On the ultimate categorical independence ratio

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Abstract

Brown, Nowakowski and Rall defined the ultimate categorical independence ratio of a graph G as $A(G) = \lim_{k\to\infty} i(G^{\times k})$, where $i(G) = \frac{\alpha(G)}{|V(G)|}$ denotes the independence ratio of a graph G, and $G^{\times k}$ is the kth categorical power of G. Let $a(G) = \max\{\frac{|U|}{|U|+|N_G(U)|} : U$ is an independent set of $G\}$, where $N_G(U)$ is the neighborhood of U in G. In this paper we answer a question of Alon and Lubetzky, namely we prove that $A(G) = a(G)$ if $a(G) \leq \frac{1}{2}$, and $A(G) = 1$ otherwise. We also discuss some other open problems related to $A(G)$ which are immediately settled by this result.

1 Introduction

The *independence ratio* of a graph G is defined as $i(G) = \frac{\alpha(G)}{|V(G)|}$, that is, as the ratio of the independence number and the number of vertices. For two graphs G and H, their *categorical product* (also called as direct or tensor product) $G\times H$ is defined on the vertex set $V(G\times H) = V(G)\times V(H)$ with edge set $E(G \times H) = \{\{(x_1, y_1), (x_2, y_2)\} : \{x_1, x_2\} \in E(G)$ and $\{y_1, y_2\} \in E(H)\}$. The kth categorical power $G^{\times k}$ is the k-fold categorical product of G. The *ultimate categorical independence ratio* of a graph G is defined as

$$A(G) = \lim_{k\to\infty} i(G^{\times k}).$$

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This parameter was introduced by Brown, Nowakowski and Rall in [2] where they proved that for any independent set U of G the inequality $A(G) \geq \frac{|U|}{|U|+|N_G(U)|}$ holds, where $N_G(U)$ denotes the neighborhood of U in G. Furthermore, they showed that $A(G) > \frac{1}{2}$ implies $A(G) = 1$.

Motivated by these results, Alon and Lubetzky [1] defined the parameters $a(G)$ and $a^*(G)$ as follows

$$a(G) = \max_{U \text{ is independent set of } G} \frac{|U|}{|U| + |N_G(U)|} \qquad \text{and} \qquad a^*(G) = \begin{cases} a(G) & \text{if } a(G) \leq \frac{1}{2} \\ 1 & \text{if } a(G) > \frac{1}{2} \end{cases},$$

and they proposed the following two questions.

Question 1 ([1]). Does every graph G satisfy $A(G) = a^*(G)$? Or, equivalently, does every graph G satisfy $a^*(G^{\times 2}) = a^*(G)$?

Question 2 ([1]). Does the inequality $i(G \times H) \leq \max\{a^*(G), a^*(H)\}$ hold for every two graphs G and H?

The above results from [2] give us the inequality $A(G) \geq a^*(G)$. One can easily see the equivalence between the two forms of Question 1; moreover, it is not hard to show that an affirmative answer to Question 1 would imply the same for Question 2 (see [1]).

Following [2] a graph G is called *self-universal* if $A(G) = i(G)$. As a consequence, the equality $A(G) = a^*(G)$ in Question 1 is also satisfied for these graphs according to the chain inequality $i(G) \leq a(G) \leq a^*(G) \leq A(G)$. Regular bipartite graphs, cliques and Cayley graphs of Abelian groups belong to this class (see [2]). In [4] the author proved that a complete multipartite graph G is self-universal, except for the case when $i(G) > \frac{1}{2}$. Therefore the equality $A(G) = a^*(G)$ is also verified for this class of graphs. (In the latter case $A(G) = a^*(G) = 1$.) In [1] it is shown that the graphs which are disjoint union of cycles and complete graphs satisfy the inequality in Question 2.

