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Ph.D. Thesis

Colouring problems related to graph products and coverings

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Chapter 1

Introduction

In this thesis we concentrate on two topics of graph colouring problems. We investigate the asymptotic behaviour of colouring-related graph parameters for different graph powers. In addition, we discuss problems on coverings with monochromatic components in edge-coloured graphs. In the next two sections we give a short introduction to the two topics and state our results.

1.1 Asymptotic values of graph parameters

Several graph parameters show an interesting behaviour when they are investigated for different powers of graphs. One of the most famous examples of such behaviour is that of the Shannon capacity of graphs (introduced by Shannon [47], see Körner and Orlitsky [41] for a survey of related topics) which is the theoretical upper limit of channel capacity for error-free coding in information theory. This graph parameter is defined as the normalized limit of the independence number under the so-called normal power and its exact value is not known even for small, simple graphs (for example odd cycles with length more than five).
The normalized asymptotic value of the chromatic number with respect to the normal power is the Witsenhausen rate. It is introduced by Witsenhausen in [52], where its information theoretic relevance is also explained. If we investigate the chromatic number for the co-normal power we get the fractional chromatic number as the corresponding limit by a famous theorem of McEliece and Posner [46], cf. also Berge and Simonovits [14].
Similar questions arise when investigating the independence ratio and the Hall-ratio of a graph.

The independence ratio of a graph is the ratio of the independence number and the number of vertices. Its asymptotic value with respect to the so-called Cartesian power is the ultimate independence ratio which is introduced by Hell, Yu and Zhou [37]. Motivated by this concept Brown, Nowakowski and Rall [15] considered the analogous, but significantly different parameter, the

ultimate categorical independence ratio which is defined with respect to the so-called categorical power. This parameter was also investigated by Alon and Lubetzky [11]. Based on the lower bounds proven in [15] they settled a relatively easy general lower bound for the parameter, and asked whether this bound always coincides with the ultimate categorical independence ratio.
In this thesis we answer this question affirmatively, and we obtain a solution for further open problems related to this concept. For instance, for the conjecture of Brown, Nowakowski and Rall, stating that the ultimate categorical independence ratio of the disjoint union of two graphs is the maximum of the value of the parameter for the two graphs.

The Hall-ratio is closely related to the independence ratio, this parameter is the ratio of the number of vertices and the independence number maximized over all subgraphs of the graph. It was introduced in [21, 20] motivated by problems of list colouring. The (appropriately normalized) asymptotic values of this graph parameter for different graph powers were investigated by Simonyi in [49]. Considering for normal and co-normal power he proved that the corresponding limit equals to the similar limit one obtains for the chromatic number.
In this thesis we prove that the asymptotic value of the Hall-ratio with respect to both the categorical power and the lexicographic power is equal to the fractional chromatic number, proving the conjectures of Simonyi.

The ultimate categorical independence ratio is investigated in Chapter 2 which is based on [1] and [2]. We deal with the asymptotic value of the Hall-ratio for the lexicographic and the categorical power in Chapter 3, based on [3] and [4].

1.2 Monochromatic coverings in edge-coloured graphs

An equivalent form of Ryser's conjecture [38] due to Gyárfás [30], states that if the edges of a graph G are coloured with k colours then the vertex set can be covered by the vertices of at most $\alpha(G)(k-1)$ monochromatic components, where $\alpha(G)$ denotes the independence number. (Given an edge colouring, a monochromatic component means a connected component of the subgraph of any given colour.) It is known to be true for $k = 2$ (when it is equivalent to König's theorem). After partial results [36, 50], the case $k = 3$ was solved by Aharoni [8], relying on an interesting topological method established in [9]. The important special case of Ryser's conjecture when the graph is complete is open for $k \geq 6$.
Recently Király [40] showed, somewhat surprisingly, that an analogue of Ryser's conjecture holds for hypergraphs: for $r \geq 3$, in every k-colouring of the edges of a complete r-uniform hypergraph, the vertex set can be covered by at most $\lfloor \frac{k}{r} \rfloor$ monochromatic components, and this bound is sharp.

Here we investigate similar covering problems of edge-coloured graphs. The first result is about edge-colourings of graphs where the number of colours is not restricted but 3-edge-coloured triangles are forbidden. We give a (finite) upper bound on the number of monochromatic components needed in the covering in terms of the independence number of the graph. The second problem is about coverings of complete bipartite graphs. In this case we try to give the best upper bound on the size of the covering in terms of the number of colours used on the edges.

An edge-colouring of a graph is called a Gallai colouring if there is no completely multi-coloured triangle. A basic property of Gallai-coloured complete graphs is that at least one of the colour classes spans a connected subgraph on the entire vertex set. Gyárfás and Sárközy proved that if we colour the edges of a not necessarily complete graph G so that no 3-coloured triangles appear then there is still a large monochromatic component whose size is proportional to the vertex number of G where the proportion depends on the independence number. In view of this result it is natural to ask whether one can also span the whole vertex set with a constant number of connected monochromatic subgraphs where the constant depends only on the independence number of G. This question led to the following problem.
Assume that D is a digraph without cyclic triangles and its vertices are partitioned into classes $A_1, \ldots, A_t$ of independent vertices. A set $U = \cup_{i \in S} A_i$ is called a dominating set of size $|S|$ if for any vertex $v \in \cup_{i \notin S} A_i$ there is a $w \in U$ such that $(w, v) \in E(D)$. Let $\beta(D)$ be the cardinality of the largest independent set of D whose vertices are from different partite classes of D. We show that there exists a $h = h(\beta(D))$ such that D has a dominating set of size at most h. From this result we get an affirmative answer to the previous question.
We also extend the covering problem of Gallai-coloured graphs to partitioning.

We also address a conjecture of Gyárfás and Lehel (a variant of Ryser's conjecture), stating that in every r-colouring of the edges of a complete bipartite graph $[X, Y]$, the vertex set can be covered by the vertices of at most $2r - 2$ monochromatic components. We reduce this conjecture to design-like conjectures, where the monochromatic components of the colour classes are complete bipartite graphs $[X', Y']$ with nonempty blocks X' and Y'. It can also be assumed that each colour class covers $X \cup Y$, moreover, no two blocks properly contain each other. We prove this reduced conjecture for $r \leq 5$.
We also discuss about the possibilty of coverings with components in the same colour, and the dual form of the conjecture which relates to transversals of hypergraphs.

The problems about Gallai colourings and domination in multipartite digraphs are discussed in Chapter 4 based on [5] and [6]. The results on monochromatic coverings in complete bipartite graphs are presented in Chapter 5 which is based on [7].

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Chapter 2

The ultimate categorical independence ratio

The *independence ratio* of a graph G is defined as $i(G) = \frac{\alpha(G)}{|V(G)|}$, that is, as the ratio of the independence number and the number of vertices.

Its asymptotic value with respect to what is called Cartesian graph exponentiation is the ultimate independence ratio which was introduced by Hell, Yu and Zhou [37] and further investigated by Hahn, Hell and Poljak [35] and by Zhu [53]. Motivated by this concept Brown, Nowakowski and Rall [15] considered the analogous, but significantly different parameter, the ultimate categorical independence ratio which is defined with respect to the categorical power of graphs.

For two graphs G and H, their *categorical product* (also called as direct or tensor product) $G \times H$ is defined on the vertex set $V(G \times H) = V(G) \times V(H)$ with edge set $E(G \times H) = \{\{(x_1, y_1), (x_2, y_2)\} : \{x_1, x_2\} \in E(G) \text{ and } \{y_1, y_2\} \in E(H)\}$. The kth categorical power $G^{\times k}$ is the k-fold categorical product of G.

Definition ([15])**.** The *ultimate categorical independence ratio* of a graph G is defined as

$$A(G) = \lim_{k \to \infty} i(G^{\times k}).$$

Brown, Nowakowski and Rall in [15] proved that for any independent set U of G the inequality $A(G) \geq \frac{|U|}{|U|+|N_G(U)|}$ holds, where $N_G(U)$ denotes the neighbourhood of U in G. Furthermore, they showed that $A(G) > \frac{1}{2}$ implies $A(G) = 1$.
The ultimate categorical independence ratio was also investigated by Alon and Lubetzky in [11], where they defined the parameters $a(G)$ and $a^*(G)$ as follows

$$a(G) = \max_{U \text{ is independent set of } G} \frac{|U|}{|U| + |N_G(U)|} \qquad \text{and} \qquad a^*(G) = \begin{cases} a(G) & \text{if } a(G) \leq \frac{1}{2} \\ 1 & \text{if } a(G) > \frac{1}{2} \end{cases},$$

and they proposed the following two questions.

Question 1 (Alon, Lubetzky [11])**.** *Does every graph G satisfy $A(G) = a^*(G)$? Or, equivalently, does every graph G satisfy $a^*(G^{\times 2}) = a^*(G)$?*

Question 2 (Alon, Lubetzky [11])**.** *Does the inequality $i(G \times H) \leq \max\{a^*(G), a^*(H)\}$ hold for every two graphs G and H?*

The mentioned lower bounds from [15] give us the inequality $A(G) \geq a^*(G)$. One can easily see the equivalence between the two forms of Question 1, moreover it is not hard to show that an affirmative answer to Question 1 would imply the same for Question 2 (see [11]).

Following [15] a graph G is called self-universal if $A(G) = i(G)$. As a consequence, the equality $A(G) = a^*(G)$ in Question 1 is also satisfied for these graphs according to the chain of inequalities $i(G) \leq a(G) \leq a^*(G) \leq A(G)$. Cliques, regular bipartite graphs, and Cayley graphs of Abelian groups belong to this class (see [15]). In [1] the author proved that a complete multipartite graph G is self-universal, except for the case when $i(G) > \frac{1}{2}$. Therefore the equality $A(G) = a^*(G)$ is also verified for this class of graphs. (In the latter case $A(G) = a^*(G) = 1$.) In [11] it is shown that the graphs which are disjoint unions of cycles and complete graphs satisfy the inequality in Question 2.

In this chapter first, in Section 2.1, we give a proof for the results of Brown, Nowakowski and Rall about the lower bounds on $A(G)$. Then, in Section 2.2 we sum up the results of the author about the values of $A(G)$ when G is a complete multipartite graph. The main result of this chapter is that we answer Question 1 affirmatively. Thereby a positive answer also for Question 2 is obtained. Moreover, it solves some other open problems related to $A(G)$. In the proofs we exploit an idea of Zhu [54] that he used on the way when proving the fractional version of Hedetniemi's conjecture. This tool is presented in Section 2.3. Then, in Section 2.4 first we prove the inequality

$$i(G \times H) \leq \max\{a(G), a(H)\}, \quad \text{for every two graphs } G \text{ and } H,$$

and give a positive answer to Question 2 (using $a(G) \leq a^*(G)$). Afterwards we prove that

$$a(G \times H) \leq \max\{a(G), a(H)\}, \quad \text{provided that } a(G) \leq \frac{1}{2} \text{ or } a(H) \leq \frac{1}{2},$$

and from this result we conclude the affirmative answer to Question 1. (If $a(G) > \frac{1}{2}$ then $a^*(G^{\times 2}) = a^*(G) = 1$. Otherwise applying the above result for $G = H$ we get $a(G^{\times 2}) \leq a(G)$, while the reverse inequality clearly holds for every G. Thus we can conclude that $a^*(G^{\times 2}) = a^*(G)$ for every graph G.) Finally, in Section 2.5, we discuss further open problems which are solved by our result. For instance, we get a proof for the conjecture of Brown, Nowakowski and Rall, stating that $A(G \uplus H) = \max\{A(G), A(H)\}$, where $G \uplus H$ denotes the disjoint union of the graphs G and H. We also give a characterization of self-universal graphs.

2.1 Results of Brown, Nowakowski and Rall: lower bounds on the ultimate categorical independence ratio

Brown, Nowakowski and Rall [15] proved the following lower bounds on the ultimate categorical independence ratio. These results are important also in our work, for this reason we give here their proof. The following short arguments are essentially from [11].

Theorem 1 (Brown, Nowakowski, Rall [15])**.** *For any independent set U of G we have*

$$A(G) \geq \frac{|U|}{|U| + |N_G(U)|}.$$

Proof. Let U_k be the set of those vertices of $G^{\times k}$ which have a coordinate in U and before the first such position all the coordinates are in $V(G) \setminus (U \cup N_G(U))$. (Note that $U_1 = U$.) It is easy to see that U_k is an independent set of $G^{\times k}$. Indeed, any two elements of U_k are non-adjacent in the first position where one of them contains an element of U. The ratio of the size of U_k and the number of those vertices which have a coordinate in $N_G(U)$ and before the first such position all the coordinates are in $V(G) \setminus (U \cup N_G(U))$ is clearly $\frac{|U|}{|N_G(U)|}$. The remaining vertices have all coordinates in $V(G) \setminus (U \cup N_G(U))$, hence their ratio to the vertex number of $G^{\times k}$ tends to zero as k approaches infinity. Therefore $\frac{|U_k|}{|V(G^{\times k})|}$ tends to $\frac{|U|}{|U|+|N_G(U)|}$, thus $A(G) \geq \frac{|U|}{|U|+|N_G(U)|}$.
□

Theorem 2 (Brown, Nowakowski, Rall [15])**.** *If $A(G) > \frac{1}{2}$ then $A(G) = 1$.*

Proof. The condition $A(G) > \frac{1}{2}$ implies that $i(G^{\times \ell}) > \frac{1}{2}$ for some positive integer ℓ. Set $H = G^{\times \ell}$, we have $A(H) = A(G)$. So there exists an independent set U of H such that $\frac{|U|}{|V(H)|} > \frac{1}{2}$, thus $|U| > |N_G(U)|$. Let U_k be the set of those vertices of $H^{\times k}$ all coordinates of whose belong to U. Then $N_{H^{\times k}}(U_k)$ consists of the vertices of $H^{\times k}$ all coordinates of whose are in $N_G(U)$. Since the ratio $\frac{|U_k|}{|U_k|+|N_{G^{\times k}}(U_k)|} = \frac{|U|^k}{|U|^k+|N_G(U)|^k}$ tends to 1 as k approaches infinity, from Theorem 1 we obtain that $A(H) = A(H^{\times k}) = 1$, implying also $A(G) = 1$. □

2.2 The ultimate categorical independence ratio for complete multipartite graphs

As we mentioned at the beginning of this chapter, Brown, Nowakowski and Rall in [15] investigated graphs for which $A(G) = i(G)$ holds and they called such graphs self-universal. In that article it is proven that some interesting graph families, for example Cayley graphs of Abelian groups, have this property. The paper [15] mentions complete multipartite graphs as one of those families of graphs for which the determination of the ultimate categorical independence ratio remained an open problem. It follows from Theorem 1 and 2 that if the largest partite class contains more than half of the vertices then the ultimate categorical independence ratio equals to 1. In [1] it was proven that in all other cases, i.e., when none of the parts of the complete multipartite graph has size greater than half the number of vertices, the graph is self-universal. In particular, the author proved the following theorem, from which the result on complete multipartite graphs can be obtained easily. We denote by $d(v)$ the degree of the vertex v.

