MŰEGYETEM 1782

Ph.D. Thesis Booklet

Colouring problems related to graph products and coverings

Ágnes Tóth

Supervisor: Prof. Gábor Simonyi

Department of Computer Science and Information Theory
Budapest University of Technology and Economics

June 2012

In the thesis we concentrate on two topics of graph colouring problems. We investigate the asymptotic behaviour of colouring-related graph parameters for different graph powers. In addition, we discuss problems on coverings with monochromatic components in edge-coloured graphs. In this booklet we give a short introduction to the two topics and state our results.

Asymptotic values of graph parameters

Several graph parameters show an interesting behaviour when they are investigated for different powers of graphs. One of the most famous examples of such behaviour is that of the Shannon capacity of graphs which is the theoretical upper limit of channel capacity for error-free coding in information theory. It was introduced by Shannon in [38] (see Körner and Orlitsky [33] for a survey of related topics). This graph parameter is defined as the normalized limit of the independence number under the so-called normal power and its exact value is not known even for small, simple graphs (for example odd cycles with length more than five).

The normalized asymptotic value of the chromatic number with respect to the normal power is the Witsenhausen rate. It is introduced by Witsenhausen in [42], where its information theoretic relevance is also explained. If we investigate the chromatic number for the co-normal power we get the fractional chromatic number as the corresponding limit by a famous theorem of McEliece and Posner [37] (cf. also Berge and Simonovits [14]).

Similar questions arise when investigating the independence ratio and the Hall-ratio of a graph.

1 The asymptotic value of the independence ratio for categorical graph power

The results of this section are based on [1] and [2].

The *independence ratio* of a graph G is defined as $i(G) = \frac{\alpha(G)}{|V(G)|}$, that is, as the ratio of the independence number and the number of vertices.
For two graphs F and G, their *categorical product* (also called as direct or tensor product) $F \times G$ is defined on the vertex set $V(F \times G) = V(F) \times V(G)$ with edge set $E(F \times G) = \{\{(u_1, v_1), (u_2, v_2)\} : \{u_1, u_2\} \in E(F) \text{ and } \{v_1, v_2\} \in E(G)\}$. The kth categorical power $G^{\times k}$ is the k-fold categorical product of G with itself, i.e, it can be defined on the k-length

sequences over the vertex set of G, and two such sequences are connected iff their elements form an edge in G at every coordinate.

Brown, Nowakowski and Rall in [15] considered the asymptotic value of the independence ratio for the categorical graph power.

Definition ([15])**.** The *ultimate categorical independence ratio* of a graph G is defined as

$$A(G) = \lim_{k\to\infty} i(G^{\times k}).$$

The authors of [15] proved that for any independent set U of G the inequality $A(G) \geq \frac{|U|}{|U|+|N_G(U)|}$ holds, where $N_G(U)$ denotes the neighborhood of U in G. Furthermore, they showed that $A(G) > \frac{1}{2}$ implies $A(G) = 1$.

The ultimate categorical independence ratio was also investigated by Alon and Lubetzky in [11], where they defined the parameters $i_{\max}(G)$ and $i^*_{\max}(G)$ as follows

$$i_{\max}(G) = \max_{U \text{ is indep. set of } G} \frac{|U|}{|U|+|N_G(U)|} \qquad \text{and} \qquad i^*_{\max}(G) = \begin{cases} i_{\max}(G) & \text{if } i_{\max}(G) \leq \frac{1}{2} \\ 1 & \text{if } i_{\max}(G) > \frac{1}{2} \end{cases},$$

and they proposed the following two questions.

Question 1 (Alon, Lubetzky [11])**.** *Does every graph G satisfy $A(G) = i^*_{\max}(G)$? Or, equivalently, does every graph G satisfy $i^*_{\max}(G^{\times 2}) = i^*_{\max}(G)$?*

Question 2 (Alon, Lubetzky [11])**.** *Does the inequality $i(F\times G) \leq \max\{i^*_{\max}(F), i^*_{\max}(G)\}$ hold for every two graphs F and G?*

The above results from [15] give us the inequality $A(G) \geq i^*_{\max}(G)$. One can easily see the equivalence between the two forms of Question 1, moreover it is not hard to show that an affirmative answer to Question 1 would imply the same for Question 2 (see [11]).