In this paper we answer Question 1 affirmatively, thereby also obtaining a positive answer for Question 2. Moreover it solves some other open problems related to $A(G)$. In the proofs we exploit an idea of Zhu [3] that he used on the way when proving the fractional version of Hedetniemi's conjecture. In Section 2 this tool is presented. Then, in Section 3, first we prove the inequality

$$i(G \times H) \leq \max\{a(G), a(H)\}, \quad \text{for every two graphs } G \text{ and } H,$$

and give a positive answer to Question 2 (using $a(G) \leq a^*(G)$). Afterwards we prove that

$$a(G \times H) \leq \max\{a(G), a(H)\}, \quad \text{provided that } a(G) \leq \frac{1}{2} \text{ or } a(H) \leq \frac{1}{2},$$

and from this result we conclude the affirmative answer to Question 1. (If $a(G) > \frac{1}{2}$ then $a^*(G^{\times 2}) = a^*(G) = 1$. Otherwise applying the above result for $G = H$ we get $a(G^{\times 2}) \leq a(G)$, while the reverse inequality clearly holds for every G. Thus we can conclude that $a^*(G^{\times 2}) = a^*(G)$ for every graph G.) Finally, in Section 4, we discuss further open problems which are solved by our result. For instance, we get a proof for the conjecture of Brown, Nowakowski and Rall, stating that $A(G + H) = \max\{A(G), A(H)\}$, where $G + H$ denotes the disjoint union of the graphs G and H.

2 Zhu's lemma

Recently Zhu [3] proved the fractional version of Hedetniemi's conjecture, that is, he showed that for every graph G and H we have $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$, where $\chi_f(G)$ denotes the fractional chromatic number of the graph G. During the proof he showed an interesting property of the independent sets of categorical product of graphs. We discuss this result in detail, because this will be the key idea also for our work.

In the sequel, we keep using the following notations for any $Z \subseteq V(G \times H)$.
For any $y \in V(H)$, let

$$Z(y) = \{x \in V(G) \,:\, (x, y) \in Z\},$$

and similarly, for any $x \in V(G)$, let

$$Z(x) = \{y \in V(H) \,:\, (x, y) \in Z\}.$$

In addition, let

$$N^G(Z) = \{(x, y) \in V(G \times H) \,:\, x \in N_G(Z(y))\}.$$

In words, $N^G(Z)$ means that we decompose Z into sections corresponding to the elements of $V(H)$, and in each section we pick those points which are neighbors of the elements of $Z(y)$ in the graph G. Similarly, let

$$N^H(Z) = \{(x, y) \in V(G \times H) \,:\, y \in N_H(Z(x))\}.$$

Keep in mind, that $Z(y) \subseteq V(G)$ and $Z(x) \subseteq V(H)$, while $N^G(Z), N^H(Z) \subseteq V(G \times H)$.

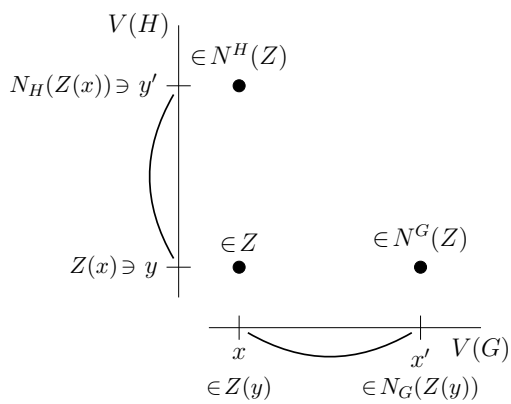


Figure 1: The elements of Z, $Z(x)$, $Z(y)$, $N^G(Z)$ and $N^H(Z)$.

Let U be an independent set of $G \times H$, and consider a partition of U into two sets, let

$$\begin{aligned} A &= \{(x, y) \in U \,:\, \nexists (x', y) \in U \text{ s.t. } \{x, x'\} \in E(G)\}, \\ B &= \{(x, y) \in U \,:\, \exists (x', y) \in U \text{ s.t. } \{x, x'\} \in E(G)\}. \end{aligned} \tag{1}$$

We have $U = A \uplus B$, where $A \uplus B$ denotes the disjoint union of the sets A and B.