Theorem 3 ([1])**.** *Let G be a graph for which $d(v) \geq |V(G)| - \alpha(G)$ holds for all vertices v of G and $i(G) \leq \frac{1}{2}$ holds. Then $i(G^{\times k}) = i(G)$ holds for every integer $k \geq 1$.*

Here we omit the proof of this theorem given in [1], but in Section 2.5 we will obtain this statement as a consequence of a more general result on self-universal graphs.

Corollary 4 ([1])**.** *Let $G = K_{\ell_1,\ell_2,\ldots,\ell_m}$ be a complete multipartite graph. Let $n = \sum_{i=1}^{m} \ell_i$ be the number of vertices and let $\ell = \max_{1\leq i\leq m} \ell_i$ be the size of the largest partite class. If $\ell \leq \frac{n}{2}$ then $A(G) = i(G) = \frac{\ell}{n}$, so G is self-universal, otherwise $A(G) = 1$.*

We remark that there are graphs which satisfy the conditions of Theorem 3 other than complete multipartite graphs. An example is given by the graph consisting of a 5-length cycle and three additional points joint to every vertex of the cycle.

2.3 A result of Zhu: nice partition of the independent sets of the product graph

In this section we present a result of Zhu [54] about the independent sets of categorical product of graphs. This will be a key tool for answering the questions of Alon and Lubetzky.

Let U be an independent set of $G \times H$. Zhu considered the partition of U into two sets, let

$$\begin{aligned} A &= \{(x, y) \in U \,:\, \nexists (x', y) \in U \text{ s.t. } \{x, x'\} \in E(G)\}, \\ B &= \{(x, y) \in U \,:\, \exists (x', y) \in U \text{ s.t. } \{x, x'\} \in E(G)\}. \end{aligned} \tag{2.1}$$

We have $U = A \uplus B$, where $A \uplus B$ denotes the disjoint union of the sets A and B.

In the sequel, we keep using the following notations for any $Z \subseteq V(G \times H)$.
For any $y \in V(H)$, let

$$Z^G(y) = \{x \in V(G) \,:\, (x, y) \in Z\}.$$

Similarly, for any $x \in V(G)$, let

$$Z^H(x) = \{y \in V(H) \,:\, (x, y) \in Z\}.$$

And, let

$$M_G(Z) = \{(x, y) \in V(G \times H) \,:\, x \in N_G(Z^G(y))\}.$$

In words, $M_G(Z)$ means that we decompose $V(G \times H)$ into sections corresponding to the elements of $V(H)$, and for each $y \in V(H)$ we pick those points from the corresponding section which are neighbours of the elements of $Z^G(y)$ in the graph G. Similarly, let

$$M_H(Z) = \{(x, y) \in V(G \times H) \,:\, y \in N_H(Z^H(x))\}.$$

Keep in mind, that $Z^G(y) \subseteq V(G)$ and $Z^H(x) \subseteq V(H)$, while $M_G(Z), M_H(Z) \subseteq V(G \times H)$. (See Figure 1.)

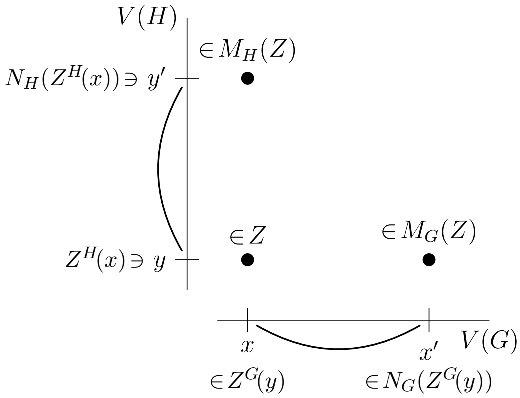

Figure 1: The elements of Z, $Z(x)$, $Z(y)$, $M_G(Z)$ and $M_H(Z)$.

For the partition of U defined in (2.1) Zhu showed the following properties.

Lemma 5 (Zhu [54])**.** *The following holds:*

(i) For every $y \in V(H)$, $A^G(y)$ is an independent set of G. For every $x \in V(G)$, $B^H(x)$ is an independent set of H.

(ii) A, B, $M_G(A)$ and $M_H(B)$ are pairwise disjoint subsets of $V(G \times H)$.

For the sake of completeness we prove this lemma.

Proof. We show the statements in (i). $A^G(y)$ is independent for every $y \in V(H)$ by definition. If for any $x \in V(G)$ the set $B^H(x)$ was not independent in H, that is $\exists y, y' \in B^H(x)$, $\{y, y'\} \in E(H)$, then from $(x, y') \in B$ we would get that $\exists (x', y') \in U$, $\{x, x'\} \in E(G)$. But this would be a contradiction, because $(x, y) \in B$ and $(x', y') \in U$ were two adjacent elements of the independent set U.

We turn to the proof of (ii). By definition $A \cap B = \emptyset$. The first part of the lemma implies that the pair $(A, M_G(A))$ is also disjoint, as well as the pair $(B, M_H(B))$.

We shall see that the pairs $(A, M_H(B))$, $(M_G(A), M_H(B))$ and $(B, M_G(A))$ are also disjoint. (See also Figures 2, 3 and 4.)

If $(x, y) \in A \cap M_H(B)$ then (by the definition of $M_H(B)$) $\exists (x, y') \in B$, $\{y, y'\} \in E(H)$, and so (by the definition of B) $\exists (x', y') \in U$, $\{x, x'\} \in E(G)$, which is a contradiction: $(x, y) \in A$ and $(x', y') \in U$ are adjacent vertices in the independent set U.

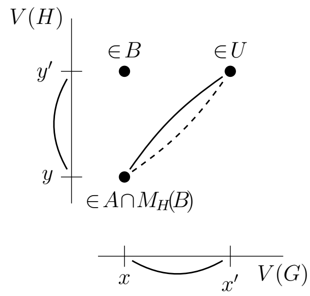

Figure 2: $A \cap M_H(B) = \emptyset$.

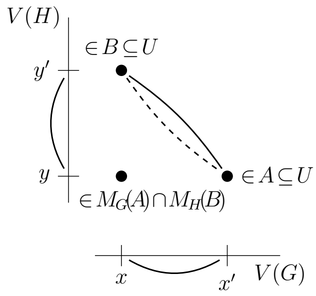

Figure 3: $M_G(A) \cap M_H(B) = \emptyset$.

Similarly, if $(x, y) \in M_G(A) \cap M_H(B)$ then (by the definition of $M_G(A)$) $\exists (x', y) \in A \subseteq U$, $\{x, x'\} \in E(G)$ while (by the definition of $M_H(B)$) $\exists (x, y') \in B \subseteq U$, $\{y, y'\} \in E(H)$, which contradicts the independence of U.

Finally, $(x, y) \in B \cap M_G(A)$ implies that $\exists (x', y) \in A$, $\{x, x'\} \in E(G)$ (by the definition of $M_G(A)$), which is in contradiction with the definition of A: there should not be an $(x, y) \in B \subseteq U$ satisfying $\{x, x'\} \in E(G)$.

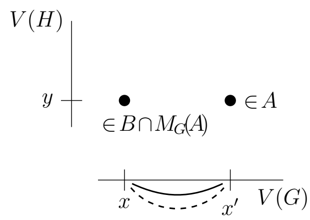

Figure 4: $B \cap M_G(A) = \emptyset$.

$\square$

2.4 Answer to the questions of Alon and Lubetzky

In this section we answer Question 2 and 1 from the beginning of this chapter. In Subsection 2.4.1 we give an upper bound for $i(G \times H)$ in terms of $a(G)$ and $a(H)$. In Subsection 2.4.2 we prove that the same upper bound holds also for $a(G \times H)$ provided that $a(G) \leq \frac{1}{2}$ or $a(H) \leq \frac{1}{2}$. Thereby we will obtain our main result, which states that $A(G) = a^*(G)$ for every graph G.

2.4.1 Upper bound for $i(G \times H)$

As a simple consequence of Zhu's result the following inequality is obtained.

Theorem 6. *For every two graphs G and H we have*

$$i(G \times H) \leq \max\{a(G), a(H)\}.$$

Proof. Let U be a maximum-size independent set of $G \times H$, then we have

$$i(G \times H) = \frac{\alpha(G \times H)}{|V(G \times H)|} = \frac{|U|}{|V(G \times H)|}. \tag{2.2}$$

We partition U into $U = A \uplus B$ according to (2.1). We also use the notations $A^G(y)$ for every $y \in V(H)$, $B^H(x)$ for every $x \in V(G)$, and $M_G(A)$, $M_H(B)$ defined in the previous section.
It is clear that $|U| = |A| + |B|$. From the second part of Lemma 5 we have that $|A| + |B| + |M_G(A)| + |M_H(B)| \leq |V(G \times H)|$. Observe that $|M_G(A)| = \sum_{y \in V(H)} |N_G(A^G(y))|$ and $|M_H(B)| = \sum_{x \in V(G)} |N_H(B^H(x))|$. Hence we get

$$\begin{aligned} \frac{|U|}{|V(G \times H)|} &\leq \frac{|A| + |B|}{|A| + |B| + |M_G(A)| + |M_H(B)|} = \\ &= \frac{\sum_{y \in V(H)} |A^G(y)| + \sum_{x \in V(G)} |B^H(x)|}{\sum_{y \in V(H)}(|A^G(y)| + |N_G(A^G(y))|) + \sum_{x \in V(G)}(|B^H(x)| + |N_H(B^H(x))|)}. \end{aligned} \tag{2.3}$$

From the first part of Lemma 5 and by the definition of $a(G)$, $a(H)$ we have $\frac{|A^G(y)|}{|A^G(y)|+|N_G(A^G(y))|} \leq a(G)$ for every $y \in V(H)$, and $\frac{|B^H(x)|}{|B^H(x)|+|N_H(B^H(x))|} \leq a(H)$ for every $x \in V(G)$, respectively.
Using the fact that if $\frac{t_1}{s_1} \leq r$ and $\frac{t_2}{s_2} \leq r$ then $\frac{t_1+t_2}{s_1+s_2} \leq r$, this yields

$$\frac{\sum_{y \in V(H)} |A^G(y)| + \sum_{x \in V(G)} |B^H(x)|}{\sum_{y \in V(H)}(|A^G(y)| + |N_G(A^G(y))|) + \sum_{x \in V(G)}(|B^H(x)| + |N_H(B^H(x))|)} \leq \max\{a(G), a(H)\}. \tag{2.4}$$

Equality (2.2) and inequalities (2.3), (2.4) together give us the stated inequality,

$$i(G \times H) \leq \max\{a(G), a(H)\}.$$

□

From Theorem 6 it follows that the answer to Question 2 is positive, as it is already stated.

2.4.2 Upper bound for $a(G \times H)$

In this subsection we answer Question 1 affirmatively. To show that $a^*(G^{\times 2}) = a^*(G)$ holds for every graph G it is enough to prove that $a(G^{\times 2}) \leq a(G)$ if $a(G) \leq \frac{1}{2}$. (This is because every G satisfies $a(G^{\times 2}) \geq a(G)$, and in addition if $a(G) > \frac{1}{2}$ then $a^*(G^{\times 2}) = a^*(G) = 1$.) A bit more general, we prove the following theorem.

Theorem 7. *If $a(G) \leq \frac{1}{2}$ or $a(H) \leq \frac{1}{2}$ then*

$$a(G \times H) \leq \max\{a(G), a(H)\}.$$

Proof. Let G and H be two graphs satisfying $a(G) \leq \frac{1}{2}$ or $a(H) \leq \frac{1}{2}$. Without loss of generality, we may assume that $a(G) \geq a(H)$. Therefore $a(H) \leq \frac{1}{2}$.
We need to show that for every independent set U of $G \times H$ we have

$$\frac{|U|}{|U| + |N_{G\times H}(U)|} \leq a(G).$$

Observe that it can be rewritten as follows. Set $b(G) = \frac{1-a(G)}{a(G)}$. It is enough to prove that

$$|N_{G\times H}(U)| \geq b(G)|U|.$$

The definition of $a(G)$ means that $|N_G(P)| \geq b(G)|P|$ for any independent set P of G (and there is an independent set R of G such that $|N_G(R)| = b(G)|R|$). Similarly, using $b(H) = \frac{1-a(H)}{a(H)}$ we have $|N_H(Q)| \geq b(H)|Q|$ for any independent set Q of H.

First, we need some notations. Let $\hat{A}$, $\hat{B}$ and C be the following subsets of U.

$$\hat{A} = \{(x,y) \in U \,:\, \nexists (x',y) \in U \text{ s.t. } \{x,x'\} \in E(G), \text{ but } \exists (x,y') \in U \text{ s.t. } \{y,y'\} \in E(H)\},$$
$$\hat{B} = \{(x,y) \in U \,:\, \nexists (x,y') \in U \text{ s.t. } \{y,y'\} \in E(H), \text{ but } \exists (x',y) \in U \text{ s.t. } \{x,x'\} \in E(G)\},$$
$$C = \{(x,y) \in U \,:\, \nexists (x',y) \in U \text{ s.t. } \{x,x'\} \in E(G), \text{ and } \nexists (x,y') \in U \text{ s.t. } \{y,y'\} \in E(H)\}.$$

(See the rules for these sets on Figure 5.)

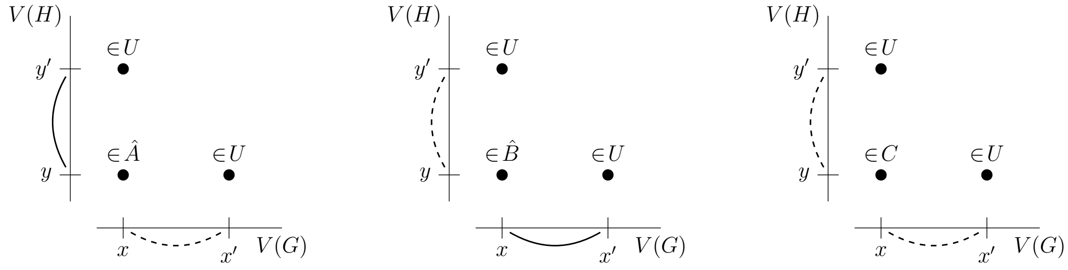

Figure 5: The elements of sets $\hat{A}$, $\hat{B}$ and C.

