbb{P}(A) = 0.5 \qquad \mathbb{P}(B) = 0.4 \qquad \mathbb{P}(C) = 0.3 \qquad \mathbb{P}(A \cap B) = 0.3$$
$$\mathbb{P}(B \cap C) = 0.2 \qquad \mathbb{P}(C \cap A) = 0.1 \qquad \mathbb{P}(A \cap B \cap C) = 0.1$$

Let Y denote the number of events that occur among A, B, and C. What is $\mathbb{P}(0 < Y < 3)$? Give the values of the pmf $k \mapsto p_Y(k) = \mathbb{P}(Y = k)$ for every k with positive probability.

6. Roll two 10-sided dice; denote the outcomes by X and Y. Compute $\mathbb{P}(X \leq Y)$.

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7. A store sells light bulbs. One percent of the bulbs are defective. If we buy 100 bulbs, then

1. What is the probability that at most three are defective?
2. What number of defective bulbs is most probable?

8. How many times should we roll a die to ensure that the probability that the number of sixes is at least two is not less than 0.5?

9. An outdated website is served by 10 microservices, each independently available with probability 80%. The system is considered to operate optimally as long as at most 3 services are down. What is the probability that the site operates optimally?

10. Choose points independently and uniformly at random from the unit interval. Continue choosing points until one falls into the middle third of the interval. Let X be the number of points chosen. What is $\mathbb{P}(X < 5)$?

11. Roll a fair die until you obtain a number less than 3. Let X be the number of rolls needed. Which probability is larger: $\mathbb{P}(2 \leq X \leq 3)$ or $\mathbb{P}(X \geq 3)$?

12. Flip a fair coin repeatedly until you obtain the second head. What is the probability that the number of flips needed *after* the first head to reach the second head equals the number of flips needed to obtain the first head?