Zhu [3] proved the following statement.

Lemma 1 ([3])**.** *Let G and H be graphs, U an independent set of $G \times H$. Then for the partition of U into $A \uplus B$ defined in (1), we have the following properties.*

(i) For every $y \in V(H)$, $A(y)$ is an independent set of G. For every $x \in V(G)$, $B(x)$ is an independent set of H.

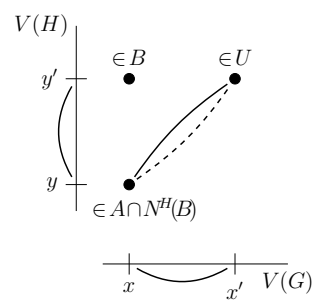
(ii) A, B, $N^G(A)$ and $N^H(B)$ are pairwise disjoint subsets of $V(G \times H)$.

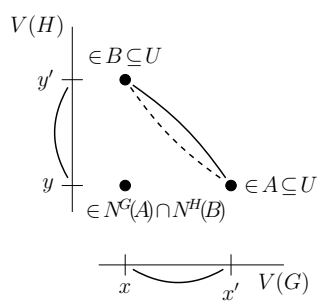
For the sake of completeness we prove this lemma.

Proof. We show the statements in (i). $A(y)$ is independent for every $y \in V(H)$ by definition. If for any $x \in V(G)$ the set $B(x)$ was not independent in H, that is $\exists y, y' \in B(x)$, $\{y, y'\} \in E(H)$, then from $(x, y') \in B$ we would get that $\exists (x', y') \in U$, $\{x, x'\} \in E(G)$. But this would be a contradiction, because $(x, y) \in B$ and $(x', y') \in U$ were two adjacent elements of the independent set U.

We turn to the proof of (ii). By definition, $A \cap B = \emptyset$. The first part of the lemma implies that the pair $(A, N^G(A))$ is also disjoint, as well as the pair $(B, N^H(B))$.

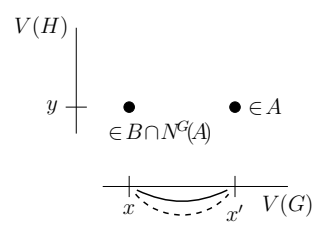
If $(x, y) \in A \cap N^H(B)$ then (by the definition of $N^H(B)$) $\exists (x, y') \in B$, $\{y, y'\} \in E(H)$, and so (by the definition of B) $\exists (x', y') \in U$, $\{x, x'\} \in E(G)$, which is a contradiction: $(x, y) \in A$ and $(x', y') \in U$ are adjacent vertices in the independent set U.





Similarly, if $(x, y) \in N^G(A) \cap N^H(B)$ then (by the definition of $N^G(A)$) $\exists (x', y) \in A \subseteq U$, $\{x, x'\} \in E(G)$ while (by the definition of $N^H(B)$) $\exists (x, y') \in B \subseteq U$, $\{y, y'\} \in E(H)$, which contradicts the independence of U.

Finally, $(x, y) \in B \cap N^G(A)$ implies that $\exists (x', y) \in A$, $\{x, x'\} \in E(G)$ (by the definition of $N^G(A)$), which is in contradiction with the definition of A: there should not be an $(x, y) \in B \subseteq U$ satisfying $\{x, x'\} \in E(G)$.



□

3 Proofs

In this section we prove the statements declared in the Introduction. In Subsection 3.1 we give an upper bound for $i(G \times H)$ in terms of $a(G)$ and $a(H)$. In Subsection 3.2 we prove that the same upper bound holds also for $a(G \times H)$ provided that $a(G) \leq \frac{1}{2}$ or $a(H) \leq \frac{1}{2}$. Thereby we will obtain our main result, which states that $A(G) = a^*(G)$ for every graph G.

3.1 Upper bound for $i(G \times H)$

As a simple consequence of Zhu's result the following inequality is obtained.