We will also use the notations $Z^H(x)$, $Z^G(y)$, $M_G(Z)$ and $M_H(Z)$ for any $Z \subseteq V(G \times H)$, $x \in V(G)$, $y \in V(H)$ defined in Section 2.3. We partition $M_G(\hat{A} \cup C)$ into two parts, let

$$N_1 = M_G(\hat{A} \cup C) \cap N_{G\times H}(U) \quad \text{and} \quad L = M_G(\hat{A} \cup C) \setminus N_{G\times H}(U).$$

And let

$$N_2 = M_H(\hat{B} \cup L).$$

The above subsets of $V(G \times H)$ will play an important role in the proof. (See the rules for them on Figure 6.)

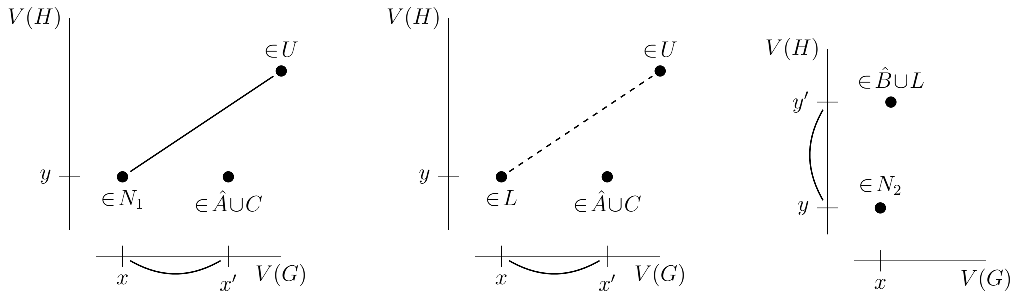

Figure 6: The elements of N_1, L and N_2.

We obtain the desired lower bound for $N_{G\times H}(U)$ in the following main steps.

(i) We show that U is partitioned into $U = \hat{A} \uplus \hat{B} \uplus C$.

(ii) We consider the elements of $\hat{A}$ and C for every $y \in V(H)$, and prove that

- **(a)** $(\hat{A} \cup C)^G(y)$ is independent in G,
- **(b)** $|N_1| \geq b(G)\big(|\hat{A}| + |C|\big) - |L|$.

(iii) We consider the elements of $\hat{B}$ and L for every $x \in V(G)$, and prove that

- **(a)** $\hat{B}(x) \cap L(x) = \emptyset$,
- **(b)** $(\hat{B} \cup L)^H(x)$ is independent in H,
- **(c)** $|N_2| \geq b(H)\big(|\hat{B}| + |L|\big)$.

(iv) For the sets N_1, N_2 we show that

- **(a)** $N_1, N_2 \subseteq N_{G\times H}(U)$,
- **(b)** $N_1 \cap N_2 = \emptyset$,
- **(c)** $|N_{G\times H}(U)| \geq |N_1| + |N_2|$.

(v) Finally, we prove that

$$|N_{G\times H}(U)| \geq b(G)|U|.$$

Now we prove the statements above.

(i) It is clear that $\hat{A}$, $\hat{B}$ and C are pairwise disjoint. In addition, there is no $(x, y) \in U$ for which $\exists (x', y), (x, y') \in U$ such that $\{x, x'\} \in E(G)$ and $\{y, y'\} \in E(H)$, because this would imply that $\{(x', y), (x, y')\} \in E(G \times H)$, but U is an independent set. Hence U is partitioned into $U = \hat{A} \uplus \hat{B} \uplus C$. (The connection with the partition of Zhu described in (2.1) is clearly the following, $A = \hat{A} \uplus C$ and $B = \hat{B}$.)

(ii) We consider the elements of $\hat{A}$ and C for every $y \in V(H)$.
(ii/a) By definition $(\hat{A} \cup C)^G(y)$ is independent in G.
(ii/b) From (ii/a) and by the definition of $b(G)$ it follows that $|N_G((\hat{A} \cup C)^G(y))| \geq b(G)|(\hat{A} \cup C)^G(y)|$. Considering the sum for all $y \in V(H)$ we have $|M_G(\hat{A} \cup C)| \geq b(G)\big(|\hat{A}| + |C|\big)$. By the definition of N_1 and L this yields $|N_1| \geq b(G)\big(|\hat{A}| + |C|\big) - |L|$.

(iii) We consider the elements of $\hat{B}$ and L for every $x \in V(G)$.
(iii/a) By the definition of $\hat{A}$ and C, the sets $\hat{B}^H(x)$ and $L^H(x)$ are disjoint. Indeed, if $(x, y) \in M \subseteq M_G(\hat{A} \cup C)$ then $\exists (x', y) \in \hat{A} \cup C, \{x, x'\} \in E(G)$ and so (x, y) cannot be in $\hat{B} \subseteq U$.

(iii/b) We claim that $(\hat{B} \cup L)^H(x)$ is independent in H for every $x \in V(G)$. Clearly, $\hat{B}^H(x)$ is independent by definition. Furthermore, if $y, y' \in L^H(x), \{y, y'\} \in E(H)$ then from $(x, y) \in L$ we get that $\exists (x', y) \in \hat{A} \cup C, \{x, x'\} \in E(G)$, hence $(x, y') \in L$ is a neighbour of $(x', y) \in U$ which contradicts to $L \cap N_{G \times H}(U) = \emptyset$. Similarly if $y \in \hat{B}^H(x), y' \in M^H(x), \{y, y'\} \in E(H)$ then from $(x, y) \in \hat{B}$ it follows that $\exists (x', y) \in U, \{x, x'\} \in E(G)$, but again, as $(x, y') \in L$ is a neighbour of $(x', y) \in U$ it is in contradiction with the definition of L. (Figure 7 illustrates the steps of the argument of this part.)

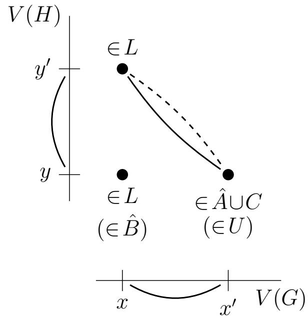

Figure 7: $(\hat{B} \cup L)^H(x)$ is independent.

(iii/c) From (iii/b) and by the definition of $b(H)$ it follows that $|N_H((\hat{B} \cup M)^H(x))| \geq b(H)|(\hat{B} \cup M)^H(x)|$. Considering the sum for all $x \in V(G)$ we get that $|M_H(\hat{B} \cup L)| \geq b(H)|\hat{B} \cup L|$. By the definition of N_2 and the statement (iii/a) we obtain $|N_2| \geq b(H)\big(|\hat{B}| + |L|\big)$.

(iv) Next, we investigate the sets N_1, N_2.

(iv/a) We have $N_1 \subseteq N_{G\times H}(U)$, by definition. We claim that $N_2 \subseteq N_{G\times H}(U)$. On the one hand, $M_H(\hat{B}) \subseteq N_{G\times H}(U)$. Indeed, if $y \in \hat{B}^H(x)$ and y' is a neighbour of y in H, and so $(x,y') \in M_H(\hat{B})$ then by the definition of $\hat{B}$, $\exists(x',y) \in U, \{x,x'\} \in E(G)$. Hence (x,y') is a neighbour of $(x',y) \in U$, that is, $(x,y') \in N_{G\times H}(U)$. On the other hand, if $y \in L^H(x)$ and y' is a neighbour of y in H, and so $(x,y') \in M_H(L)$ then by the definition of L, $\exists(x',y) \in \hat{A} \cup C, \{x,x'\} \in E(G)$, therefore $\{(x',y),(x,y')\} \in E(G\times H)$, thus $(x,y') \in N_{G\times H}(U)$. This yields $M_H(L) \subseteq N_{G\times H}(U)$. (See Figure 8.)

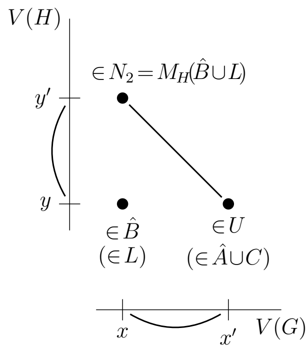

Figure 8: $N_2 \subseteq N_{G\times H}(U)$.

(iv/b) We claim that $N_1 \cap N_2 = \emptyset$. Suppose indirectly, that $\exists(x,y) \in N_1 \cap N_2$. Then $(x,y) \in N_1$ implies that $\exists(x',y) \in \hat{A} \cup C, \{x,x'\} \in E(G)$. While from $(x,y) \in N_2$ we get that $\exists(x,y') \in \hat{B}$ or $\exists(x,y') \in L$, $\{y,y'\} \in E(H)$. It is a contradiction since (x',y) and (x,y') are adjacent in $G \times H$, but no edge can go between $\hat{A} \cup C$ and $\hat{B} \cup L$ by the independence of U and the definition of L.
(iv/c) The statements (iv/a), (iv/b) give $|N_{G\times H}(U)| \geq |N_1| + |N_2|$.

(v) From (ii/b), (iii/c) and (iv/c) we get that

$$|N_{G\times H}(U)| \geq |N_1| + |N_2| \geq \Big(b(G)\big(|\hat{A}| + |C|\big) - |L|\Big) + \Big(b(H)\big(|\hat{B}| + |L|\big)\Big). \tag{2.5}$$

From the assumption $a(G) \geq a(H)$ it follows $b(G) \leq b(H)$. We also have $a(H) \leq \frac{1}{2}$, that is $b(H) \geq 1$. Thus we obtain

$$\Big(b(G)\big(|\hat{A}| + |C|\big) - |L|\Big) + \Big(b(H)\big(|\hat{B}| + |L|\big)\Big) \geq$$
$$\geq b(G)\Big(|\hat{A}| + |\hat{B}| + |C|\Big) + \big(b(H) - 1\big)|L| \geq b(G)|U|. \tag{2.6}$$

Combining the inequalities (2.5) and (2.6) we conclude $|N_{G\times H}(U)| \geq b(G)|U|$.

Consequently, for every independent set U of $G \times H$ we showed that

$$\frac{|U|}{|U| + |N_{G\times H}(U)|} \leq a(G),$$

and the proof is complete. $\square$

We mentioned in the introduction part of this chapter that the two forms of Question 1 are equivalent. Hence from the equality $a^*(G^{\times 2}) = a^*(G)$ for every graph G we obtain the following corollary. (Indeed, suppose on the contrary that G is a graph with $a^*(G) < A(G)$ then $\exists k$ such that $a^*(G) < i(G^{\times k}) \leq a^*(G^{\times k})$, and as the sequence $\{a^*(G^{\times \ell})\}_{\ell=1}^{\infty}$ is monotone increasing, it follows that $\exists m$ for which $a^*(G^{\times m}) < a^*((G^{\times m})^{\times 2})$, giving a contradiction.)

Corollary 8. *For every graph G we have $A(G) = a^*(G)$, that is*

$$A(G)(=\lim_{k\to\infty} i(G^{\times k})) = \begin{cases} a(G)\big(=\max_{U \text{ is independent in } G} \frac{|U|}{|U|+|N_G(U)|}\big), & \text{if } a(G) \leq \frac{1}{2}, \\ 1, & \text{otherwise.} \end{cases}$$

□

2.5 Further consequences

Brown, Nowakowski and Rall in [15] asked whether $A(G \uplus H) = \max\{A(G), A(H)\}$, where $G \uplus H$ is the disjoint union of G and H. This equality immediately follows from Corollary 8 since the analogous statement, $a^*(G \uplus H) = \max\{a^*(G), a^*(H)\}$ is straightforward. In [11] it is shown that $A(G \uplus H) = A(G \times H)$, therefore we get the following result.

Corollary 9. *For every two graphs G and H we have*

$$A(G \uplus H) = A(G \times H) = \max\{A(G), A(H)\}.$$

□

The authors of [15] also addressed the question whether $A(G)$ is computable, and if so what is its complexity. They showed that in the case when G is bipartite then $A(G) = \frac{1}{2}$ if G has a perfect matching, and $A(G) = 1$ otherwise. Hence for bipartite graphs $A(G)$ can be determined in polynomial time. Furthermore, it is proven in [11] that $a(G) \leq \frac{1}{2}$ if and only if G contains a fractional perfect matching. Therefore given an input graph G, determining whether $A(G) = 1$ or $A(G) \leq \frac{1}{2}$ can be done in polynomial time. From Corollary 8 we can conclude that the problem of deciding whether $A(G) > t$ for a given graph G and a given value t, is in NP. Moreover it is not hard to prove that it is in fact NP-complete. (The maximum independent set problem has a Karp-reduction to this problem, by adding sufficiently many vertices to the graph which are connected to each other and every other vertex of the graph, and choosing t appropriately.)

Any rational number in $(0, \frac{1}{2}] \cup \{1\}$ is the ultimate categorical independence ratio for some graph G, as it is shown in [15]. Here we remark that we obtained that $A(G)$ cannot be irrational, solving another problem mentioned in [15].

As a consequence of Corollary 8 we also have the following characterization of self-universal graphs. We call a graph empty if it has no edge. For every other graph G it holds that $i(G) < 1$.

Corollary 10. *A non-empty graph G is self-universal if and only if $a(G) = i(G)$ and $i(G) \leq \frac{1}{2}$.* $\square$

In other words, a nonempty graph G is self-universal iff the expression $\frac{|U|}{|U|+|N_G(U)|}$ reach its maximum (also) for maximum-sized independent sets among all independent sets of G and this maximum is at most $\frac{1}{2}$. Clearly, from this result Theorem 3 of Section 2.2 also follows.

Chapter 3

The asymptotic value of the Hall-ratio for categorical and lexicographic power

The *Hall-ratio* of a graph G was investigated in [20, 21] where it is defined as

$$\rho(G) = \max\left\{\frac{|V(H)|}{\alpha(H)} : H \subseteq G\right\},$$

that is, as the ratio of the number of vertices and the independence number maximized over all subgraphs of G. (See also [23] and some of the references therein for an earlier appearance of the same notion on a different name.) The asymptotic values of the Hall-ratio for different graph powers were investigated by Simonyi [49]. He considered the (appropriately normalized) asymptotic values of the Hall-ratio for the exponentiations called normal, co-normal, lexicographic and categorical, respectively.