Following [15] a graph G is called self-universal if $A(G) = i(G)$. As a consequence, the equality $A(G) = i^*_{\max}(G)$ in Question 1 is also satisfied for these graphs according to the chain of inequalities $i(G) \leq i_{\max}(G) \leq i^*_{\max}(G) \leq A(G)$. Cliques, regular bipartite graphs, and Cayley graphs of Abelian groups belong to this class (see [15]). In [1] the author proved that a complete multipartite graph G is self-universal, except for the case when $i(G) > \frac{1}{2}$. Therefore the equality $A(G) = i^*_{\max}(G)$ is also verified for this class of graphs. (In the latter case $A(G) = i^*_{\max}(G) = 1$.) In [11] it is shown that the graphs which are disjoint unions of cycles and complete graphs satisfy the inequality in Question 2.

1.1 Answer to the questions of Alon and Lubetzky

In the thesis we answer Question 1 affirmatively. Thereby a positive answer also for Question 2 is obtained. Moreover it solves some other open problems related to $A(G)$. In the proofs we exploit an idea of Zhu [43] that he used on the way when proving the fractional version of Hedetniemi's conjecture, i.e., that $\chi_f(F \times G) = \min\{\chi_f(F), \chi_f(G)\}$.

First we give an upper bound for $i(F \times G)$ in terms of $i_{\max}(F)$ and $i_{\max}(G)$. From the inequality the positive answer to Question 2 follows (using $i_{\max}(G) \leq i^*_{\max}(G)$).

Theorem 1. *For every two graphs F and G we have*

$$i(F \times G) \leq \max\{i_{\max}(F), i_{\max}(G)\}.$$

Next, we prove that the same upper bound holds also for $i_{\max}(F \times G)$ provided that $i_{\max}(F) \leq \frac{1}{2}$ or $i_{\max}(G) \leq \frac{1}{2}$.

Theorem 2. *If $i_{\max}(F) \leq \frac{1}{2}$ or $i_{\max}(G) \leq \frac{1}{2}$ then*

$$i_{\max}(F \times G) \leq \max\{i_{\max}(F), i_{\max}(G)\}.$$

From this result we conclude the affirmative answer to Question 1. (If $i_{\max}(G) > \frac{1}{2}$ then $i^*_{\max}(G^{\times 2}) = i^*_{\max}(G) = 1$. Otherwise applying the above result for $F = G$ we get $i_{\max}(G^{\times 2}) \leq i_{\max}(G)$, while the reverse inequality clearly holds for every G. Thus we can conclude that $i^*_{\max}(G^{\times 2}) = i^*_{\max}(G)$ for every graph G.)

As we mentioned, the two forms of Question 1 are equivalent. Hence from the equality $i^*_{\max}(G^{\times 2}) = i^*_{\max}(G)$ for every graph G we obtain the following main result. (Indeed, suppose on the contrary that G is a graph with $i^*_{\max}(G) < A(G)$ then $\exists k$ such that $i^*_{\max}(G) < i(G^{\times k}) \leq i^*_{\max}(G^{\times k})$, and as the sequence $\{i^*_{\max}(G^{\times \ell})\}_{\ell=1}^{\infty}$ is monotone increasing, it follows that $\exists m$ for which $i^*_{\max}(G^{\times m}) < i^*_{\max}((G^{\times m})^{\times 2})$, giving a contradiction.)

Theorem 3. *For every graph G we have $A(G) = i^*_{\max}(G)$.*

1.2 Further consequences

We also discuss some other open problems related to the ultimate categorical independence ratio which are settled by our main result.

Brown, Nowakowski and Rall in [15] asked whether $A(F \uplus G) = \max\{A(F), A(G)\}$, where $F \uplus G$ is the disjoint union of F and G. From Theorem 3 we immediately receive this equality since the analogous statement, $i^*_{\max}(F \uplus G) = \max\{i^*_{\max}(F), i^*_{\max}(G)\}$ is straightforward. In [11] it is shown that $A(F \uplus G) = A(F \times G)$, therefore we get the following result.

Corollary 4. *For every two graphs F and G we have*

$$A(F \uplus G) = A(F \times G) = \max\{A(F), A(G)\}.$$

The authors of [15] also addressed the question whether $A(G)$ is computable, and if so what is its complexity. They showed that for bipartite graphs $A(G)$ can be determined in polynomial time. (This is because in the case when G is bipartite, we have $A(G) = \frac{1}{2}$ if G has a perfect matching, and $A(G) = 1$ otherwise.) Furthermore, it is proven in [11] that given an input graph G, determining whether $A(G) = 1$ or $A(G) \leq \frac{1}{2}$ can also be done in polynomial time. (They showed that $i_{\max}(G) \leq \frac{1}{2}$ if and only if G contains a fractional perfect matching.) From Theorem 3 we can conclude that the problem of deciding whether $A(G) > t$ for a given graph G and a given value t, is in NP. Moreover it is not hard to prove that it is in fact NP-complete.