Theorem 2. *For every two graphs G and H we have*

$$i(G \times H) \leq \max\{a(G), a(H)\}.$$

Proof. Let U be a maximum-size independent set of $G \times H$, then we have

$$i(G \times H) = \frac{\alpha(G \times H)}{|V(G \times H)|} = \frac{|U|}{|V(G \times H)|}. \tag{2}$$

We partition U into $U = A \uplus B$ according to (1). We also use the notations $A(y)$ for every $y \in V(H)$, $B(x)$ for every $x \in V(G)$, and $N^G(A)$, $N^H(B)$ defined in the previous section.

It is clear that $|U| = |A| + |B|$. From the second part of Lemma 1 we have that $|A| + |B| + |N^G(A)| + |N^H(B)| \leq |V(G \times H)|$. Observe that $|N^G(A)| = \sum_{y \in V(H)} |N_G(A(y))|$ and $|N^H(B)| = \sum_{x \in V(G)} |N_H(B(x))|$. Hence we get

$$\frac{|U|}{|V(G \times H)|} \leq \frac{|A| + |B|}{|A| + |B| + |N^G(A)| + |N^H(B)|} =$$
$$= \frac{\sum_{y \in V(H)} |A(y)| + \sum_{x \in V(G)} |B(x)|}{\sum_{y \in V(H)} (|A(y)| + |N_G(A(y))|) + \sum_{x \in V(G)} (|B(x)| + |N_G(B(x))|)}. \tag{3}$$

From the first part of Lemma 1 and by the definition of $a(G)$ and $a(H)$ we have $\frac{|A(y)|}{|A(y)| + |N_G(A(y))|} \leq a(G)$ for every $y \in V(H)$, and $\frac{|B(x)|}{|B(x)| + |N_H(B(x))|} \leq a(H)$ for every $x \in V(G)$, respectively. Using the fact that if $\frac{t_1}{s_1} \leq r$ and $\frac{t_2}{s_2} \leq r$ then $\frac{t_1 + t_2}{s_1 + s_2} \leq r$, this yields

$$\frac{\sum_{y \in V(H)} |A(y)| + \sum_{x \in V(G)} |B(x)|}{\sum_{y \in V(H)} (|A(y)| + |N_G(A(y))|) + \sum_{x \in V(G)} (|B(x)| + |N_H(B(x))|)} \leq \max\{a(G), a(H)\}. \tag{4}$$

Equality (2) and inequalities (3), (4) together give us the stated inequality,

$$i(G \times H) \leq \max\{a(G), a(H)\}.$$

□

From Theorem 2 it follows that the answer to Question 2 is positive, as it is already stated.

3.2 Upper bound for $a(G \times H)$

In this subsection we answer Question 1 affirmatively. To show that $a^*(G^{\times 2}) = a^*(G)$ holds for every graph G it is enough to prove that $a(G^{\times 2}) \leq a(G)$ if $a(G) \leq \frac{1}{2}$. Because every G satisfies $a(G^{\times 2}) \geq a(G)$, and, in addition, if $a(G) > \frac{1}{2}$ then $a^*(G^{\times 2}) = a^*(G) = 1$. (The condition $a(G) \leq \frac{1}{2}$ is necessary, since otherwise $A(G) = 1$, and therefore $i(G^{\times k})$, and $a(G^{\times k})$ as well, can be arbitrarily close to 1 for sufficiently large k.) A bit more general, we prove the following theorem.