All the above four graph powers of the graph G are defined on the k-length sequences over $V(G)$. In the normal power $G^{\odot k}$ two sequences are adjacent iff their elements at every coordinate are either equal or form an edge in G. In the co-normal power G^k two such sequences are connected iff there is some coordinate where the corresponding elements of the two sequences form an edge of G. The asymptotic value of the Hall-ratio with respect to the co-normal power is defined as $h(G) = \lim_{k\to\infty} \sqrt[k]{\rho(G^k)}$, the analogous asymptotic value for the normal power is denoted by $h_{\odot}(G)$. Simonyi [49] proved that $h(G) = \chi_f(G)$, where $\chi_f(G)$ is the fractional chromatic number of graph G, while $h_{\odot}(G) = R(G)$, where $R(G)$ denotes the so-called Witsenhausen rate. The latter is the normalized asymptotic value of the chromatic number with respect to the normal power and is introduced by Witsenhausen in [52] where its information theoretic relevance is also explained. The *fractional chromatic number* is the well-known graph invariant one obtains from the fractional relaxation of the integer program defining the chromatic number.

That is,

$$\chi_f(G) = \inf\Big\{ \sum_{U \in S(G)} f(U) \,:\, f \text{ is a fractional colouring of } G \Big\}, \text{ where}$$

$$f \text{ is a fractional colouring of } G \text{ if } f : S(G) \to [0,1] \text{ and}$$

$$\forall v \in V(G) \,:\, \sum_{v \in U \in S(G)} f(U) \geq 1,$$

$$S(G) \text{ denotes the set of the independent sets of } G.$$

In the lexicographic power $G^{\circ k}$ two sequences of the original vertices are adjacent iff they are adjacent in the first coordinate where they differ. The ultimate lexicographic Hall-ratio of graph G is $h_\circ(G) = \lim_{k\to\infty} \sqrt[k]{\rho(G^{\circ k})}$. In the categorical power $G^{\times k}$ two sequences of the original vertices are connected iff their elements form an edge in G at every coordinate. The ultimate categorical Hall-ratio of graph G is $h_\times(G) = \lim_{k\to\infty} \rho(G^{\times k})$. (Note, that we do not need any normalization here.) Simonyi [49] conjectured that also for the lexicographic power and for the categorical power we get the fractional chromatic number as the asymptotic value. In this chapter we prove both of his conjectures. In Section 3.1 and 3.2 the ultimate lexicographic Hall-ratio, in Section 3.3, 3.4 and 3.5 the ultimate categorical Hall-ratio is discussed.

In the proofs we will also need the fractional relaxation of the clique number. A function $g : V(G) \to [0,1]$ for which $\forall U \in S(G) : \sum_{v \in U} g(v) \leq 1$ is a *fractional clique* of G with value $z(g) = \sum_{v \in V(G)} g(v)$. The *fractional clique number* of G is $\omega_f(G) = \sup\{z(g) :$ g is a fractional clique of G with value $z(g)\}$. A fractional clique of G is called optimal if its value is $\omega_f(G)$. (Figure 9 illustrates a fractional colouring and a fractional clique of a graph.) The duality theorem of linear programming implies that $\chi_f(G) = \omega_f(G)$ for every graph G. See [48] for more details.

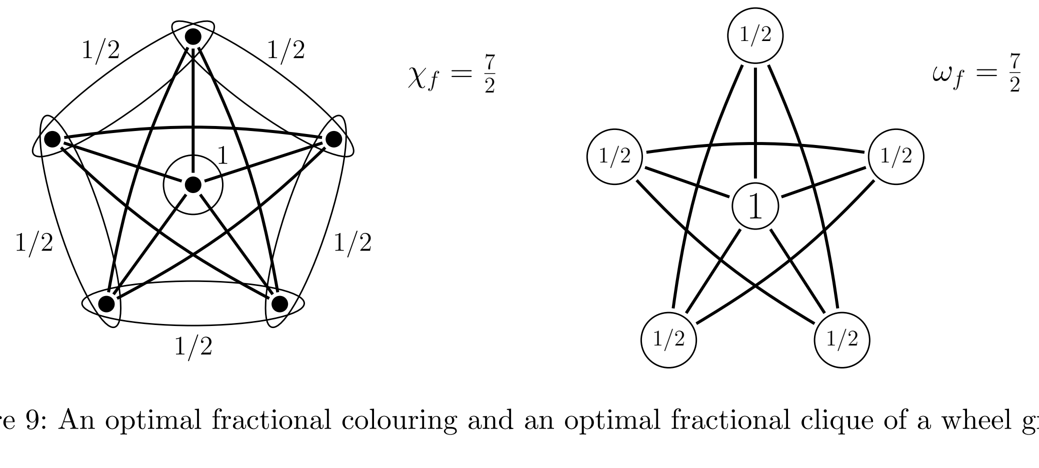

Figure 9: An optimal fractional colouring and an optimal fractional clique of a wheel graph.

3.1 The ultimate lexicographic Hall-ratio

For two graphs F and G, their *lexicographic product* $F \circ G$ is defined on the vertex set $V(F \circ G) = V(F) \times V(G)$ with edge set $E(F \circ G) = \{\{(u_1, v_1), (u_2, v_2)\} : \{u_1, u_2\} \in E(F)$, or $u_1 = u_2$ and $\{v_1, v_2\} \in E(G)\}$. The lexicographic product $F \circ G$ is also known as the substitution of G into all vertices of F, the name we use follows the book [39]. The nth lexicographic power $G^{\circ n}$ is the n-fold lexicographic product of G. That is, the lexicographic power is defined on the vertex sequences of the original graph and we connect two such sequences iff they are adjacent in the first coordinate where they differ. (See Figure 10.)

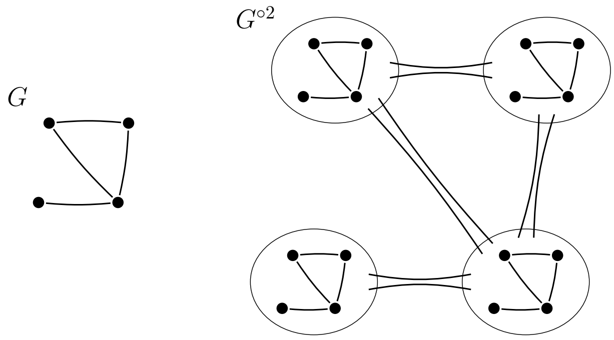

Figure 10: The lexicographic square of a graph. (Double lines mean that the corresponding vertex classes are totally connected.)

Definition ([49])**.** The *ultimate lexicographic Hall-ratio* of graph G is

$$h_\circ(G) = \lim_{n\to\infty} \sqrt[n]{\rho(G^{\circ n})}.$$

The normal and co-normal products of two graphs F and G are also defined on $V(F) \times V(G)$ as vertex sets and their edge sets are such that $E(F \odot G) \subseteq E(F \circ G) \subseteq E(F \cdot G)$ holds, where $F \odot G$ denotes the normal, $F \cdot G$ the co-normal product of F and G. (In particular, $\{(u_1, v_1), (u_2, v_2)\} \in E(F \odot G)$ if $\{u_1, u_2\} \in E(F)$ and $\{v_1, v_2\} \in E(G)$, or $\{u_1, u_2\} \in E(F)$ and $v_1 = v_2$, or $u_1 = u_2$ and $\{v_1, v_2\} \in E(G)$, while $\{(u_1, v_1), (u_2, v_2)\} \in E(F \cdot G)$ if $\{u_1, u_2\} \in E(F)$ or $\{v_1, v_2\} \in E(G)$.)
As we have seen at the beginning of this chapter, denoting by $h_\odot(G)$ and $h(G)$ the normalized asymptotic values analogous to $h_\circ(G)$ for the normal and co-normal power, respectively, Simonyi [49] proved that $h(G) = \chi_f(G)$, where $\chi_f(G)$ is the fractional chromatic number of graph G, while $h_\odot(G) = R(G)$, where $R(G)$ denotes the Witsenhausen rate.

It follows from the above discussion that the value of $h_\circ(G)$ falls into the interval $[R(G), \chi_f(G)]$. We remark that the lower bound $R(G)$ is sometimes better but sometimes worse than the easy lower bound $\rho(G)$, cf. [49]. Thus we know that

$$\max\{\rho(G), R(G)\} \leq h_\circ(G) \leq \chi_f(G).$$

For some types of graphs the upper and lower bounds are equal, so this formula gives the exact value of the ultimate lexicographic Hall-ratio. For instance, if G is a perfect graph, then $\chi_f(G) = \chi(G) = \omega(G) \leq \rho(G)$. If G is a vertex-transitive graph, then $\chi_f(G) = \frac{|V(G)|}{\alpha(G)} \leq \rho(G)$. (The proof of the fact that $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$ holds for vertex-transitive graphs, can be found for example in [48].)

The length of the interval $[\max\{\rho(G), R(G)\}, \chi_f(G)]$ is positive in general. An example is the 5-wheel, W_5 consisting of a 5-length cycle and an additional point joint to every vertex of the cycle. It is clear that $\rho(W_5) = 3$. To get an upper bound for $R(W_5)$, one can find a colouring of $C_5^{\odot 2}$ with 5 colours (see [52]) which can be completed to a colouring of $W_5^{\odot 2}$ with 12 colours, so $\chi(W_5^{\odot 2}) \leq 12$. Since $\chi(G^{\odot n}) \leq (\chi(G))^n$ (see, e.g., [39] for the easy proof) and by the definition of $R(G)$ we get $R(W_5) \leq \sqrt{12}$. Furthermore, $\chi_f(W_5) = \chi_f(C_5) + 1 = \frac{7}{2} > \max\{3, \sqrt{12}\}$.

It was conjectured in [49], that in fact, $h_\circ(G)$ always coincides with the larger end of the above interval. The goal of this part is to prove this conjecture.

Theorem 11. *The ultimate lexicographic Hall-ratio equals to the fractional chromatic number for every graph G, that is*

$$h_\circ(G) = \chi_f(G).$$

3.2 Proof of the result in Section 3.1

We know $h_\circ(G) \leq \chi_f(G)$ thus it is enough to prove the reverse inequality.

3.2.1 Definition of $p_G(k, \alpha)$ and $q_G(k, \alpha)$, formula for $h_\circ(G)$ in terms of $q_G(k, \alpha)$

Preparing for the proof we introduce some notations. Let k be a positive integer and let α be a positive real number. Denote by $p_G(k, \alpha)$ the number of vertices maximized over all subgraphs of $G^{\circ k}$ with independence number at most α, that is

$$p_G(k, \alpha) = \max\left\{|V(H)| \,:\, H \subseteq G^{\circ k}, \alpha(H) \leq \alpha\right\}$$

and let

$$q_G(k, \alpha) = \frac{p_G(k, \alpha)}{\alpha}.$$

Clearly, $p_G(k, \alpha) = p_G(k, \lfloor \alpha \rfloor)$ and $q_G(k, \alpha) \leq q_G(k, \lfloor \alpha \rfloor)$. In spite of this fact it will be useful that $p_G(k, \alpha)$ is defined also for non-integral α values.

Now we are going to prove some technical lemmas.

Lemma 12. *The ultimate lexicographic Hall-ratio can be expressed by the values of $q_G(k,\alpha)$ as follows.*

$$h_\circ(G) = \lim_{k\to\infty} \max\left\{ \sqrt[k]{q_G(k,\alpha)} \,:\, \alpha \in \mathbb{R}_+ \right\} \tag{3.1}$$

Proof. The Hall-ratio of the kth lexicographic power of G can be calculated by the above terms the following simple way:

$$\rho(G^{\circ k}) = \sup\{q_G(k,\alpha) \,:\, \alpha \in \mathbb{R}_+\}.$$

Since $p_G(k,\alpha)$ is a bounded, monotone increasing function and $q_G(k,\alpha)$ is the ratio of this and the strictly monotone increasing identity function, the above supremum is always reached. Since $q_G(k,\alpha) \le q_G(k,\lfloor\alpha\rfloor)$, it is reached at some integer value of α, so the maximum value belongs to one of the subgraphs of $G^{\circ k}$.

Thus we get $h_\circ(G) = \lim\limits_{k\to\infty} \sqrt[k]{\rho(G^{\circ k})} = \lim\limits_{k\to\infty} \max\left\{ \sqrt[k]{q_G(k,\alpha)} \,:\, \alpha \in \mathbb{R}_+ \right\}$. □

Thus our aim is to show that $\lim\limits_{k\to\infty} \max\left\{ \sqrt[k]{q_G(k,\alpha)} \,:\, \alpha \in \mathbb{R}_+ \right\} \ge \chi_f(G)$.

3.2.2 Recursive lower bound for $q_G(k,\alpha)$ from an optimal fractional clique

Let $g : V(G) \to \mathbb{R}_{+,0}$ be an optimal fractional clique of G. That is, (denoting the set of independent sets in G by $S(G)$) it is a fractional clique:

$$\forall U \in S(G) : \sum_{v\in U} g(v) \le 1, \tag{3.2}$$

and it is optimal:

$$\sum_{v\in V(G)} g(v) = \chi_f(G). \tag{3.3}$$

(See Figure 11.)

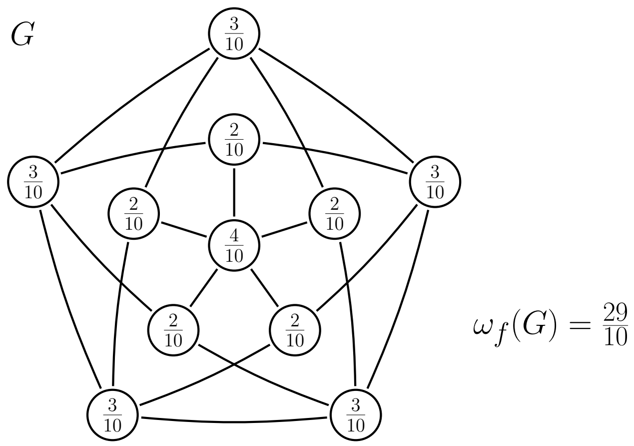

Figure 11: An optimal fractional clique of a graph.

We may assume that $g(v) \neq 0$ for any $v \in V(G)$. (Otherwise we can consider the subgraph G' of G induced by those vertices v of G for which $g(v) \neq 0$. As $\omega_f(G') = \omega_f(G)$ and $h_\circ(G') \leq h_\circ(G)$, if we show $h_\circ(G') \geq \chi_f(G')$ then $h_\circ(G) \geq \chi_f(G)$ also follows.)