Any rational number in $(0, \frac{1}{2}] \cup \{1\}$ is the ultimate categorical independence ratio for some graph G, as it is shown in [15]. Here we remark that we obtained that $A(G)$ cannot be irrational, solving another problem mentioned in [15].

As a consequence of Theorem 3 we also have the following characterization of self-universal graphs. We call a graph empty if it has no edge. For every other graph G it holds that $i(G) < 1$.

Corollary 5. *A non-empty graph G is self-universal if and only if $i_{\max}(G) = i(G)$ and $i(G) \leq \frac{1}{2}$.*

In other words, a nonempty graph G is self-universal iff the expression $\frac{|U|}{|U|+|N_G(U)|}$ reach its maximum (also) for maximum-sized independent sets among all independent sets of G and this maximum is at most $\frac{1}{2}$.

2 The asymptotic value of the Hall-ratio for lexicographic and categorical powers

The results of this section are based on [3] and [4].

The Hall-ratio is closely related to the independence ratio. It was introduced in [18, 19] motivated by problems of list colouring. The *Hall-ratio* of a graph G is defined as

$$\rho(G) = \max\left\{\frac{|V(H)|}{\alpha(H)} \,:\, H \subseteq G\right\},$$

that is, as the ratio of the number of vertices and the independence number maximized over all subgraphs of G. The asymptotic values of the Hall-ratio for different graph powers were investigated by Simonyi [39]. He considered the (appropriately normalized) asymptotic values of the Hall-ratio for the exponentiations called normal, co-normal, lexicographic and categorical, respectively.

All the above four graph powers of the graph G are defined on the k-length sequences over $V(G)$. In the normal power $G^{\odot k}$ two sequences are adjacent iff their elements at every coordinate are either equal or form an edge in G. In the co-normal power G^k two such sequences are connected iff there is some coordinate where the corresponding elements of the two sequences form an edge of G. The asymptotic value of the Hall-ratio with respect to the co-normal power is defined as $h(G) = \lim_{k\to\infty} \sqrt[k]{\rho(G^k)}$, the analogous asymptotic value for the normal power is denoted by $h_{\odot}(G)$. Simonyi [39] proved that $h(G) = \chi_f(G)$, where $\chi_f(G)$ is the fractional chromatic number of graph G, while $h_{\odot}(G) = R(G)$, where $R(G)$ denotes the Witsenhausen rate that was already mentioned in the introduction. Recall that the latter is the normalized asymptotic value of the chromatic number with respect to the normal power and is introduced by Witsenhausen in [42] where its information theoretic relevance is also explained. The *fractional chromatic number* is the well-known graph invariant one obtains from the fractional relaxation of the integer program defining the chromatic number. That is,

$$\chi_f(G) = \inf\left\{ \sum_{U\in S(G)} f(U) \ : \ f \text{ is a fractional colouring of } G \right\}, \text{ where}$$

$$f \text{ is a fractional colouring of } G \text{ if } f : S(G) \to [0,1] \text{ and } \forall v \in V(G) : \sum_{v\in U\in S(G)} f(U) \geq 1,$$

and $S(G)$ denotes the set of the independent sets of G.

Simonyi [39] conjectured that also for the lexicographic and for the categorical powers we get the fractional chromatic number as the (appropriately normalized) asymptotic value. In the thesis we prove both of his conjectures.

2.1 The ultimate lexicographic Hall-ratio

For two graphs F and G, their *lexicographic product* $F \circ G$ is defined on the vertex set $V(F \circ G) = V(F) \times V(G)$ with edge set $E(F \circ G) = \{\{(u_1, v_1), (u_2, v_2)\} \ : \ \{u_1, u_2\} \in E(F), \text{ or } u_1 = u_2 \text{ and } \{v_1, v_2\} \in E(G)\}$. (The lexicographic product $F \circ G$ is also known as the substitution of G into all vertices of F. The name we use follows the book [31].) The kth lexicographic power $G^{\circ k}$ is the k-fold lexicographic product of G. That is, the lexicographic power is defined on the vertex sequences of the original graph and we connect two such sequences iff they are adjacent in the first coordinate where they differ.