Theorem 3. *If $a(G) \leq \frac{1}{2}$ or $a(H) \leq \frac{1}{2}$ then*

$$a(G \times H) \leq \max\{a(G), a(H)\}.$$

Proof. Let G and H be two graphs satisfying $a(G) \leq \frac{1}{2}$ or $a(H) \leq \frac{1}{2}$. Without loss of generality, we may assume that $a(G) \geq a(H)$. Therefore $a(H) \leq \frac{1}{2}$.
We need to show that for every independent set U of $G \times H$ we have

$$\frac{|U|}{|U| + |N_{G\times H}(U)|} \leq a(G).$$

Observe that the above inequality can be rewritten as follows. Set $b(G) = \frac{1-a(G)}{a(G)} = \frac{1}{a(G)} - 1$. It is enough to prove that

$$|N_{G\times H}(U)| \geq b(G)|U|.$$

The definition of $a(G)$ means that $|N_G(P)| \geq b(G)|P|$ for any independent set P of G (and there is an independent set R of G such that $|N_G(R)| = b(G)|R|$). Similarly, using $b(H) = \frac{1-a(H)}{a(H)}$ we have $|N_H(Q)| \geq b(H)|Q|$ for any independent set Q of H.

First, we need some notations. Let $\hat{A}$, $\hat{B}$ and C be the following subsets of U.

$$\hat{A} = \{(x,y) \in U \,:\, \nexists (x',y) \in U \text{ s.t. } \{x,x'\} \in E(G), \text{ but } \exists (x,y') \in U \text{ s.t. } \{y,y'\} \in E(H)\},$$
$$\hat{B} = \{(x,y) \in U \,:\, \nexists (x,y') \in U \text{ s.t. } \{y,y'\} \in E(H), \text{ but } \exists (x',y) \in U \text{ s.t. } \{x,x'\} \in E(G)\},$$
$$C = \{(x,y) \in U \,:\, \nexists (x',y) \in U \text{ s.t. } \{x,x'\} \in E(G), \text{ and } \nexists (x,y') \in U \text{ s.t. } \{y,y'\} \in E(H)\}.$$

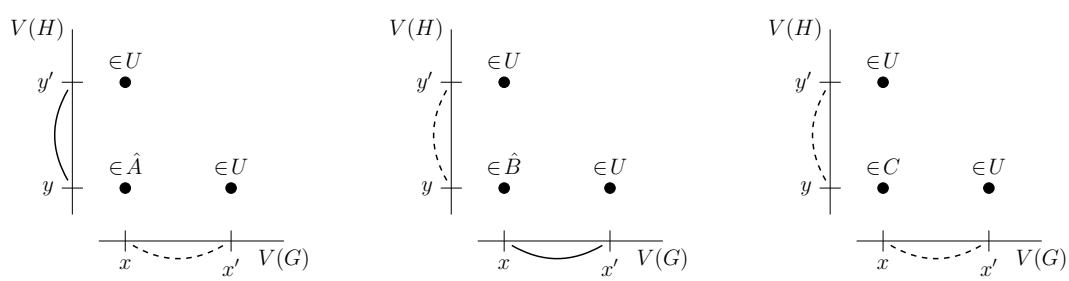


Figure 2: The elements of sets $\hat{A}$, $\hat{B}$ and C.

We will also use the notations $Z(x)$, $Z(y)$, $N^G(Z)$ and $N^H(Z)$ for any $Z \subseteq V(G \times H)$, $x \in V(G)$, $y \in V(H)$ defined in Section 2. We partition $N^G(\hat{A} \cup C)$ into two parts, let

$$N_1 = N^G(\hat{A} \cup C) \cap N_{G\times H}(U) \quad \text{and} \quad M = N^G(\hat{A} \cup C) \setminus N_{G\times H}(U).$$

Let

$$N_2 = N^H(\hat{B} \cup M).$$

The above subsets of $V(G \times H)$ will play an important role in the proof.