Lemma 13.

$$q_G(k,\alpha) \geq \sum_{v \in V(G)} g(v) q_G\left(k-1, g(v)\alpha\right)$$

Proof. Every subgraph of $G^{\circ k}$ can be imagined as if the vertices of G would be substituted by subgraphs of $G^{\circ(k-1)}$. Furthermore, every independent set of $G^{\circ k}$ can be thought of as having the vertices of an independent set of G substituted by independent sets of (the above subgraphs of) $G^{\circ(k-1)}$.

If we substitute every vertex v of G by a subgraph of $G^{\circ(k-1)}$ with independence number at most $g(v)\alpha$, then we get a subgraph of $G^{\circ k}$ with independence number at most $\max\limits_{U \in S(G)} \sum\limits_{v \in U} g(v)\alpha \leq \alpha \cdot \max\limits_{U \in S(G)} \sum\limits_{v \in U} g(v) \leq \alpha$, because of (3.2). (See Figure 12.)

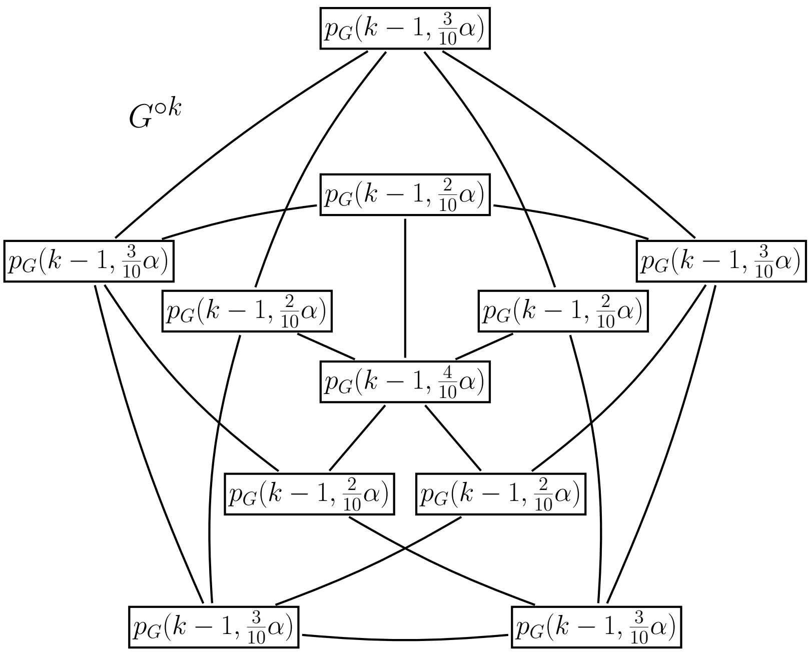

Figure 12: Number of vertices of the subgraphs of $G^{\circ(k-1)}$ substituted in the vertices of G.

Thus we get

$$p_G(k,\alpha) \geq \sum_{v \in G} p_G\left(k-1, g(v)\alpha\right).$$

It follows from this inequality and the definition of $q_G(k,\alpha)$ that

$$q_G(k,\alpha) = \frac{p_G(k,\alpha)}{\alpha} \geq \frac{1}{\alpha}\sum_{v\in G} p_G\left(k-1, g(v)\alpha\right) =$$
$$= \sum_{v\in G} \frac{g(v)\alpha}{\alpha}\frac{p_G\left(k-1, g(v)\alpha\right)}{g(v)\alpha} = \sum_{v\in V(G)} g(v) q_G\left(k-1, g(v)\alpha\right).$$

□

3.2.3 Further transformations, definition of $r_G(k,\alpha)$ and $s_G(k,\alpha)$

Next we bound the $q_G(k,\alpha)$ function from below, it will be important for later calculations. Let us define function $r_G(k,\alpha)$ as follows.

$$r_G(1,\alpha) = \begin{cases} c_G, & \text{if } 1 \leq \alpha \leq m = |V(G)| \\ 0, & \text{otherwise} \end{cases}$$

where c_G is a positive constant, which bounds $q_G(1,\alpha)$ from below for all $1 \leq \alpha \leq m = |V(G)|$. Such c_G exists, for example $c_G = \frac{1}{m}$ is a good choice.
For $k \geq 2$ let

$$r_G(k,\alpha) = \sum_{v\in V(G)} g(v) r_G\left(k-1, g(v)\alpha\right).$$

By Lemma 13 and by the construction of $r_G(k,\alpha)$ it holds for all positive integer k and all positive real number α that

$$r_G(k,\alpha) \leq q_G(k,\alpha). \tag{3.4}$$

Thus it is enough to show that $\limsup_{k\to\infty} \max\left\{ \sqrt[k]{r_G(k,\alpha)} \,:\, \alpha \in \mathbb{R}_+ \right\} \geq \chi_f(G)$.

To make the calculations simpler, we express α as m^β, that is $\beta = \log_m \alpha$ and introduce

$$s_G(k,\beta) = r_G(k, m^\beta),$$

where k is a positive integer, β is a real number. Since this transformation does not change the maximum value of the function (only its place), it holds that

$$\max\left\{ \sqrt[k]{r_G(k,\alpha)} \,:\, \alpha \in \mathbb{R}_+ \right\} = \max\left\{ \sqrt[k]{s_G(k,\beta)} \,:\, \beta \in \mathbb{R} \right\}. \tag{3.5}$$

Thus it is enough to prove that $\limsup_{k\to\infty} \max\left\{ \sqrt[k]{s_G(k,\beta)} \,:\, \beta \in \mathbb{R} \right\} \geq \chi_f(G)$.

3.2.4 Recursive formula for $s_G(k, \alpha)$ and the desired lower bound

Observe that the following equalities hold.

$$s_G(1, \beta) = \begin{cases} c_G, & \text{if } 0 \le \beta \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$s_G(k, \beta) = \sum_{v \in V(G)} g(v) s_G\left(k-1, \log_m g(v) + \beta\right), \quad k \ge 2.$$

We get the formula for $s_G(1, \beta)$ from the definition of the function $s_G(k, \beta)$. The second equality follows by writing

$$\begin{aligned} s_G(k, \beta) = r_G(k, m^\beta) &= \sum_{v \in V(G)} g(v) r_G\left(k-1, g(v) m^\beta\right) = \\ &= \sum_{v \in V(G)} g(v) s_G\left(k-1, \log_m(g(v) m^\beta)\right) = \sum_{v \in V(G)} g(v) s_G\left(k-1, \log_m g(v) + \beta\right). \end{aligned}$$

Lemma 14. *It holds for all graphs G that*

$$\limsup_{k \to \infty} \max\left\{ \sqrt[k]{s_G(k, \beta)} \ : \ \beta \in \mathbb{R} \right\} \ge \chi_f(G). \tag{3.6}$$

Proof. Let us determine the integral of the function $s_G(k, \beta)$.

$$\int_{\beta=-\infty}^{\infty} s_G(1, \beta)\, \mathrm{d}\beta = c_G$$

$$\begin{aligned} \int_{\beta=-\infty}^{\infty} s_G(k, \beta)\, \mathrm{d}\beta &= \int_{\beta=-\infty}^{\infty} \sum_{v \in V(G)} g(v) s_G\left(k-1, \log_m g(v) + \beta\right)\, \mathrm{d}\beta = \\ &= \sum_{v \in V(G)} \left(g(v) \int_{\beta=-\infty}^{\infty} s_G(k-1, \log_m g(v) + \beta)\, \mathrm{d}\beta \right) = \\ &= \sum_{v \in V(G)} \left(g(v) \int_{\beta=-\infty}^{\infty} s_G(k-1, \beta)\, \mathrm{d}\beta \right) = \\ &= \left(\sum_{v \in V(G)} g(v) \right) \int_{\beta=-\infty}^{\infty} s_G(k-1, \beta)\, \mathrm{d}\beta = \\ &= \chi_f(G) \int_{\beta=-\infty}^{\infty} s_G(k-1, \beta)\, \mathrm{d}\beta, \qquad k \ge 2, \end{aligned}$$

where in the last equation we used (3.3). Hence,

$$\int\limits_{\beta=-\infty}^{\infty} s_G(k,\beta)\,\mathrm{d}\beta = c_G(\chi_f(G))^{k-1}.$$

For a function $f(x)$ we call the support of $f(x)$, denoted by $T(f(x))$, the set of reals x for which $f(x) \neq 0$. Let us determine $T(s_G(k,\beta))$.
$T(s_G(1,\beta)) = [0,1]$. Let g_G be any real value satisfying $g_G \leq \log_m g(v) \leq 0$ for all $v \in V(G)$. Such g_G exists, for example $g_G = \min\{\log_m g(v) : v \in V(G)\}$ is a good choice. Thus $T(s_G(k,\beta)) \subseteq [0, 1-(k-1)g_G]$.

It is clear from the above discussion that $\int_{\beta=-\infty}^{\infty} s_G(k,\beta)\,\mathrm{d}\beta$ asymptotically equals to $(\chi_f(G))^k$, i.e., the limit of their ratio equals 1 as k goes to infinity. The length of the support of $s_G(k,\beta)$ can be bounded from above by a linear function of k, let this function be $d_G k$ where d_G is a constant. These facts imply that $\limsup\limits_{k\to\infty} \max\left\{\sqrt[k]{s_G(k,\beta)} : \beta \in \mathbb{R}\right\} \geq \chi_f(G)$. Suppose indirectly that there is an $\varepsilon > 0$ and $N \in \mathbb{N}_+$, for which $\forall k > N$, $\forall \beta \in \mathbb{R}$: $s_G(k,\beta) < (\chi_f(G)-\varepsilon)^k$, then $\int_{\beta=-\infty}^{\infty} s_G(k,\beta)\,\mathrm{d}\beta < d_G k(\chi_f(G)-\varepsilon)^k$. Since $\lim\limits_{k\to\infty} \frac{d_G k(\chi_f(G)-\varepsilon)^k}{\chi_f(G)^k} = \lim\limits_{k\to\infty}(1-\frac{\varepsilon}{\chi_f(G)})^k = 0$, it is in contradiction with the statement at the beginning of this paragraph. □

3.2.5 Summary of the proof

By now we have essentially proved Theorem 11, it needs only to be summarized.

Proof of Theorem 11. The preceding lemmas imply that

$$h_\circ(G) = \lim_{k\to\infty} \max\left\{\sqrt[k]{q_G(k,\alpha)} : \alpha \in \mathbb{R}_+\right\} \geq \limsup_{k\to\infty} \max\left\{\sqrt[k]{r_G(k,\alpha)} : \alpha \in \mathbb{R}_+\right\} =$$
$$= \limsup_{k\to\infty} \max\left\{\sqrt[k]{s_G(k,\beta)} : \beta \in \mathbb{R}\right\} \geq \chi_f(G),$$

where the stated relations follow from (3.1), (3.4), (3.5) and (3.6), respectively.

Thus we have proved

$$h_\circ(G) = \chi_f(G).$$

□

3.2.6 Remarks

Concerning Theorem 11 we have the following remarks.

Remark 1. There are graphs for which the sequence $\left\{\sqrt[k]{\rho(G^{\circ k})}\right\}_{k=1}^{\infty}$ does not reach its limit $\chi_f(G)$ for any finite k. The 5-wheel is an example for which no t attains $\sqrt[t]{\rho(W_5^{\circ t})} = \chi_f(W_5) = \frac{7}{2}$.

This is because if there was such a t then there must be a subgraph H of $W_5^{\circ t}$ for which $\frac{|V(H)|}{\alpha(H)} = \left(\frac{7}{2}\right)^t = \frac{7^t}{2^t}$, but this fraction is irreducible and $|V(H)| \leq |V(W_5^{\circ t})| = 6^t$. $\Diamond$

Remark 2. It is known from the theorem of McEliece and Posner [46] (cf. also in [48]) that the normalized asymptotic value of the chromatic number with respect to the co-normal product is the fractional chromatic number. This theorem with the result proven here implies that the normalized asymptotic value of each of the Hall-ratio, the fractional chromatic number and the chromatic number with respect to both the co-normal and the lexicographic power equals to the fractional chromatic number. This is because $\rho(G) \leq \chi_f(G) \leq \chi(G)$ holds for every graph G and the lexicographic power of a graph is a subgraph of its co-normal power. These relations were already known except for the asymptotic value of the Hall-ratio for the lexicographic power. As we mentioned, it is proven in [49] that the normalized asymptotic value of the Hall-ratio for the co-normal power equals to the fractional chromatic number. The multiplicativity of the fractional chromatic number for the lexicographic product is a theorem in [39]. $\Diamond$

3.3 The ultimate categorical Hall-ratio

Recall that, for two graphs F and G, their *categorical product* (also called direct product) $F \times G$ is defined on the vertex set $V(F \times G) = V(F) \times V(G)$ with edge set $E(F \times G) = \{\{(u_1, v_1), (u_2, v_2)\} : \{u_1, u_2\} \in E(F) \text{ and } \{v_1, v_2\} \in E(G)\}$. The kth categorical power $G^{\times k}$ is the k-fold categorical product of G. That is, the categorical power is defined on the vertex sequences of the original graph and we connect two such sequences iff they are adjacent in every coordinate. (See Figure 13.)

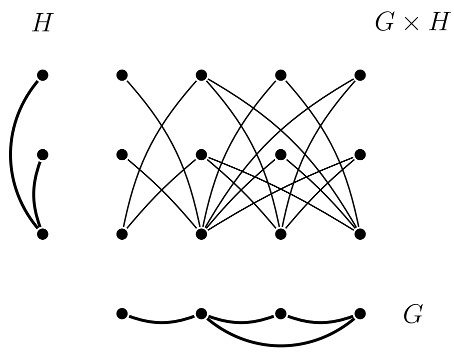

Figure 13: The categorical product of two graphs.

Definition. ([49]) The *ultimate categorical Hall-ratio* of graph G is

$$h_\times(G) = \lim_{k\to\infty} \rho(G^{\times k}).$$

Note that in this case we do not need any normalization on the sequence. It is shown in [49] that this graph parameter is bounded from above by the fractional chromatic number and conjectured that equality holds for all graphs. This conjecture can be shown easily for perfect and for vertex-transitive graphs. It is proven in [49] that it is also true for wheel graphs constructed from a cycle and an additional point joint to every vertex of the cycle. Using a similar argument which was used in the proof of that result the following generalization was also proven by the author in [3]. Let G be a graph for which $h_\times(G) = \chi_f(G)$ holds and let $\hat{G}$ be the graph we obtain from G by connecting each of its vertices to an additional vertex, then $h_\times(\hat{G}) = \chi_f(\hat{G})$ holds, too. Here we prove the above conjecture in general.