Definition ([39])**.** The *ultimate lexicographic Hall-ratio* of graph G is

$$h_\circ(G) = \lim_{k\to\infty} \sqrt[k]{\rho(G^{\circ k})}.$$

As we can see from their definitions, the normal and co-normal power of a graph G satisfy that $E(G^{\odot k}) \subseteq E(G^{\circ k}) \subseteq E(G^k)$, therefore the value of $h_\circ(G)$ falls into the interval $[R(G), \chi_f(G)]$, using the mentioned results from [39]. We remark that the lower bound $R(G)$ is sometimes better but sometimes worse than the easy lower bound $\rho(G)$, cf. [39]. Thus we know that

$$\max\{\rho(G), R(G)\} \le h_\circ(G) \le \chi_f(G).$$

For some types of graphs the upper and lower bounds are equal, so this formula gives the exact value of the ultimate lexicographic Hall-ratio. (For instance, if G is a perfect graph, then $\chi_f(G) = \chi(G) = \omega(G) \le \rho(G)$. If G is a vertex-transitive graph, then $\chi_f(G) = \frac{|V(G)|}{\alpha(G)} \le \rho(G)$.) The length of the interval $[\max\{\rho(G), R(G)\}, \chi_f(G)]$ is positive in general. (An example is the 5-wheel, W_5 consisting of a 5-length cycle and an additional point joint to every vertex of the cycle. One can show that $\rho(W_5) = 3$, $\chi_f(W_5) = \frac{7}{2}$ and $R(W_5) \le \sqrt{12}$, cf. [39].)

It was conjectured in [39], that in fact, $h_\circ(G)$ always coincides with the larger end of the above interval. In the thesis we prove this conjecture.

Theorem 6. *The ultimate lexicographic Hall-ratio equals to the fractional chromatic number for every graph G, that is*

$$h_\circ(G) = \chi_f(G).$$

2.2 The ultimate categorical Hall-ratio

Recall that, the categorical power of a graph is defined on the vertex sequences of the original graph and we connect two such sequences iff they are adjacent in every coordinate.

Definition. ([39]) The *ultimate categorical Hall-ratio* of graph G is

$$h_\times(G) = \lim_{k\to\infty} \rho(G^{\times k}).$$

Note that in this case we do not need any normalization on the sequence.

It is shown in [39] that this graph parameter is bounded from above by the fractional chromatic number and conjectured that equality holds for all graphs.

The conjecture can be shown easily for perfect and for vertex-transitive graphs. It is proven in [39] that it is also true for wheel graphs constructed from a cycle and an additional point joint to every vertex of the cycle. By using a similar argument which was used in the proof of that result the following generalization was also proven by the author in [3]. Let G be a graph for which $h_\times(G) = \chi_f(G)$ holds and let $\hat{G}$ be the graph we obtain from G by connecting each of its vertices to an additional vertex. Then $h_\times(\hat{G}) = \chi_f(\hat{G})$ also holds.

In the thesis we prove the above conjecture in general.

Theorem 7. *The ultimate categorical Hall-ratio equals to the fractional chromatic number for every graph G, that is*

$$h_\times(G) = \chi_f(G).$$

The proof uses a recent result of Zhu [43] that he proved on the way when proving the fractional version of Hedetniemi's conjecture.

Monochromatic coverings in edge-coloured graphs

An equivalent form of Ryser's conjecture [30] due to Gyárfás [25], states that if the edges of a graph G are coloured with k colours then the vertex set can be covered by the vertices of at most $\alpha(G)(k-1)$ monochromatic components, where $\alpha(G)$ denotes the independence number. It is known to be true for $k=2$ (when it is equivalent to König's theorem). After partial results [29, 40], the case $k=3$ was solved by Aharoni [8], relying on an interesting topological method established in [9]. The important special case of Ryser's conjecture when the graph is complete is open for $k \geq 6$.

Recently Király [32] showed, somewhat surprisingly, that an analogue of Ryser's conjecture holds for hypergraphs. For $r \geq 3$, in every k-colouring of the edges of a complete r-uniform hypergraph, the vertex set can be covered by at most $\lfloor \frac{k}{r} \rfloor$ monochromatic components, and this bound is sharp.

In the thesis we investigate similar covering problems of edge-coloured graphs.

3 Gallai colourings and domination in multipartite digraphs

The results of this section are based on [5] which is joint work with A. Gyárfás and G. Simonyi and on [6] which is joint work with S. Fujita, M. Furuya and A. Gyárfás.

Investigating comparability graphs Gallai [23] proved an interesting theorem about edge-colourings of complete graphs that contain no triangle for which all three of its edges receive distinct colours. (Note that here and in the sequel edge-colouring just means a partition of the edge set rather than a proper colouring of it.) Such colourings turned out to be relevant and Gallai's theorem proved to be useful also in other contexts, see e.g., [13, 16, 17, 22, 24, 27, 28, 34, 35]. Honoring the above mentioned work of Gallai, an edge-colouring of the complete graph is called a Gallai colouring if there is no completely multicoloured triangle. Recently this notion was extended to other (not necessarily complete) graphs in [26].