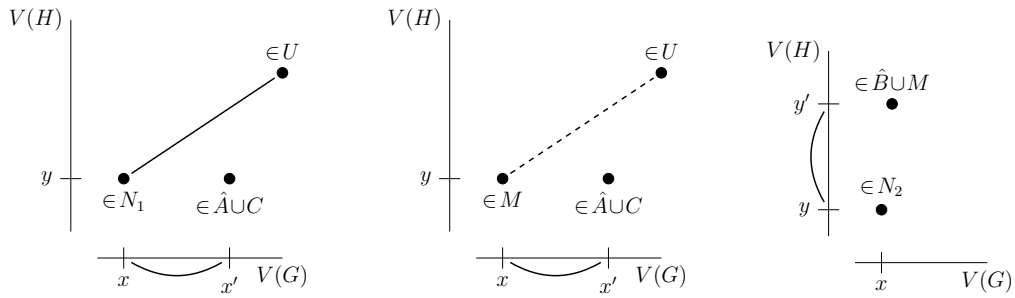


Figure 3: The elements of N_1, M and N_2.

We obtain the desired lower bound for $N_{G\times H}(U)$ in the following main steps.

(i) We show that U is partitioned into $U = \hat{A} \uplus \hat{B} \uplus C$.

(ii) We consider the elements of $\hat{A}$ and C for every $y \in V(H)$, and prove that

- **(a)** $(\hat{A} \cup C)(y)$ is independent in G,
- **(b)** $|N_1| \geq b(G)\big(|\hat{A}| + |C|\big) - |M|$.

(iii) We consider the elements of $\hat{B}$ and M for every $x \in V(G)$, and prove that

- **(a)** $\hat{B}(x) \cap M(x) = \emptyset$,
- **(b)** $(\hat{B} \cup M)(x)$ is independent in H,
- **(c)** $|N_2| \geq b(H)\big(|\hat{B}| + |M|\big)$.

(iv) For the sets N_1, N_2 we show that

- **(a)** $N_1, N_2 \subseteq N_{G\times H}(U)$,
- **(b)** $N_1 \cap N_2 = \emptyset$,
- **(c)** $|N_{G\times H}(U)| \geq |N_1| + |N_2|$.

(v) Finally, we prove that

$$|N_{G\times H}(U)| \geq b(G)|U|.$$

Now we prove the statements above.

(i) It is clear that $\hat{A}$, $\hat{B}$ and C are pairwise disjoint. In addition, there is no $(x,y) \in U$ for which $\exists (x',y),(x,y') \in U$ such that $\{x,x'\} \in E(G)$ and $\{y,y'\} \in E(H)$, because this would imply that $\{(x',y),(x,y')\} \in E(G \times H)$, but U is an independent set. Hence U is partitioned into $U = \hat{A} \uplus \hat{B} \uplus C$. (Clearly, the connection with the partition of Zhu described in (1) is $A = \hat{A} \uplus C$ and $B = \hat{B}$.)

(ii) We consider the elements of $\hat{A}$ and C for every $y \in V(H)$.

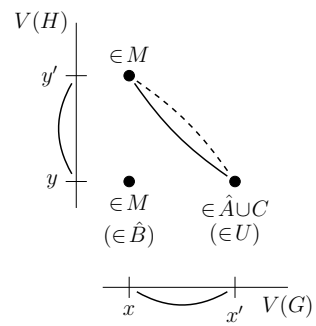
(ii/a) By definition $(\hat{A} \cup C)(y)$ is independent in G.

(ii/b) From (ii/a) and by the definition of $b(G)$ it follows that $|N_G((\hat{A} \cup C)(y))| \geq b(G)|(\hat{A} \cup C)(y)|$. Considering the sum for all $y \in V(H)$ we have $|N^G(\hat{A} \cup C)| \geq b(G)\big(|\hat{A}| + |C|\big)$. By the definition of N_1 and M this yields $|N_1| \geq b(G)\big(|\hat{A}| + |C|\big) - |M|$.

(iii) We consider the elements of $\hat{B}$ and M for every $x \in V(G)$.

(iii/a) By the definition of $\hat{A}$ and C, the sets $\hat{B}(x)$ and $M(x)$ are disjoint. Indeed, if $(x,y) \in M \subseteq N^G(\hat{A} \cup C)$ then $\exists (x',y) \in \hat{A} \cup C, \{x,x'\} \in E(G)$ and so (x,y) cannot be in $\hat{B} \subseteq U$.