Theorem 15. *The ultimate categorical Hall-ratio equals to the fractional chromatic number for every graph G, that is*

$$h_\times(G) = \chi_f(G).$$

The proof uses a recent result of Zhu [54] that he proved on the way when proving the fractional version of Hedetniemi's conjecture, i.e., that $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$.

3.4 A result of Zhu: a nice fractional clique of the product graph

In this section we present the result of Zhu that we will use in the proof of Theorem 15. (In the proof of this lemma we apply Lemma 5 that we used in the previous chapter, in Section 2.3, on page 12.)

Let $g : V(G) \to [0,1]$ and $h : V(H) \to [0,1]$ be an optimal fractional clique of G and H, respectively. Define $f : V(G \times H) \to [0,1]$ as follows,

$$f((x,y)) = \frac{g(x)h(y)}{\max\{\omega_f(G), \omega_f(H)\}}. \tag{3.7}$$

Lemma 16 (Zhu [54])**.** *The function defined in (3.7) is a fractional clique of $G \times H$ with value $\min\{\omega_f(G), \omega_f(H)\}$.*

For the sake of completeness we give a proof for this result. Let $f : X \to \mathbb{R}$. For any $Y \subseteq X$ we use the notation $f(Y) = \sum_{y\in Y} f(y)$.

Proof of Lemma 16. Let f be the weight function on $V(G \times H)$ defined in (3.7). Firstly, we prove that the sum of the weights on the vertex set is $\min\{\omega_f(G), \omega_f(H)\}$. Indeed,

$$\begin{aligned} f(V(G \times H)) = \sum_{(x,y) \in V(G \times H)} f((x,y)) = \sum_{(x,y) \in V(G \times H)} \frac{g(x)h(y)}{\max\{\omega_f(G), \omega_f(H)\}} = \\ = \frac{\omega_f(G)\omega_f(H)}{\max\{\omega_f(G), \omega_f(H)\}} = \min\{\omega_f(G), \omega_f(H)\}. \end{aligned} \tag{3.8}$$

Secondly, we show that the function f is a fractional clique of $G \times H$. To this, we shall use the following claim.

Claim. Let $g : V(G) \to [0,1]$ be an optimal fractional clique of G. Then

$$g(U) \leq \frac{g(U \cup N_G(U))}{\omega_f(G)},$$

for any independent set U of G.

Proof. Assume on the contrary that there exists an independent set U of G such that $g(U \cup N_G(U)) < g(U)\omega_f(G)$. This means that $g(V(G) \setminus (U \cup N_G(U))) > \omega_f(G)(1 - g(U))$, using $g(V(G)) = \omega_f(G)$. Set $G' = G[V(G) \setminus (U \cup N_G(U))]$, and consider the values of g on the vertices of G'. From $g(V(G')) > \omega_f(G)(1 - g(U))$ and $\omega_f(G') \leq \omega_f(G)$ it follows that there is an independent set U' of G' with $g(U') > 1 - g(U)$. (Otherwise, assuming $g(U) \neq 1$, the function $g'(v) = \frac{g(v)}{1-g(U)}$ would be a fractional clique of G' with value greather than $\omega_f(G')$. If $g(U) = 1$ then $U' = \{v\}$ is a good choice for any vertex v of G' with $g(v) > 0$.) As $U \cup U'$ is an independent set of G, the inequality $g(U \cup U') > 1$ gives a contradiction, proving the Claim. □

Let U be an independent set of $G \times H$. We prove that $f(U) \leq 1$. Partition U according to (2.1) into $U = A \cup B$, recall that

$$\begin{aligned} A &= \{(x,y) \in U \,:\, \nexists (x', y) \in U \text{ s.t. } \{x, x'\} \in E(G)\}, \\ B &= \{(x,y) \in U \,:\, \exists (x', y) \in U \text{ s.t. } \{x, x'\} \in E(G)\}. \end{aligned}$$

We have

$$f(U) = f(A) + f(B). \tag{3.9}$$

Using the Claim and the statement (i) of Lemma 5 we obtain the following upper bound for $f(A)$.

(See the meaning of the notation of $A^G(y)$, $M_G(A)$ and $M_H(B)$ in Section 2.3.)

$$\begin{aligned} f(A) &= \sum_{y\in V(H), x\in A^G(y)} f(x,y) = \sum_{y\in V(H), x\in A^G(y)} \frac{1}{\max\{\omega_f(G),\omega_f(H)\}} h(y)g(x) = \\ &= \frac{1}{\max\{\omega_f(G),\omega_f(H)\}} \sum_{y\in V(H)} h(y)g(A^G(y)) \le \\ &\le \frac{1}{\max\{\omega_f(G),\omega_f(H)\}} \sum_{y\in V(H)} h(y)\frac{g(A^G(y)\cup N_G(A^G(y)))}{\omega_f(G)} = \\ &= \frac{1}{\omega_f(G)} f(A\cup M_G(A)), \end{aligned} \tag{3.10}$$

Similarly,

$$f(B) \le \frac{1}{\omega_f(H)} f(B\cup M_H(B)). \tag{3.11}$$

From (3.9), (3.10), (3.11) and the statement (ii) of Lemma 5 we get that

$$\begin{aligned} f(U) &\le \frac{1}{\omega_f(G)} f(A\cup M_G(A)) + \frac{1}{\omega_f(H)} f(B\cup M_H(B)) \le \\ &\le \frac{1}{\min\{\omega_f(G),\omega_f(H)\}} \left(f(A\cup M_G(A)) + f(B\cup M_H(B))\right) = \\ &= \frac{1}{\min\{\omega_f(G),\omega_f(H)\}} f(A\cup M_G(A)\cup B\cup M_H(B)) \le \\ &\le \frac{1}{\min\{\omega_f(G),\omega_f(H)\}} f(V(G\times H)) = 1, \end{aligned}$$

as we needed, using also (3.8). □

From this lemma, the fractional version of Hedetniemi's conjecture easily follows.

Theorem 17 (Zhu [54]). *For every two graphs G and H we have*

$$\chi_f(G\times H) = \min\{\chi_f(G),\chi_f(H)\}.$$

Proof. Clearly, we have $\chi_f(G\times H) \le \min\{\chi_f(G),\chi_f(H)\}$. The reverse inequality, in the form $\omega_f(G\times H) \ge \min\{\omega_f(G),\omega_f(H)\}$, follows from Lemma 16. □

3.5 Proof of the result in Section 3.3

It is shown in [49] that the sequence $\{\rho(G^{\times k})\}_{k=1}^{\infty}$ is monotone increasing (we get this from $G^{\times k} \subseteq G^{\times(k+1)}$) and is bounded from above by the fractional chromatic number (which is a consequence of the easy facts that $\rho(G) \le \chi_f(G)$ and $\chi_f(G^{\times k}) = \chi_f(G)$). Thus finding a finite k_0 for which $\rho(G^{\times k_0}) \geq \chi_f(G)$ proves that the limit of the sequence equals to the fractional chromatic number.

Let $g : V(G) \to \mathbb{R}_{+,0}$ be an optimal fractional clique of G. We may assume that the weights of the vertices are rationals, moreover $g(v) = \frac{s(v)}{r}$, where $s(v)$ for $\forall v \in V(G)$ and r are integrals. (See Figure 14.) Set $s = \sum_{v \in V(G)} s(v)$ and let F be the induced subgraph of $G^{\times s}$ on the vertices which have exactly $s(v)$ coordinates equal to v for every vertex v of G. F is vertex-transitive, that is its automorphism group acts transitively upon its vertices. We will show that $\chi_f(F) = \chi_f(G)$, and this will imply that $\rho(G^{\times s}) \geq \frac{|V(F)|}{\alpha(F)} = \chi_f(F) = \chi_f(G)$ using the well-known fact that for every vertex-transitive graph H we have $\chi_f(H) = \frac{|V(H)|}{\alpha(H)}$. This idea of the proof was suggested by Simonyi [49].

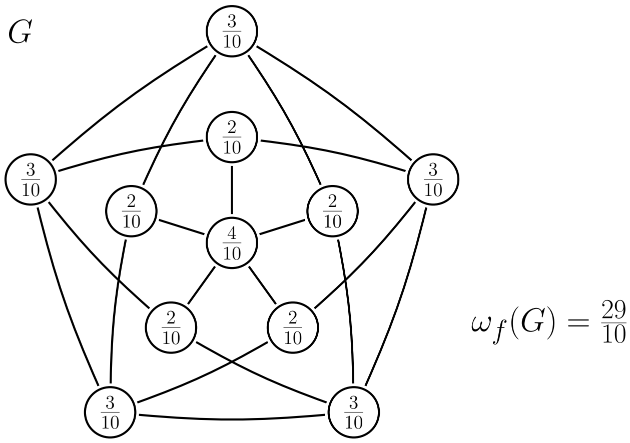

Figure 14: An optimal fractional clique of a graph G with value $\frac{29}{10}$.
The resulting graph F has $\frac{29!}{(2!)^5(3!)^5 4!}$ vertices.

Thus it remains to prove the following lemma. (Actually we only need that $\chi_f(F) \geq \chi_f(G)$. Nevertheless, the reverse inequality clearly holds since $\chi_f(F) > \chi_f(G)$ would imply $h_\times(G) > \chi_f(G)$ contradicting the upper bound of the monotone increasing sequence defining $h_\times(G)$.)

Lemma 18. *For F obtained from G as described above we have*

$$\chi_f(F) = \chi_f(G).$$

We split the proof into the following smaller parts. In the first two subsections we will define operations on fractional cliques. Then using them (aside from some technical details described in the third subsection) we will construct a fractional clique of F with value $\chi_f(G)$ in the last subsection. From this we will conclude that $\chi_f(F) = \omega_f(F) \geq \chi_f(G)$.

3.5.1 Equally distributed fractional cliques

We will use Lemma 16 in the following special case. Denote by $\mathcal{I}_X$ the indicator function on the set X, that is $\mathcal{I}_X(x) = 1$ if $x \in X$, $\mathcal{I}_X(x) = 0$ otherwise.

Lemma 19. *Assume that for $i = 1, 2$ the function $g_{H_i} = c_i \cdot \mathcal{I}_{U_i}$ is an optimal fractional clique of H_i with value z, where c_i is a constant number ($c_i = \frac{z}{|U_i|}$), U_i is a subset of $V(H_i)$. Then $g_{H_1 \times H_2} = c_{12} \cdot \mathcal{I}_{U_1 \times U_2}$ is a fractional clique of $H_1 \times H_2$ with value z for $c_{12} = \frac{c_1 c_2}{z} = \frac{z}{|U_1||U_2|}$.* $\square$

In other words this lemma states that if the value of the optimal fractional clique g_{H_1} and g_{H_2} is equally distributed on the vertices of U_1 in H_1 and U_2 in H_2, respectively, then distributing the same value to the vertices of $U_1 \times U_2$ in $H_1 \times H_2$ also with equal portions we get a(n optimal) fractional clique of $H_1 \times H_2$.

3.5.2 Further operations with fractional cliques

As a consequence of the fact that the graph H is a subgraph of $H^{\times m}$ we get a fractional clique of $H^{\times m}$ concentrated on its diagonal in the following way.

Lemma 20. *If $g_H(u) = c, \forall u \in V(H)$ is a fractional clique of H with value z for some constant $c(= \frac{z}{|V(H)|})$ then the function $g_{H^{\times m}}(u_1, u_2, \ldots, u_m) = c$ for $u_1 = u_2 = \ldots = u_m$, and $g_{H^{\times m}}(u_1, u_2, \ldots, u_m) = 0$ otherwise, is a fractional clique of $H^{\times m}$ also with value z.*

We need yet another operation.

Lemma 21. *If $g_H^{(1)} = c_1 \cdot \mathcal{I}_{U_1}$, $g_H^{(2)} = c_2 \cdot \mathcal{I}_{U_2}$ are fractional cliques of H with value $z(g_H^{(1)}) = z(g_H^{(2)}) = z$ and $U_1 \subsetneq U_2 \subseteq V(H)$ then $g_H^{(3)} = c_3 \cdot \mathcal{I}_{U_2 \setminus U_1}$, where $c_3 = \frac{c_1 c_2}{c_2 - c_1}$, is also a fractional clique of H with value z.*

Indeed, let $g_H^{(3)}$ be a linear combination of $g_H^{(1)}$ and $g_H^{(2)}$, such that $g_H^{(3)} = \alpha g_H^{(1)} + (1-\alpha) g_H^{(2)}$. Choosing $\alpha = \frac{c_2}{c_2 - c_1}$ it satisfies $\alpha c_1 + (1-\alpha)c_2 = 0$. (We know $c_1 \neq c_2$ from $|U_1| \neq |U_2|$.) Thus $g_H^{(3)}(u) = 0$ if $u \in U_1 \cup (V(H) \setminus U_2) = V(H) \setminus (U_2 \setminus U_1)$ and $g_H^{(3)}(u) = \frac{c_1 c_2}{c_2 - c_1} = c_3$ if $u \in U_2 \setminus U_1$. The value of $g_H^{(3)}$ is clearly $\alpha z(g_H^{(1)}) + (1-\alpha) z(g_H^{(2)}) = z$.

3.5.3 Graph products and blown up graphs

To prove Lemma 18 technically it is easier to estimate the fractional chromatic number of a blown up version of F which one gets by substituting every vertex of F by $\prod\limits_{v \in V(G)} s(v)!$ independent copies of it. The graph so obtained we denote by $\hat{F}$.

Blowing up vertices does not change the fractional chromatic number. (The blown up graph contains the original graph as an induced subgraph, while from any fractional colouring of the original graph we can get a fractional colouring of the blown up graph with the same value just by replacing the blown up vertices with all their independent copies in the weighted independent sets.) Thus we have $\chi_f(F) = \chi_f(\hat{F})$.

We will also use the blown up version of the original graph, so let $\hat{G}$ be the graph which one gets from G by substituting every vertex v by $s(v)$ independent copies of it. Similarly, $\chi_f(\hat{G}) = \chi_f(G)$. Furthermore the constant function $g_{\hat{G}}(v) = \frac{1}{r}$, for $\forall v \in V(\hat{G})$ is an optimal fractional clique of $\hat{G}$. (See Figure 15.)