A basic property of Gallai-coloured complete graphs is that at least one of the colour classes spans a connected subgraph on the entire vertex set. In [26] it was proved that if we colour the edges of a not necessarily complete graph G so that no completely multicoloured triangles appear then there is still a large monochromatic component whose size is proportional to the number of vertices of G where the proportion depends on the independence number, $\alpha(G)$. Another, in a sense stronger possible generalization of the above basic property of Gallai colourings is also suggested by this result. Gyárfás proposed the following problem at a workshop at Fredericia, Denmark in November, 2009.

Problem 3 (Gyárfás)**.** *Suppose that the edges of a graph G are coloured so that no triangle is coloured with three distinct colours. Is it true that the vertices of G can be covered by the vertices of at most k monochromatic components where k depends only on $\alpha(G)$?*

This question led to a problem about the existence of dominating sets in directed graphs that we believe to be interesting in itself. In the thesis we solve this latter problem thereby giving an affirmative answer to the previous question.

3.1 Dominating multipartite digraphs

We consider multipartite digraphs, i.e., digraphs D whose vertices are partitioned into classes $A_1, \dots, A_t$ of independent vertices. (Note that here we consider directed graphs without pairs of edges connecting the same two vertices in opposite direction.) Suppose that $S \subseteq [t]$. A set $U = \cup_{i \in S} A_i$ is called a *dominating set* of size $|S|$ if for any vertex $v \in \cup_{i \notin S} A_i$ there is a $w \in U$ such that $(w, v) \in E(D)$. The smallest $|S|$ for which a multipartite digraph D has a dominating set $U = \cup_{i \in S} A_i$ is denoted by $k(D)$. Let $\beta(D)$ be the cardinality of the largest independent set of D whose vertices are from different partite classes of D. An important special case is when $|A_i| = 1$ for each $i \in [t]$. In this case $\beta(D) = \alpha(D)$ and $k(D) = \gamma(D)$, the usual domination number of D, the smallest number of vertices in D whose closed outneighborhoods cover $V(D)$.

Our main result is the following theorem.

Theorem 8. *For every integer β there exists an integer $h = h(\beta)$ such that the following holds. If D is a multipartite digraph without cyclic triangles and $\beta(D) = \beta$, then $k(D) \leq h$. If $\beta(D) = 1$ then $k(D) = 1$ and if $\beta(D) = 2$ then $k(D) \leq 4$.*

Notice that the condition forbidding cyclic triangles in D is important even when $|A_i| = 1$ for all i and $\beta(D) = 1$, i.e. for tournaments. It is well known that $\gamma(D)$ can be arbitrarily large for tournaments (see, e.g., in [12]), so $h(1)$ would not exist without excluding cyclic triangles.

From the proof of Theorem 8 we get a factorial upper bound for $k(D)$ from the recurrence formula $h(\beta) = 3\beta + (2\beta + 1)h(\beta - 1)$. Though this upper bound on $h(\beta)$ is much weaker we could not even rule out the existence of a bound that is linear in β. We cannot prove a linear upper bound even in the special case when every partite class consists of only one vertex. Nevertheless, we treat this case also separately and provide a slightly better bound than the one following from Theorem 8. The class of digraphs we have here, i.e., those with no directed triangles, is called the class of *clique-acyclic digraphs*, see [10]. The union of t vertex disjoint cyclic pentagons shows that we can have $\alpha(D) = 2t$ and $\gamma(D) = 3t$. Thus in case a linear upper bound would be valid at least in the special case of clique-acyclic digraphs, it could not be smaller than $\frac{3}{2}\alpha(D)$.

One can see from the proofs that the dominating sets we find there contain two kinds of partite classes. The first kind could be substituted by just one vertex in it, while the second kind is chosen not so much to dominate others but because it is itself not dominated by others. That is, apart from a bounded number of exceptional partite classes we dominate the rest of our digraph with a bounded number of vertices. We also prove another theorem showing that the exceptional classes are indeed needed.

3.2 Monochromatic coverings and partitions of Gallai-coloured graphs

Theorem 8 implies an affirmative answer to Problem 3. Let $g(1) = 1$ and for $\alpha \geq 2$, let $g(\alpha) = g(\alpha - 1) + h(\alpha)$ where h is the function given by Theorem 8.

Theorem 9. *Suppose that the edges of a graph G are coloured so that no triangle is coloured with three distinct colours. Then the vertex set of G can be covered by the vertices of at most $g(\alpha(G))$ monochromatic components. In case $\alpha(G) = 2$ at most five components are enough.*

We also extend the statement of Theorem 9 from covering to partitioning. We say that the vertex set of an edge-coloured graph G can be *partitioned* into ℓ monochromatic connected parts, if there is a partition $\{V_1, \ldots, V_\ell\}$ of $V(G)$ such that every $G[V_i]$ ($1 \leq i \leq \ell$) is connected in some colour, where $G[S]$ denotes the induced subgraph by the subset S of the vertex set in G. (Note that, arbitrary subsets of the monochromatic connected components may not be used as parts of our partition because they can be disconnected in the corresponding colour.)