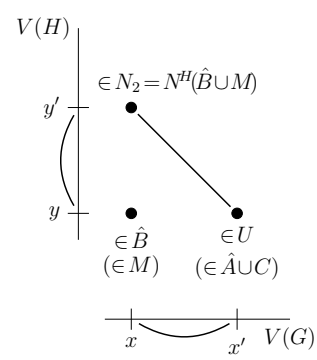
(iii/b) We claim that $(\hat{B} \cup M)(x)$ is independent in H for every $x \in V(G)$. Clearly, $\hat{B}(x)$ is independent by definition. Furthermore, if $y, y' \in M(x), \{y,y'\} \in E(H)$ then from $(x,y) \in M$ we get that $\exists (x',y) \in \hat{A} \cup C, \{x,x'\} \in E(G)$, hence $(x,y') \in M$ is a neighbor of $(x',y) \in U$ which contradicts to $M \cap N_{G \times H}(U) = \emptyset$. Similarly if $y \in \hat{B}(x), y' \in M(x), \{y,y'\} \in E(H)$ then from $(x,y) \in \hat{B}$ it follows that $\exists (x',y) \in U, \{x,x'\} \in E(G)$, but again, as $(x,y') \in M$ is a neighbor of $(x',y) \in U$ it is in contradiction with the definition of M.



(iii/c) From (iii/b) it follows that $|N_H((\hat{B} \cup M)(x))| \geq b(G)|(\hat{B} \cup M)(x)|$. Considering the sum for all $x \in V(G)$ we get that $|N^H(\hat{B} \cup M)| \geq b(G)|\hat{B} \cup M|$. By the definition of N_2 and the statement (iii/a) we obtain $|N_2| \geq b(H)\big(|\hat{B}| + |M|\big)$.

(iv) Next, we investigate the sets N_1, N_2.

(iv/a) We have $N_1 \subseteq N_{G \times H}(U)$, by definition. We claim that $N_2 \subseteq N_{G \times H}(U)$. On the one hand, $N^H(\hat{B}) \subseteq N_{G \times H}(U)$. Indeed, if $(x,y') \in N^H(\hat{B})$, that is, for a neighbor y of y' in H we have $y \in \hat{B}(x)$, then by the definition of $\hat{B}$, $\exists (x',y) \in U, \{x,x'\} \in E(G)$. Hence (x,y') is a neighbor of $(x',y) \in U$, and so $(x,y') \in N_{G \times H}(U)$. On the other hand, if $(x,y') \in N^H(M)$, that is, for a neighbor y of y' in H we have $y \in M(x)$, then by the definition of M, $\exists (x',y) \in \hat{A} \cup C, \{x,x'\} \in E(G)$, therefore $\{(x',y),(x,y')\} \in E(G \times H)$, thus $(x,y') \in N_{G \times H}(U)$. This yields $N^H(M) \subseteq N_{G \times H}(U)$.



(iv/b) We claim that $N_1 \cap N_2 = \emptyset$. Suppose indirectly, that $\exists (x,y) \in N_1 \cap N_2$. Then $(x,y) \in N_1$ implies that $\exists (x',y) \in \hat{A} \cup C, \{x,x'\} \in E(G)$. While from $(x,y) \in N_2$ we get that $\exists (x,y') \in \hat{B}$ or

$\exists (x, y') \in M$, $\{y, y'\} \in E(H)$. It is a contradiction since (x', y) and (x, y') are adjacent in $G \times H$, but no edge can go between $\hat{A} \cup C$ and $\hat{B} \cup M$ by the independence of U and the definition of M.
(iv/c) The statements (iv/a), (iv/b) give $|N_{G\times H}(U)| \geq |N_1| + |N_2|$.

