The ultimate categorical independence ratio

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Outline of the talk

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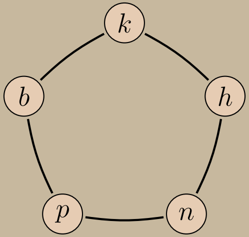
- Shannon-capacity (information theoretic background, graphtheoretical definition),
- The ultimate categorical independence ratio (definition, open problems related to the paremeter, my results),
- Comparing the two graph invariants.

Shannon capacity

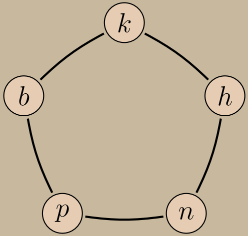
The Shannon-capacity is the theoretical upper limit of channel capacity for error-free coding in information theory.

Let us consider a channel on which one can transmit five characters. Because of the noise in the channel some pairs of characters may be confused during the transmission.
The graph on the figure represents the channel:
the vertices are the characters, and
two vertices are connected with an edge if they can be confused.



Our aim is to transmit on the channel as much information as we can (in a unit of time).

Shannon capacity



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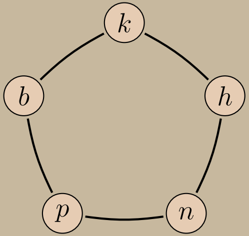
Shannon capacity



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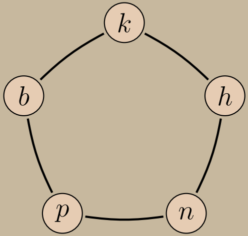
We can create 4 distinguishable **character pairs**: 'kk', 'kn', 'nk', 'nn'.

Shannon capacity



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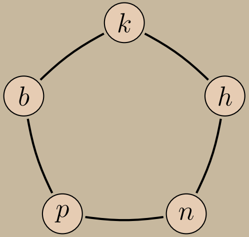
Moreover, we can create 5 distinguishable character pairs: **'kk', 'hn', 'nb', 'ph', 'bp'**.

Shannon capacity



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Using t characters we can transmit $5^{t/2}$ different messages.

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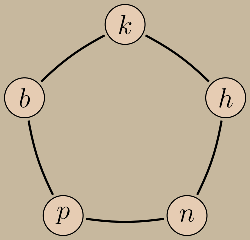

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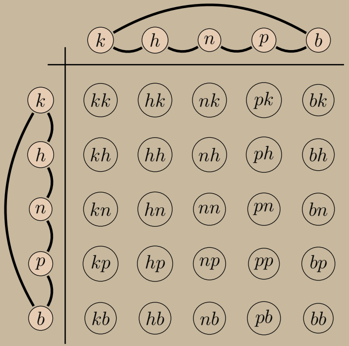
It was an open problem for a long time whether this is the optimum, and a celebrated result of Lovász is that the Shannon-capacity of the pentagon is $\sqrt{5}$, that is, on this channel one cannot send more then $\sqrt{5}^t$ distinguishable messages with t characters.

Shannon capacity

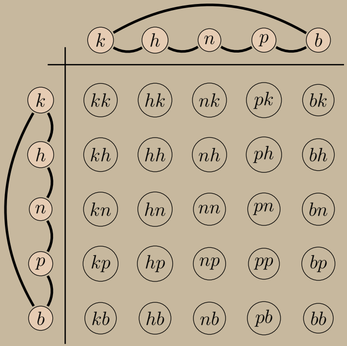
The **independence number** of a graph is the size of the largest subset of the vertex set which contains no edge. The parameter is denoted by $\alpha(G)$.



Shannon capacity

The **independence number** of a graph is the size of the largest subset of the vertex set which contains no edge. The parameter is denoted by $\alpha(G)$.

The **normal product** of two graphs is defined on set of pairs formed by the vertex set of the two base graphs, and we connect two pairs if the corresponding elements are equal or form an edge in both coordinates.



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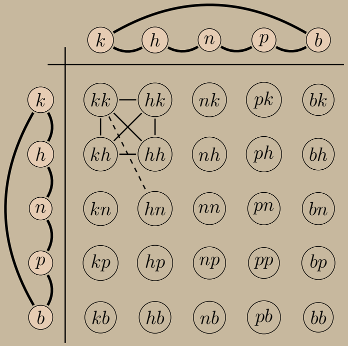

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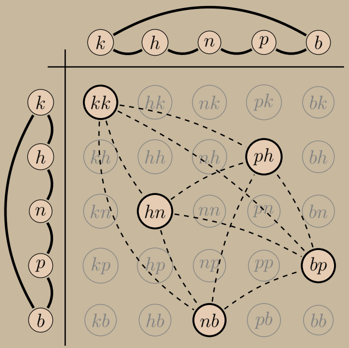

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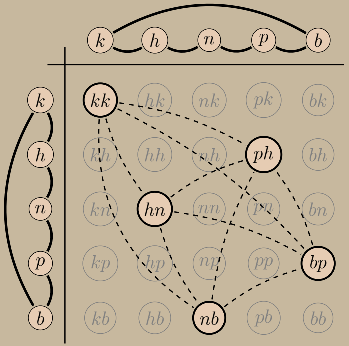

The **normal product** of two graphs is defined on set of pairs formed by the vertex set of the two base graphs, and we connect two pairs if the corresponding elements are equal or form an edge in both coordinates.