Let $\hat{g}(1) = 1$ and for $\alpha \geq 2$, let $\hat{g}(\alpha) = \max\{h(\alpha)(\alpha^2+\alpha-1), 2h(\alpha)\hat{g}(\alpha-1)+h(\alpha)+1\}$ where h is the function given by Theorem 8.

Theorem 10. *Suppose that the edges of a graph G are coloured so that no triangle is coloured with three distinct colours. Then, the vertex set of G can be partitioned into at most $\hat{g}(\alpha(G))$ monochromatic connected parts.*

4 Monochromatic covering of complete bipartite graphs

The results of this section are based on [7] which is joint work with G. Chen, S. Fujita, A. Gyárfás and J. Lehel.

A special case of Ryser's conjecture states that intersecting r-partite hypergraphs have a transversal of at most $r-1$ vertices. This conjecture is open for $r \geq 6$. It is trivially true for $r=2$, the cases $r=3,4$ are solved in [25] and in [20], and for the case $r=5$, see [20] and [41]. The following equivalent formulation is from [25],[21]. In the sequel let $r \geq 2$.

Conjecture 4 ([25], [21])**.** *In every r-colouring of the edges of a complete graph, the vertex set can be covered by the vertices of at most $r-1$ monochromatic components.*

Gyárfás and Lehel proposed a bipartite version of this conjecture [25], [36]. A complete bipartite graph G with nonempty vertex classes X and Y is referred to here as a *biclique* $[X, Y]$.

Conjecture 5 (Gyárfás [25], Lehel [36])**.** *In every r-colouring of the edges of a biclique, the vertex set can be covered by the vertices of at most $2r-2$ monochromatic components.*

Gyárfás showed in [25] that if Conjecture 5 is true, it is best possible. It is also worth noting that the statement becomes obviously true if the number of monochromatic components is just one larger than stated in the conjecture.

In the thesis we show that Conjecture 5 can be reduced to design-like conjectures. For example, one can assume that all components of all colour classes are complete bipartite graphs. (Similar reduction is not known for Conjecture 4.) We prove the conjecture for $r=2,3,4,5$, in fact in a stronger form. The possibility of coverings with components in the same colour, and the dual form of the conjecture which relates to transversals of hypergraphs are also discussed in the thesis.

4.1 An equivalent formulation

Let us call a graph partition $G_1, \ldots, G_r$ of biclique G a *spanning partition* if each vertex $v \in V(G)$ is included in every $V(G_i)$, $i=1,\ldots,r$. A *bi-equivalence graph* is a bipartite graph whose connected components are bicliques. The width of a bi-equivalence graph is the number of its components. (A graph whose connected components are complete graphs, i.e. cliques, is usually called equivalence graph, that is the reason of this name.) Let a biclique $[X, Y]$ be partitioned into the bi-equivalence graphs $G_1, G_2, \ldots, G_r$. Any connected component of G_i is a biclique, its vertex classes will be called *blocks* in colour i. Let us call a spanning bi-equivalence graph partition $G_1, \ldots, G_r$ of biclique G an *antichain partition* if no blocks (in different colours) properly contain each other.

We show that the following is an equivalent form of Conjecture 5.

Conjecture 6. *If a biclique has an antichain partition into r bi-equivalence graphs, then its vertex set can be covered by at most $2r-2$ biclique components.*

4.2 Bi-equivalence partitions for small r values

In the thesis we prove Conjecture 6 for $r = 2, 3, 4, 5$, in fact in the following stronger forms.

Theorem 11. *Let $2 \leq r \leq 4$. If a biclique has an antichain partition into r bi-equivalence graphs, then its vertex set can be covered by at most r monochromatic components of the same colour, or equivalently, one of the bi-equivalence graphs has width at most r.*

Theorem 12. *If a biclique has an antichain partition into 5 bi-equivalence graphs, then its vertex set can be covered by at most 8 monochromatic components of the same colour.*

4.3 Homogeneous coverings

Chen asked (in 1998) whether a stronger version of Conjecture 5 can be true, i.e. whether $2r-2$ monochromatic components of the same colour can cover the vertex set. Call such a cover a *homogeneous cover*. Although it is proven in [7] that there are no homogeneous covers with $2r-2$ bicliques in general for spanning bi-equivalence partitions, they might exist for antichain partitions.