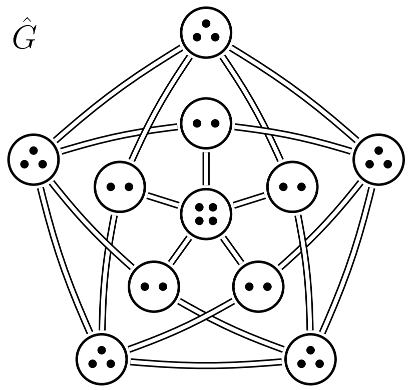

Figure 15: The graph $\hat{G}$, it has 29 vertices. The graph $\hat{F}$ has 29! vertices.

We consider the graph $\hat{F}$ as a subgraph of $\hat{G}^{\times s}$ which is induced by such vertices of $\hat{G}^{\times s}$ whose coordinate sequences are permutations of $V(\hat{G})$. ($|V(\hat{G})| = s$ by definition, and $\hat{F}$ has $\frac{s!}{\prod_{v\in V(G)} s(v)!} \prod_{v\in V(G)} s(v)! = s!$ vertices.) In the next subsection we will construct an optimal fractional clique of $\hat{G}^{\times s}$ which gets non-zero value just on the vertices of $\hat{F}$. This will imply $\omega_f(\hat{F}) \geq \omega_f(\hat{G}^{\times s})$. As $\omega_f(\hat{F}) = \chi_f(\hat{F}) = \chi_f(F)$ and $\omega_f(\hat{G}^{\times s}) = \chi_f(\hat{G}^{\times s}) = \chi_f(\hat{G}) = \chi_f(G)$ this means that $\chi_f(F) \geq \chi_f(G)$ as stated. (The fractional clique of F with value $\chi_f(G)$ can be easily derived from the above fractional clique of $\hat{G}^{\times s}$.)

3.5.4 Constructing the optimal fractional clique

For a vertex $v = (v_1, v_2, \ldots, v_k)$ of $\hat{G}^{\times k}$ we call the *type* of v the partition $P = \{P_1, P_2, \ldots, P_t\}$ of the set $\{1, 2, \ldots, k\}$ for which $v_i = v_j$ iff $\exists l : i, j \in P_l$. We denote by $V(\hat{G}^{\times k})[P]$ the vertices of $\hat{G}^{\times k}$ whose type is P. With this notation the vertex set of $\hat{F}$ is $V(\hat{G}^{\times s})[\{\{1\}, \{2\}, \ldots, \{s\}\}]$. (Note that this definition of type is similar but not equivalent to the well-known concept often used in information theory under this name [22].)

Let $\mathcal{S}$ be the set of partitions of $S = \{1, 2, \ldots, s\}$. For two partitions P and Q we say that Q is *coarser than* P (and P is a refinement of Q), if every partition class of P is a subset of some partition class of Q. The coarser-than relation is a partial order on $\mathcal{S}$ and defines a lattice. Denote by $\mathcal{S}[P^{\geq}]$ the set of partitions Q which are coarser than the partition P, and by $V(\hat{G}^{\times s})[P^{\geq}]$ the vertices of $\hat{G}^{\times s}$ whose type is in $\mathcal{S}[P^{\geq}]$.

Now we construct in two steps the promised optimal fractional clique of $\hat{G}^{\times s}$ which gets non-zero value just on the vertices of $\hat{F}$. (See also Figure 16.)

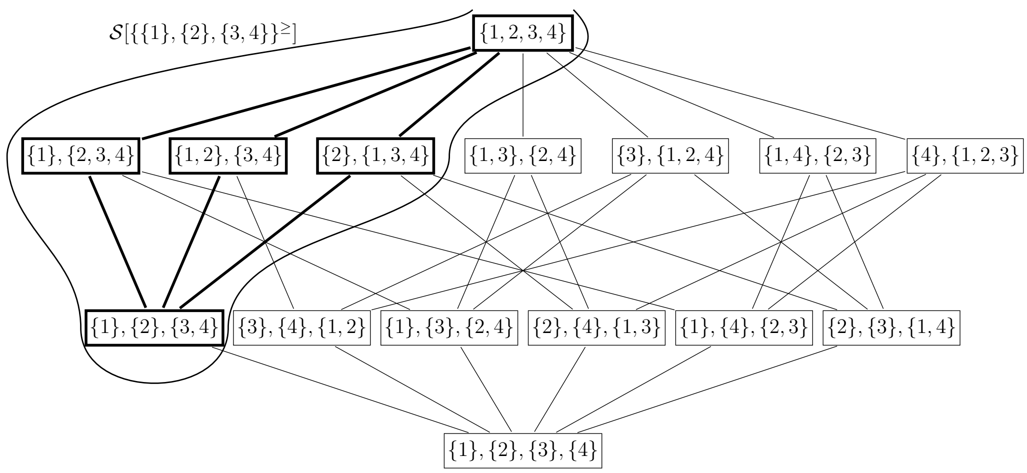

Figure 16: The lattice on $\mathcal{S}$ with the coarser-than relation for s=4.

Step 1. *We give an optimal fractional clique of $\hat{G}^{\times s}$ concentrated on $V(\hat{G}^{\times s})[P^{\geq}]$ for any $P \in \mathcal{S}$. This function will be constant on the set $V(\hat{G}^{\times s})[P^{\geq}]$ and zero outside of it.*

Let $\hat{G}_k$ for $k = 1, 2, \ldots, s$ be disjoint copies of $\hat{G}$. For every $P_m \in P$ we consider $\hat{G}^{\times P_m}$ as the categorical product of the graphs $\hat{G}_k$ for $k \in P_m$. First we construct an optimal fractional clique of $\hat{G}^{\times P_m}$ concentrated on its diagonal elements. Lemma 20 gives us this function from the constant fractional clique of $\hat{G}$. Formally for the partition class $P_m = \{i_1^m, i_2^m, \ldots, i_{t_m}^m\}$ we get $g_{\hat{G}^{\times P_m}}((v_{i_1^m}, v_{i_2^m}, \ldots, v_{i_{t_m}^m})) = \frac{1}{r} \cdot \mathcal{I}_{V[\hat{G}^{\times P_m}][\{\{i_1^m, i_2^m, \ldots, i_{t_m}^m\}\}]}$ as an optimal fractional clique of $\hat{G}^{\times P_m}$. After that from these fractional cliques we construct an optimal fractional clique of $\hat{G}^{\times s}$ concentrated on the vertices of $\hat{G}^{\times s}$ with type in $\mathcal{S}[P^{\geq}]$ using Zhu's result. So applying Lemma 19 repeatedly we get that $g_{\hat{G}^{\times s}}((v_1, v_2, \ldots, v_s)) = c \cdot \mathcal{I}_{V(\hat{G}^{\times s})[P^{\geq}]}$ is an optimal fractional clique of $\hat{G}^{\times s}$ for some constant c. (The appropriate value of c is $\frac{s}{r}\frac{1}{s^t} = \frac{1}{rs^{t-1}}$.) If the vertex $v = (v_1, v_2, \ldots, v_s)$ is an element of the categorical product of the sets $V(\hat{G}^{\times P_m})[\{\{i_1^m, i_2^m, \ldots, i_{t_m}^m\}\}]$ then $i, j \in P_l$ for $P_l \in P$ forces $v_i = v_j$, but also other equalities may arise causing the type of v to be coarser than P. Hence the support of the constructed fractional clique is not just $V(\hat{G}^{\times s})[P]$, but $V(\hat{G}^{\times s})[P^{\geq}]$ as stated.

Step 2. *We construct for any type P such an optimal fractional clique of $\hat{G}^{\times s}$ which is concentrated on just $V(\hat{G}^{\times s})[P]$ and it is constant on this set.*

We get these fractional cliques for the partitions of S in an order with increasing number of classes. We obtain the one for $P = \{\{1, 2, \dots, s\}\}$ from Lemma 20 applied for $H = \hat{G}$, $m = s$ and $g_{\hat{G}}(v) = \frac{1}{r}$, so $g_{\hat{G}^{\times s}}(v) = \frac{1}{r} \cdot \mathcal{I}_{V(\hat{G}^{\times s})[\{\{1,2,\dots,s\}\}]}$ is an optimal fractional clique of $\hat{G}^{\times s}$. For further types P we get the desired fractional clique from the ones corresponding to the partitions $Q \neq P$ which are coarser than P and from the fractional clique concentrated on $V(\hat{G}^{\times s})[P^{\geq}]$. (If $Q \neq P$ is coarser than P then Q has a smaller number of partition classes than P has.) In more detail, we have fractional cliques $g^0_{\hat{G}^{\times s}}((v_1, v_2, \dots v_s)) = c_0 \cdot \mathcal{I}_{V(\hat{G}^{\times s})[P^{\geq}]}$, and $g^i_{\hat{G}^{\times s}}((v_1, v_2, \dots v_s)) = c_i \cdot \mathcal{I}_{V(\hat{G}^{\times s})[Q_i]}$ for all partitions $Q_i \in \mathcal{S}[P^{\geq}] \setminus \{P\}$. Since $V(\hat{G}^{\times s})[P^{\geq}]$ is a disjoint union of $V(\hat{G}^{\times s})[Q_i]$ for all $Q_i \in \mathcal{S}[P^{\geq}]$ using Lemma 21 several times we get an optimal fractional clique of $\hat{G}^{\times s}$ in the form $g_{\hat{G}^{\times s}}((v_1, v_2, \dots v_s)) = c \cdot \mathcal{I}_{V(\hat{G}^{\times s})[P]}$ for some appropriate constant c. (We take the elements of $\mathcal{S}[P^{\geq}] \setminus \{P\}$ in some order: $Q_1, Q_2, \dots, Q_l$ and apply Lemma 21 with $U_1 = V(\hat{G}^{\times s})[Q_i]$ and $U_2 = V(\hat{G}^{\times s})[P^{\geq}] \setminus \bigcup_{j<i} V(\hat{G}^{\times s})[Q_j]$, for $i = 1, 2, \dots, l$.) At the end we get the optimal fractional clique of $\hat{G}^{\times s}$ concentrated on $V(\hat{G}^{\times s})[\{\{1\}, \{2\}, \dots, \{s\}\}] = V(\hat{F})$ proving that $\omega_f(\hat{F}) \geq \omega_f(\hat{G}^{\times s})$, which implies $\chi_f(F) \geq \chi_f(G)$.

Thus we have finished the proof of Lemma 18, and as we have seen before this implies that $h(G^{\times s}) \geq \chi_f(G)$ and so that the ultimate categorical Hall-ratio equals to the fractional chromatic number for every graph G, as stated in Theorem 15. $\square$

Remark 3. While in the case of lexicographic power the example of odd wheels showed that we cannot expect that $\sqrt[k]{\rho(G^{\circ k})}$ reaches its limit for any finite k, considering the categorical power we reached the corresponding limit for a finite k. $\diamondsuit$

3.6 Further remarks on the asymptotic values of the independence ratio and the Hall-ratio

The asymptotic value for the independence ratio and the Hall-ratio for the Cartesian product are closely related. For two graphs F and G, their *Cartesian product* $F \square G$ is defined on the vertex set $V(F \square G) = V(F) \times V(G)$ with edge set $E(F \square G) = \{\{(u_1, v_1), (u_2, v_2)\} : \{u_1, u_2\} \in E(F)$ and $v_1 = v_2$, or $u_1 = u_2$ and $\{v_1, v_2\} \in E(G)\}$. The kth Cartesian power $G^{\square k}$ is the k-fold Cartesian product of G. That is, the Cartesian power is also defined on the vertex sequences of the original graph and two such sequences form an edge iff they differ at exactly one place and at that place the corresponding coordinates form an edge of the original graph.

The ultimate (Cartesian) independence ratio introduced in [37] as $I(G) = \lim_{k\to\infty} i(G^{\Box k})$, and it was studied in [35]. The ultimate Cartesian Hall-ratio was introduced by Simonyi in [49] as $h_{\Box}(G) = \lim_{k\to\infty} \rho(G^{\Box k})$. It is shown in [49] (based on a result of Hahn, Hell and Poljak [35]) that $h_{\Box}(G) = \frac{1}{I(G)}$ for every graph G. (It is also known about $h_{\Box}(G)$ that $\chi_f(G) \leq h_{\Box}(G) \leq \chi(G)$ and $h_{\Box}(G)$ can be strictly between the two bounds. Furthermore $h_{\Box}(G) = \lim_{k\to\infty} \chi_f(G^{\Box k})$. See [49], [35, 37], [53].)

One can also investigate the asymptotic value of the independence number for the normal, co-normal and lexicographic power. The asymptotic value of the independence number for the normal product, $\lim_{k\to\infty} \sqrt[k]{\alpha(G^{\odot k})}$ is the well-studied graph parameter, the Shannon-capacity, denoted by $c(G)$. So the asymptotic value for the independence ratio, $\lim_{k\to\infty} \sqrt[k]{i(G^{\odot k})}$ equals to $\frac{c(G)}{|V(G)|}$. The analogous asymptotic value for the co-normal and lexicographic power equals to the independence ratio of the original graph, because the independence number for both of these products are multiplicative.

Chapter 4

Gallai colourings and domination in multipartite digraphs

Investigating comparability graphs Gallai [28] proved an interesting theorem about edge-colourings of complete graphs that contain no triangle for which all three of its edges receive distinct colours. (Note that here and in the sequel edge-colouring just means a partition of the edge set rather than a proper colouring of it.) Such colourings turned out to be relevant and Gallai's theorem proved to be useful also in other contexts, see e.g., [13, 16, 17, 27, 29, 33, 34, 42, 43].

Honoring the above mentioned work of Gallai, an edge-colouring of the complete graph is called a Gallai colouring if there is no completely multicoloured triangle. Recently this notion was extended to other (not necessarily complete) graphs in [32].

A basic property of Gallai-coloured complete graphs is that at least one of the colour classes spans a connected subgraph on the entire vertex set. In [32] it was proved that if we colour the edges of a not necessarily complete graph G so that no 3-coloured triangles appear then there is still a large monochromatic component whose size is proportional to the number of vertices of G where the proportion depends on the independence number, $\alpha(G)$.

In view of this result it is natural to ask whether one can also span the whole vertex set with a constant number of connected monochromatic subgraphs where the constant depends only on $\alpha(G)$. This question led to a problem about the existence of dominating sets in directed graphs that we believe to be interesting in itself. In this chapter we solve this latter problem thereby giving an affirmative answer to the previous question.