Shannon-capacity of a graph is defined as the normalized limit of the independence number under the normal power. That is,

$$c(G) = \lim_{t\to\infty} \sqrt[t]{\alpha(G^t)}.$$

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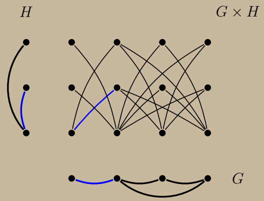
The asymptotic value of the independence ratio for categorical product

The **independence ratio** of a graph G: $i(G) = \frac{\alpha(G)}{|V(G)|}$, where
$\alpha(G)$ denotes the independence number of G.

The **categorical product** of graphs F and G:
$F \times G$, where
$V(F \times G) = V(F) \times V(G)$,
$\{(u_1, v_1), (u_2, v_2)\} \in E(F \times G)$, if
$\{u_1, u_2\} \in E(F)$ and $\{v_1, v_2\} \in E(G)\}$.



The kth categorical power of a graph G is denoted by $G^{\times k}$.

Definition (Brown, Nowakowski, Rall - 1996.):
The ultimate categorical independence ratio:

$$A(G) = \lim_{k \to \infty} i(G^{\times k})$$

$(0 < i(G) \leq A(G) \leq 1)$

The results of Brown, Nowakowski and Rall

Theorem (Brown, Nowakowski, Rall - 1996.):

$A(G) \geq \frac{|U|}{|U|+|N(U)|}$, if U is an independent set of G, and $N(U)$ denotes the neighbourhoodset of U.

Theorem (BNR): $A(G) \in \left(0, \frac{1}{2}\right] \cup \{1\}$, for any graph G.

Conjecture (BNR): $A(F \cup G) = \max\{A(F), A(G)\}$.

Further questions (BNR):

- Is it possible that $A(G)$ is irrational for some G?
- Is the problem of deciding whether $A(G) > t$, for a given graph G and value t, decidable? If so, what is its complexity?

Questions of Alon and Lubetzky

Observation (Alon, Lubetzky): $A(G) \geq i^*_{max}(G)$, where

$$i_{max}(G) = \max_{U \text{ indep. in } G} \frac{|U|}{|U|+|N(U)|} \quad \text{and} \quad i^*_{max}(G) = \begin{cases} i_{max}(G), & \text{if } i_{max}(G) \leq \frac{1}{2} \\ 1, & \text{if } i_{max}(G) > \frac{1}{2} \end{cases}$$

$$(i(G) \leq i_{max}(G) \leq i^*_{max}(G) \leq A(G))$$

Question (Alon, Lubetzky - 2007.):

1. Does every graph G satisfy $A(G) = i^*_{max}(G)$?
2. Does the inequality $i(F \times G) \leq \max\{i^*_{max}(F), i^*_{max}(G)\}$ holds for every two graphs F and G?

Theorem (AL): If F is a complete graph or a cycle (and G is arbitrary), then the inequality of Question 2 holds.

Answer to the questions of Alon and Lubetzky

Theorem (Á. Tóth):

- $i(F \times G) \leq \max\{i_{max}(F), i_{max}(G)\}$.
- $i_{max}(F \times G) \leq \max\{i_{max}(F), i_{max}(G)\}$, provided that $i_{max}(F) \leq \frac{1}{2}$ or $i_{max}(G) \leq \frac{1}{2}$.

Corollary: $A(G) = i^*_{max}(G)$, for every graph G.

Thus the answer for both questions of Alon and Lubetzky is positive.

Further consequences:

- $A(F \cup G) = \max\{A(F), A(G)\}$.
- $A(G)$ cannot be irrational.

Complexity aspects

Theorem (BNR): If G is a bipartite graph, then $A(G)$ can be determined in polynomial time.

Theorem (AL): Given an input graph G, determining whether $A(G) = 1$ or $A(G) \leq \frac{1}{2}$ can be done in polynomial time.

Theorem (Á. Tóth):
Given an input graph G and value t, deciding whether $A(G) > t$ is an NP-complete problem.

The Hedetniemi conjecture

Hedetniemi's conjecture - 1966.:
For every two graphs F and G we have

$$\chi(F \times G) = \min\{\chi(F), \chi(G)\}.$$

The fractional version of Hedetniemi's conjecture:

$$\chi_f(F \times G) = \min\{\chi_f(F), \chi_f(G)\}.$$

(χ_f denotes the fractional chromatic number)

$\chi_f(F \times G) \leq \min\{\chi_f(F), \chi_f(G)\}$ is easy.
Tardif, 2005.: $\chi_f(F \times G) \geq \frac{1}{4} \min\{\chi_f(F), \chi_f(G)\}$.

In 2010 Zhu proved the fractional version of Hedetniemi's conjecture.

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Comparing the Shannon-capacity and the ultimate categorical independence ratio

Ultimate categorical independence ratio

- the asymptotic value of the independence ratio for the categorical graph product: $A(G) = \lim_{k \to \infty} i(G^{\times k})$
- it can be expressed by a simple formula for every graph
- $A(F \cup G) = \max\{A(F), A(G)\}$
- computable (NP-complete)

Shannon capacity

- the asymptotic value of the independence number for the normal graph product: $c(G) = \lim_{t \to \infty} \sqrt[t]{\alpha(G^t)}$
- it is not known even for small simple graphs
- $c(F \cup G) \neq c(F) + c(G)$
- computable?

Thank you for your attention!

The talk is based on the following paper:
Á. Tóth, *Answer to a question of Alon and Lubetzky about the ultimate categorical independence ratio*,
submitted to Journal of Combinatorial Theory, Series B.

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