Question 7. *Suppose that a biclique has an antichain partition into r bi-equivalence graphs. Is it true that some of them has width at most $2r-2$?*

Acknowledgement

First and foremost, I would like to express my sincere gratitude to my supervisor, Gábor Simonyi for teaching me many interesting topics in graph theory from the beginning of my undergraduate studies and later on for providing me with exciting research topics. I am also grateful for our numerous and invaluable discussions, for his useful suggestions and continuous support.

I am truly grateful to András Gyárfás for giving me interesting problems, for the pleasant joint work and for the kind interest which he showed in my results. I also wish to thank Shinya Fujita for inviting me to Gunma and Tokyo for three weeks in the spring of 2011 and for the fruitful collaboration. I acknowledge Péter Csikvári for the helpful discussions and for carefully reviewing some of my manuscripts.

For the friendly atmosphere I enjoyed during my PhD years, I thank all of my colleagues at the Department of Computer Science and Information Theory, Budapest University of Technology and Economics.

Last but not least, I would like to thank my family for their constant encouragement and support.

The author's publications related to the thesis

[1] Á. Tóth, *The ultimate categorical independence ratio of complete multipartite graphs*, SIAM J. Discrete Math. **23** (2009), 1900–1904.

[2] Á. Tóth, *Answer to a question of Alon and Lubetzky about the ultimate categorical independence ratio*, submitted to J. Combin. Theory Ser. B.

[3] Á. Tóth, *On the ultimate lexicographic Hall-ratio*, Discrete Math. **309** (2009), 3992–3997.

[4] Á. Tóth, *On the ultimate direct Hall-ratio*, submitted to Graphs Combin.

[5] A. Gyárfás, G. Simonyi, and Á. Tóth, *Gallai colorings and domination in multipartite digraphs*, to appear in J. Graph Theory.

[6] S. Fujita, M. Furuya, A. Gyárfás, and Á. Tóth, *Partition of graphs and hypergraphs into monochromatic connected parts*, submitted to Electron. J. Combin.

[7] G. Chen, S. Fujita, A. Gyárfás, J. Lehel, and Á. Tóth, *Around a biclique cover conjecture*, to be submitted.

- Á. Tóth, Asymptotic values of graph parameters, Proceedings of the 6th Hungarian-Japanese Symposium on Discrete Mathematics and Its Applications, Budapest, 2009., 388–392., based on [1,3]

- Á. Tóth, On the asymptotic values of the Hall-ratio, Proceedings of the 7th Hungarian-Japanese Symposium on Discrete Mathematics and Its Applications, Kyoto, 2011., 470–472., based on [4]

Further publications of the author

- G. Brightwell, G. Cohen, E. Fachini, M. Fairthorne, J. Körner, G. Simonyi, Á. Tóth, *Permutation capacities of families of oriented infinite paths*, SIAM J. Discrete Math. **24**, (2010), 441–456.
  Conference version: in the proceedings of The Sixth European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2011, Budapest; Electron. Notes Discrete Math. 38 (2011) 195–199.

- L. Lesniak, S. Fujita, Á. Tóth, *New results on long monochromatic cycles in edge-colored complete graphs*, submitted to Discrete Math.

Further references

[8] R. Aharoni, *Ryser's conjecture for tripartite 3-graphs*, Combinatorica **21** (2001), 1–4.

[9] R. Aharoni and P. Haxell, *Hall's theorem for hypergraphs*, J. Graph Theory **35** (2000), 83–88.

[10] R. Aharoni and R. Holzman, *Fractional kernels in digraphs*, J. Combin. Theory Ser. B. **73** (1998), 1–6.

[11] N. Alon and E. Lubetzky, *Independent sets in tensor graph powers*, J. Graph Theory **54** (2007), 73–87.

[12] N. Alon and J. H. Spencer, *The probabilistic method, third edition*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley and Sons, Hoboken, NJ, 2008.

[13] R. N. Ball, A. Pultr, and P. Vojtěchovský, *Colored graphs without colorful cycles*, Combinatorica **27** (2007), 407–427.

[14] C. Berge and M. Simonovits, *The coloring numbers of the direct product of two hypergraphs*, Lecture Notes in Math. **411** (1974), 21–33, Hypergraph Seminar (Proceedings of the First Working Seminar, Ohio State University, Columbus, Ohio, 1972; dedicated to Arnold Ross).

[15] J. I. Brown, R. J. Nowakowski, and D. Rall, *The ultimate categorical independence ratio of a graph*, SIAM J. Discrete Math. **9** (1996), 290–300.

[16] K. Cameron and J. Edmonds, *Lambda composition*, J. Graph Theory **26** (1997), 9–16.

