Probability Theory and Statistics

Exercise 10

11.10. – 11.14.

Conditional expectation, law of total probability in continuous case, law of total expectation

1. We roll a fair die three times. Let X denote the smallest and Y the largest outcome. Compute the conditional expectation $\mathbb{E}(X \mid Y = 3)$.
2. We toss a fair coin four times. Let Y denote the total number of heads, and let A be the event that the *second* toss is the first head. Find the conditional expectation $\mathbb{E}(Y \mid A)$.
3. A fair die is rolled n times. Let X be the number of sixes, and Y the number of even outcomes. Compute the regression $\mathbb{E}(Y \mid X)$.

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4. A number X is chosen uniformly at random from the interval $[1, 2]$. Given X, we wait for an exponentially distributed time with parameter X; denote this waiting time by Y.
   (a) Compute $\mathbb{E}(Y \mid X)$.
   (b) Determine the joint density function of (X, Y).
5. Let the joint density function of X and Y be
$$f_{X,Y}(x, y) = \begin{cases} \frac{12}{5}(x^2 - xy + y^2), & 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$
   Find the following: a) the conditional density $f_{Y|X}(y \mid x)$, b) the conditional expectation $\mathbb{E}(Y \mid X)$, c) the covariance matrix of (X, Y), and d) the linear regression of Y on X.
6. Let the joint density function of X and Y be
$$f_{X,Y}(x, y) = \begin{cases} \frac{4}{5}(x + y + xy), & 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$
   Determine the regression $\mathbb{E}(X \mid Y)$.
7. The joint density function of two random variables X and Y is
$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{4}, & 1 < x < 3,\ 2 < y < 2x, \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Determine $f_{Y|X}(y \mid x)$ for all $x, y \in \mathbb{R}$.
   (b) Show that $\mathbb{E}(Y \mid X) = X + 1$.
   (c) Compute the probability $\mathbb{P}(X < 2)$.
8. The continuous random vector (X, Y) is uniformly distributed over the region bounded by the x-axis and the curve $y = \sin(x)$ between the points $(0, 0)$ and $(\frac{\pi}{2}, 0)$. Determine the conditional density $f_{Y|X}$ and the conditional distribution function $F_{Y|X}$ (you may use the formulas of f_X and $f_{X,Y}$ from Exercise 8).
9. Let $X \sim \mathrm{N}(0, 1)$ and $Y \sim \mathrm{Exp}(2)$ be independent random variables. Compute the following conditional expectations:

   a) $\mathbb{E}(3X - Y + 1 \mid X)$ b) $\mathbb{E}((2XY)^2 - 7Y \mid X)$

   c) $\mathbb{E}(X^2 + 2XY + Y^2 \mid X + Y)$ d) $\mathbb{E}(X^2 \tan(Y) + \frac{5X}{Y} - 2Y \mid Y)$

10. Let $X \sim \mathrm{N}(2, 4)$ and $Y \sim \mathrm{U}(0, 2)$ be independent random variables, and define

$$Z = 2X \ln Y + X^2 Y^2 - 3Y.$$

Compute the regression $\mathbb{E}(Z \mid Y)$.

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11. A file is stored on one of two drives: with probability $\frac{2}{3}$ on the first drive, and otherwise on the second. Searching the first drive takes 7 minutes, and searching the second takes 3 minutes. Assume that the file is found only after the complete scan of the drive that contains it. What is the expected search time if we start the search a) with the first drive, b) with the second drive?
12. A man is lost in a mine and stands at a junction with three possible paths. If he chooses the first path, he exits the mine after 1 hour; if he chooses the second, he returns to the same junction after 2 hours; if he chooses the third, he returns after 3 hours. Assuming that whenever he has to choose, he picks each option with equal probability, find the expected time until he exits the mine.