Colouring problems related to graph products and coverings

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In the thesis we concentrate on two topics of graph colouring problems. On the one hand, we investigate the asymptotic behaviour of colouring-related graph parameters for different graph powers. On the other hand, we discuss problems on coverings with monochromatic components in edge-coloured graphs.

Several graph parameters show an interesting behaviour when they are investigated for different powers of graphs. One of the most famous examples of such behaviour is that of the Shannon capacity of graphs which is the theoretical upper limit of channel capacity for error-free coding in information theory. This graph parameter is defined as the normalized limit of the independence number under the so-called normal power and its exact value is not known even for small, simple graphs, for example odd cycles with length more than five. Similar questions arise when investigating the independence ratio and the Hall-ratio of a graph.

The independence ratio of a graph is defined as the ratio of the independence number and the number of vertices. The categorical power of a graph is defined on the vertex sequences of the original graph and we connect two such sequences iff they are adjacent in every coordinate. The asymptotic value of the independence ratio with respect to the categorical power is the ultimate categorical independence ratio which is introduced by Brown, Nowakowski and Rall. Based on the lower bounds they gave, Alon and Lubetzky formulated a relatively easy general lower bound for the parameter, and asked whether this bound always coincides with the ultimate categorical independence ratio. In the thesis we answer this question affirmatively. From the result we immediately obtain the solution for further open problems related to this concept. For instance, we prove the conjecture of Brown, Nowakowski and Rall, stating that the ultimate categorical independence ratio of the disjoint union of two graphs is the maximum of the value of the parameter for the two graphs. We also show that the parameter is computable. The analogous question for Shannon-capacity is still open.

The Hall-ratio is closely related to the independence ratio. It is defined as the ratio of the number of vertices and the independence number maximized over all subgraphs of a graph. The appropriately normalized asymptotic values of the Hall-ratio for different graph powers were investigated by Simonyi. Considering for normal and co-normal power he proved that the corresponding limit equals to the similar limit one gets for the chromatic number. In the thesis we prove that the asymptotic value of the Hall-ratio with respect to both the categorical and the lexicographic power is equal to the fractional chromatic number, proving two conjectures of Simonyi.

An equivalent form due to Gyárfás of a celebrated conjecture of Ryser, states that if the edges of a graph with independence number α are coloured with k colours then the vertex set can be covered by the vertices of at most $(k-1)\alpha$ monochromatic components. It is known to be true for $k=2$, when it is equivalent to König's theorem. The case $k=3$ was solved by Aharoni, relying on an interesting topological method. The important special case of Ryser's conjecture when the graph is complete is open for $k \geq 6$. In the thesis we investigate similar covering problems of edge-coloured graphs.

An edge-colouring of a graph is called a Gallai-colouring if there is no completely multicoloured triangle. A basic property of Gallai-coloured complete graphs is that at least one of the colour classes spans a connected subgraph on the entire vertex set. Gyárfás and Sárközy proved that if we colour the edges of a not necessarily complete graph so that no completely multicoloured triangles appear then there is still a large monochromatic connected component whose size is proportional to the number of vertices of the graph where the proportion depends only on the independence number of the graph. In view of this result it is natural to ask whether one can also span the whole vertex set with a constant number of connected monochromatic subgraphs where the constant depends only on the independence number. This question led to a problem about the existence of dominating sets in directed graphs. Solving this we obtain an affirmative answer to the previous question.

We also address a conjecture of Gyárfás and Lehel, a variant of Ryser's conjecture, stating that in every k-colouring of the edges of a complete bipartite graph, the vertex set can be covered by the vertices of at most $2k-2$ monochromatic components. We reduce this conjecture to design-like conjectures, and prove this reduced statement for $k \leq 5$. We also discuss the possibility of coverings with components in the same colour, and the dual form of the conjecture which relates to transversals of hypergraphs.

Some of the results are joint with G. Chen, S. Fujita, M. Furuya, A. Gyárfás, J. Lehel and G. Simonyi.