[17] K. Cameron, J. Edmonds, and L. Lovász, *A note on perfect graphs*, Period. Math. Hungar. **17** (1986), 441–447.

[18] M. Cropper, A. Gyárfás, and J. Lehel, *Hall-ratio of the Mycielski graphs*, Discrete Math. **306** (2006), 1988–1990.

[19] M. Cropper, M. S. Jacobson, A. Gyárfás, and J. Lehel, *The Hall ratio of graphs and hypergraphs*, Les cahiers du Laboratoire Leibniz, Grenoble **17** (2000).

[20] P. Duchet, *Représentations, noyaux en théorie des graphes at hypergraphes*, Ph.D. thesis, Paris, 1979.

[21] P. Erdős, A. Gyárfás, and L. Pyber, *Vertex coverings by monochromatic cycles and trees*, J. Combin. Theory Ser. B **51** (1991), 90–95.

[22] S. Fujita, C. Magnant, and K. Ozeki, *Rainbow generalizations of Ramsey Theory: A Survey*, Graphs Combin. **26** (2010), 1–30.

[23] T. Gallai, *Transitiv orientierbare graphen*, Acta Math. Sci. Hungar. **18** (1976), 25–66, English translation by F. Maffray and M. Preissmann, in: J. L. Ramírez Alfonsín and B. A. Reed (editors), Perfect Graphs, John Wiley and Sons, 2001, 25–66.

[24] V. Gurvich, *Decomposing complete edge-chromatic graphs and hypergraphs. Revisited*, Discrete Applied Math. **157** (2009), 3069–3085.

[25] A. Gyárfás, *Partition coverings and blocking sets in hypergraphs (in Hungarian)*, Communications of the Computer and Automation Institute of the Hungarian Academy of Sciences **71** (1977), 62 pages.

[26] A. Gyárfás and G. N. Sárközy, *Gallai colorings of non-complete graphs*, Discrete Math. **310** (2010), 977–980.

[27] A. Gyárfás, G. N. Sárközy, A. Sebő, and S. Selkow, *Ramsey-type results for Gallai colorings*, J. Graph Theory **64** (2010), 233–243.

[28] A. Gyárfás and G. Simonyi, *Edge colorings of complete graphs without tricolored triangles*, J. Graph Theory **46** (2004), 211–216.

[29] P. Haxell, *A note on a conjecture of Ryser*, Periodica Math. Hungar. **30** (1995), 73–79.

[30] J. R. Henderson, *Permutation decomposition of (0-1)-matrices and decomposition transversals*, Ph.D. thesis, Caltech, 1971, for Ryser's conjecture see also at `http://garden.irmacs.sfu.ca/?q=op/rysers_conjecture`.

[31] W. Imrich and S. Klavžar, *Product graphs. Structure and Recognition, Chapter 8: Invariants*, Wiley-Interscience Series in Discrete Mathematics and Optimalization, John Wiley and Sons, Chichester, 2000.

[32] Z. Király, *Monochromatic components in edge-colored complete uniform hypergraphs*, Electronic Notes in Discrete Mathematics **38C** (2011), 517–521.

[33] J. Körner and A. Orlitsky, *Zero-error information theory*, IEEE Trans. Inform. Theory **44** (1998), 2207–2229.

[34] J. Körner and G. Simonyi, *Graph pairs and their entropies: Modularity problems*, Combinatorica **20** (2000), 227–240.

[35] J. Körner, G. Simonyi, and Zs. Tuza, *Perfect couples of graphs*, Combinatorica **12** (1992), 179–192.

[36] J. Lehel, *Ryser's conjecture for linear hypergraphs*, manuscript, 1998.

[37] R. J. McEliece and E. C. Posner, *Hide and seek, data storage, and entropy*, Ann. Math. Statist. **42** (1971), 1706–1716.

[38] G. Sárközy, *Monochromatic cycle partitions of edge-colored graphs*, J. Graph Theory **66** (2010), 57–64.

[39] G. Simonyi, *Asymptotic values of the Hall-ratio for graph powers*, Discrete Math. **306** (2006), 2593–2601.

[40] E. Szemerédi and Zs. Tuza, *Upper bound for transversals of tripartite hypergraphs*, Period. Math. Hungar. **13** (1982), 321–323.

[41] Zs. Tuza, *Some special cases of Ryser's conjecture*, manuscript, 1978.

[42] H. S. Witsenhausen, *The zero-error side-information problem and chromatic numbers*, IEEE Trans. Inform. Theory **22** (1976), 592–593.

[43] X. Zhu, *Fractional Hedetniemi's conjecture is true*, European J. Combin. **32** (2011), 1168–1175.