(v) From (ii/b), (iii/c) and (iv/c) we get that

$$|N_{G\times H}(U)| \geq |N_1| + |N_2| \geq \Big(b(G)\big(|\hat{A}| + |C|\big) - |M|\Big) + \Big(b(H)\big(|\hat{B}| + |M|\big)\Big). \tag{5}$$

From the assumption $a(G) \geq a(H)$ it follows $b(G) = \frac{1}{a(G)} - 1 \leq \frac{1}{a(H)} - 1 = b(H)$. We also have $a(H) \leq \frac{1}{2}$, that is $b(H) \geq 1$. Thus we obtain

$$\Big(b(G)\big(|\hat{A}| + |C|\big) - |M|\Big) + \Big(b(H)\big(|\hat{B}| + |M|\big)\Big) \geq$$
$$\geq b(G)\Big(|\hat{A}| + |\hat{B}| + |C|\Big) + \big(b(H) - 1\big)|M| \geq b(G)|U|. \tag{6}$$

Combining the inequalities (5) and (6) we conclude $|N_{G\times H}(U)| \geq b(G)|U|$.

Consequently, for every independent set U of $G \times H$ we showed that

$$\frac{|U|}{|U| + |N_{G\times H}(U)|} \leq a(G),$$

and the proof is complete. □

We mentioned in the Introduction that the two forms of Question 1 are equivalent. Hence from the equality $a^*(G^{\times 2}) = a^*(G)$ for every graph G we obtain that $A(G) = a^*(G)$ is also holds for every graph G. (Indeed, suppose on the contrary that G is a graph with $a^*(G) < A(G)$ then $\exists k$ such that $a^*(G) < i(G^{\times k}) \leq a^*(G^{\times k})$, and as the sequence $\{a^*(G^{\times \ell})\}_{\ell=1}^{\infty}$ is monotone increasing, it follows that $\exists m$ for which $a^*(G^{\times m}) < a^*((G^{\times m})^{\times 2})$, giving a contradiction.) Thus we have the following corollary.

Corollary 4. *For every graph G we have $A(G) = a^*(G)$, that is*

$$A(G)(= \lim_{k\to\infty} i(G^{\times k})) = \begin{cases} a(G)\big(= \max\limits_{U \text{ is independent in } G} \frac{|U|}{|U|+|N_G(U)|}\big), & \text{if } a(G) \leq \frac{1}{2}, \\ 1, & \text{otherwise.} \end{cases}$$

4 Further consequences

Brown, Nowakowski and Rall in [2] asked whether $A(G+H) = \max\{A(G), A(H)\}$, where $G+H$ is the disjoint union of G and H. From Corollary 4 we immediately receive this equality since the analogous statement, $a^*(G+H) = \max\{a^*(G), a^*(H)\}$ is straightforward. In [1] it is shown that $A(G+H) = A(G \times H)$, therefore we have

$$A(G+H) = A(G \times H) = \max\{A(G), A(H)\}, \text{ for every graph } G \text{ and } H.$$

The authors of [2] also addressed the question whether $A(G)$ is computable, and if so what is its complexity. They showed that if G is bipartite then $A(G) = \frac{1}{2}$ if G has a perfect matching, and $A(G) = 1$ otherwise. Hence for bipartite graphs $A(G)$ can be determined in polynomial time. Moreover, it is proven in [1] that $a(G) \leq \frac{1}{2}$ if and only if G contains a fractional perfect matching. Therefore given an input graph G, determining whether $A(G) = 1$ or $A(G) \leq \frac{1}{2}$ can be done in polynomial time. From Corollary 4 we can conclude that the problem of deciding whether $A(G) > t$ for a given graph G and a given value t, is in NP. Moreover, it is not hard to prove that it is in fact NP-complete. (The maximum independent set problem has a Karp-reduction to this problem, by adding sufficiently many vertices to the graph which are connected to each other and every other vertex of the graph, and choosing t appropriately.)

Any rational number in $(0, \frac{1}{2}] \cup \{1\}$ is the ultimate categorical independence ratio for some graph G, as it is showed [2]. Here we remark that by Corollary 4, $A(G)$ cannot be irrational, solving another problem mentioned in [2].

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