The chapter is organized as follows. In Section 4.1 we describe our digraph problem and state our results on it. In Section 4.2 the connection with Gallai colourings will be explained. Then, Section 4.3 contains the proofs of the results in Section 4.1. We finish this chapter by extending the covering problem of Gallai-coloured graphs to partitioning in Section 4.4.

4.1 Dominating multipartite digraphs

We consider multipartite digraphs, i.e., digraphs D whose vertices are partitioned into classes $A_1, \dots, A_t$ of independent vertices. (Note that here we consider directed graphs without pairs of edges connecting the same two vertices in opposite direction.) Suppose that $S \subseteq [t]$. A set $U = \cup_{i \in S} A_i$ is called a dominating set of size $|S|$ if for any vertex $v \in \cup_{i \notin S} A_i$ there is a $w \in U$ such that $(w, v) \in E(D)$. The smallest $|S|$ for which a multipartite digraph D has a dominating set $U = \cup_{i \in S} A_i$ is denoted by $k(D)$. Let $\beta(D)$ be the cardinality of the largest independent set of D whose vertices are from different partite classes of D. (Such independent sets we sometimes refer to as transversal independent sets.) An important special case is when $|A_i| = 1$ for each $i \in [t]$. In this case $\beta(D) = \alpha(D)$ and $k(D) = \gamma(D)$, the usual domination number of D, the smallest number of vertices in D whose closed outneighbourhoods cover $V(D)$. The main result in this section is the following theorem.

Theorem 22. *For every integer β there exists an integer $h = h(\beta)$ such that the following holds. If D is a multipartite digraph without cyclic triangles and $\beta(D) = \beta$, then $k(D) \leq h$.*

Notice that the condition forbidding cyclic triangles in D is important even when $|A_i| = 1$ for all i and $\beta(D) = 1$, i.e. for tournaments. It is well known that $\gamma(D)$ can be arbitrarily large for tournaments (see, e.g., in [12]), so $h(1)$ would not exist without excluding cyclic triangles.

From the proof of Theorem 22 we will get a factorial upper bound for $k(D)$ from the recurrence formula $h(\beta) = 3\beta + (2\beta + 1)h(\beta - 1)$. We have relatively small upper bounds on k only for $\beta = 1, 2$.

Theorem 23. *Suppose that D is a multipartite digraph without cyclic triangles. If $\beta(D) = 1$ then $k(D) = 1$ and if $\beta(D) = 2$ then $k(D) \leq 4$.*

Though the upper bound on $h(\beta)$ obtained from our proof of Theorem 22 is much weaker we could not even rule out the existence of a bound that is linear in β. We cannot prove a linear upper bound even in the special case when every partite class consists of only one vertex. Nevertheless, we treat this case also separately and provide a slightly better bound than the one following from Theorem 22. The class of digraphs we have here, i.e., those with no directed triangles, is called the class of *clique-acyclic digraphs*, see [10]. These digraphs have been well-studied also because of the Caccetta-Häggkvist Conjecture, see, e.g., in [18].

Theorem 24. *Let $f(1) = 1$ and for $\alpha \geq 2$, $f(\alpha) = \alpha + \alpha f(\alpha - 1)$. If D is a clique-acyclic digraph then $\gamma(D) \leq f(\alpha(D))$.*

Apart from the obvious case $\alpha(D) = 1$ (when D is a transitive tournament) we know the best possible bound only for $\alpha(D) = 2$.

Theorem 25. *If D is a clique-acyclic digraph with $\alpha(D) = 2$, then $\gamma(D) \leq 3$.*

Note that Theorem 25 is sharp as shown by the cyclically oriented pentagon. Moreover, the union of t vertex disjoint cyclic pentagons shows that we can have $\alpha(D) = 2t$ and $\gamma(D) = 3t$. Thus in case a linear upper bound would be valid at least in the special case of clique-acyclic digraphs, it could not be smaller than $\frac{3}{2}\alpha(D)$. There are some easy subcases though when the bound is simply $\alpha(D)$.

Proposition 26. *If D is an acyclically oriented graph or a clique-acyclic perfect graph then $\gamma(D) \leq \alpha(D)$.*

Note that Proposition 26 is sharp in the sense that every graph G has a clique-acyclic orientation resulting in digraph D with $\gamma(D) = \alpha(G) = \alpha(D)$. Indeed, an acyclic orientation of G where every vertex of a fixed maximum independent set has indegree zero shows this. It is worth noting the interesting result of Aharoni and Holzman [10] stating that a clique-acyclic digraph always has a fractional kernel, i.e., a fractional independent set, which is also fractionally dominating.

We will see in Section 4.3 from the proof of Theorems 22 and 23 that the dominating sets we find there contain two kinds of partite classes. The first kind could be substituted by just one vertex in it, while the second kind is chosen not so much to dominate others but because it is itself not dominated by others. That is, apart from a bounded number of exceptional partite classes we will dominate the rest of our digraph with a bounded number of vertices. In the last subsection of Section 4.3 we will prove another theorem showing that the exceptional classes are indeed needed.

4.2 Monochromatic coverings of Gallai-coloured graphs

Recall that Gallai colourings are originally defined as edge-colourings of complete graphs where no triangle gets three different colours. As already mentioned earlier, one of the basic properties of Gallai colourings is that at least one colour spans a connected subgraph, i.e. forms a component covering all vertices of the underlying complete graph. In [32] the notion was extended to arbitrary graphs and it was proved that in this setting there is still a large monochromatic component. More precisely the following was proved.

Theorem 27 (Gyárfás, Sárközy [32]). *Suppose that the edges of a graph G are coloured so that no triangle is coloured with three distinct colours. Then there is a monochromatic component in G with at least $\frac{|V(G)|}{\alpha^2(G)+\alpha(G)-1}$ vertices.*

Another, in a sense stronger possible generalization of the above basic property of Gallai colourings is also suggested by Theorem 27. Gyárfás proposed the following problem at a workshop at Fredericia, Denmark in November, 2009.

Problem 3. *Suppose that the edges of a graph G are coloured so that no triangle is coloured with three distinct colours. Is it true that the vertices of G can be covered by the vertices of at most k monochromatic components where k depends only on $\alpha(G)$?*

We remark that an example in [32] shows that even if the k of Problem 3 exists, it must be at least $\frac{c\alpha^2(G)}{\log \alpha(G)}$ where c is a small constant.

Theorem 22 implies an affirmative answer to Problem 3. Let $g(1) = 1$ and for $\alpha \geq 2$, let $g(\alpha) = g(\alpha - 1) + h(\alpha)$ where h is the function given by Theorem 22.

Theorem 28. *Suppose that the edges of a graph G are coloured so that no triangle is coloured with three distinct colours. Then the vertex set of G can be covered by the vertices of at most $g(\alpha(G))$ monochromatic components. In case $\alpha(G) = 2$ at most five components are enough.*

Note that the last statement of Theorem 28 generalizes Theorem 27 in the case $\alpha(G) = 2$.
In the sequel we will use the notation $G[A]$ that denotes the subgraph of graph G induced by $A \subseteq V(G)$.

Proof of Theorem 28. For $\alpha(G) = 1$ the result is obvious by the mentioned property of Gallai-coloured complete graphs. For $\alpha(G) \geq 2$, suppose that $v \in V(G)$ and let X be the set of vertices in G that are not adjacent to v. By induction, the subgraph $G[X]$ can be covered by the vertices of $g(\alpha(G) - 1)$ monochromatic components. Let ℓ be the number of colours used

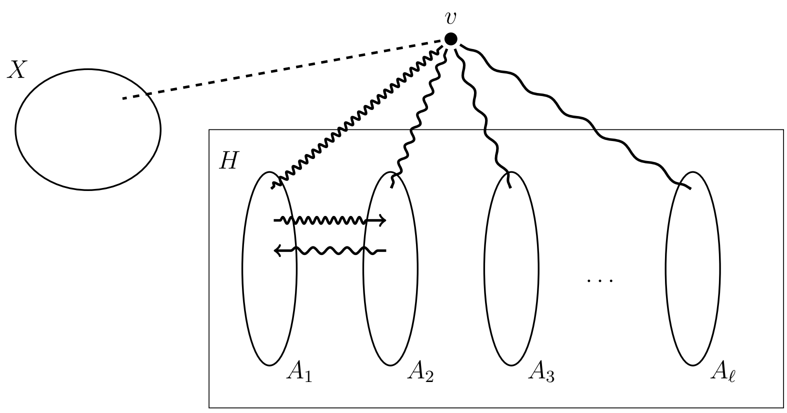

Figure 17: Construction of H. (The waved lines with different frequencies mean different colours.)

on edges of G incident to v and let A_i be the set of vertices incident to v in colour i. Observe that the condition on the colouring implies that edges of G between A_i, A_j are coloured with either colour i or colour j whenever $1 \leq i < j \leq \ell$. (See Figure 17.) Thus orienting all edges of colour i outward from A_i for every i, all edges of G between different classes A_j are oriented. Moreover, in this orientation there are no cyclic triangles. Thus Theorem 22 is applicable to the oriented subgraph H spanned by the union of the classes A_j after the edges inside the A_j's are removed. As $\beta(H) \leq \alpha(G)$, we obtain at most $h(\alpha(G))$ dominating sets A_i and each set $v \cup A_i$ together with the vertices that A_i dominates form a connected subgraph of G in colour i. Thus all vertices of G can be covered by at most $g(\alpha(G)-1) + h(\alpha(G)) = g(\alpha(G))$ connected components. In case of $\alpha(G) = 2$ we can use Theorem 23 to get a covering with at most five monochromatic components. $\square$

Remark 4. Gyárfás and Simonyi in [34] proved that in a Gallai colouring of a complete graph there is a monochromatic spanning tree with height at most two. This result can also be generalized for non-complete graphs. From the previous proof we easily obtain that each of the $g(\alpha(G))$ monochromatic components which cover the vertex set of G have a spanning tree with height at most two. $\Diamond$

In Section 4.4 we will extend the statement of Theorem 28 from covering to partitioning.

4.3 Proofs of the results in Section 4.1

We will use the following notation throughout. If D is a digraph and $U \subseteq V(D)$ is a subset of its vertex set then $N_+(U) = \{v \in V(D) : \exists u \in U \ (u,v) \in E(D)\}$ is the *outneighbourhood* of U. The *closed outneighbourhood* $\hat{N}_+(U)$ of U is meant to be the set $U \cup N_+(U)$. When $U = \{u\}$ is a single vertex we also write $N_+(u)$ and $\hat{N}_+(u)$ for $N_+(U)$ and $\hat{N}_+(U)$, respectively. When $(u,v) \in E(D)$, we will often say that u *sends an edge to* v.

We first deal with the case $\beta(D) = 1$ and prove the first statement of Theorem 23. As it will be used several times later, we state it separately as a lemma.

Lemma 29. *Let D be a multipartite digraph with no cyclic triangle. If $\beta(D) = 1$ then $k(D) = 1$.*

Proof. Let K be a partite class for which $|\hat{N}_+(K)|$ is largest. We claim that K is a dominating set. Suppose on the contrary, that there is a vertex l in a partite class $L \neq K$, which is not dominated by K. Since all edges between distinct partite classes are present in D with some orientation, l must send an edge to all vertices of K. Furthermore, if a vertex m in a partite class $M \neq K, L$ is an outneighbour of some $k \in K$ then it is also an outneighbour of l, otherwise

m, l and k would form a cyclic triangle. Thus $\hat{N}_+(K) \subseteq \hat{N}_+(L)$. Moreover, $l \in \hat{N}_+(L) \setminus \hat{N}_+(K)$, so $|\hat{N}_+(L)| > |\hat{N}_+(K)|$ contradicting the choice of K. This completes the proof of the lemma. □

In the following two subsections we prove Theorems 23 and 22, respectively.

4.3.1 At most 2 independent vertices

To prove the second statement of Theorem 23 we will need the following stronger variant of Lemma 29.

Lemma 30. *Let D be a multipartite digraph with no cyclic triangle and $\beta(D) = 1$. Then there is a partite class K which is a dominating set, and there is a vertex $k \in K$ such that $V(D) \setminus (K \cup L) \subseteq N_+(k)$ for some partite class $L \neq K$.*

Thus Lemma 30 states that the dominating partite class K has an element that alone dominates almost the whole of D, there may be only one exceptional partite class L whose vertices are not dominated by this single element of K.

For proving Lemma 30, the following observations will be used, where X, Y, Z will denote partite classes.

Observation 31. *Let D be a multipartite digraph with no cyclic triangle and $\beta(D) = 1$. Suppose that for vertices $x_1, x_2 \in X$ and $y \in Y$ the edges (x_2, y) and (y, x_1) are present in D. Then for every $z \in Z \neq X, Y$ with $(x_1, z) \in E(D)$ we also have $(x_2, z) \in E(D)$.*

Proof. Assume on the contrary that for some $z \in Z$ the orientation is such that we have (x_1, z), $(z, x_2) \in E(D)$. Then the edge connecting z and y cannot be oriented either way: $(z, y) \in E(D)$ would give a cyclic triangle on vertices z, y, x_1, while $(y, z) \in E(D)$ would create one on y, z, x_2. (Figure 18 illustrates the statement of this observation.)

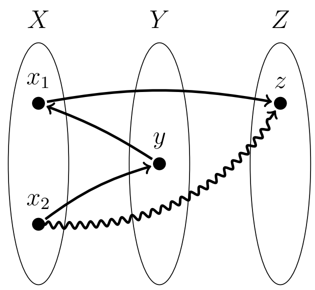

Figure 18: A simple configuration: if x_1 dominates z then x_2 also dominates z.

□

Observation 32. *Let D be a multipartite digraph with no cyclic triangle and $\beta(D) = 1$. Suppose that for vertices $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ the edges (x_1, y_2), (y_2, x_2), $(x_2, y_1), (y_1, x_1)$ are present in D forming a cyclic quadrangle. Then in every partite class $Z \neq X, Y$ the outneighbourhood of these four vertices is the same.*

Proof. Let z be an element of $Z \cap N_+(x_1)$. By $(y_1, x_1) \in E(D)$ we must have $z \in Z \cap N_+(y_1)$, otherwise y_1, x_1, z would form a cyclic triangle. Thus we have $Z \cap N_+(x_1) \subseteq Z \cap N_+(y_1)$. Now shifting the role of vertices along the oriented quadrangle backwards we similarly get $Z \cap N_+(x_1) \subseteq Z \cap N_+(y_1) \subseteq Z \cap N_+(x_2) \subseteq Z \cap N_+(y_2) \subseteq Z \cap N_+(x_1)$ proving that we have equality everywhere. (Figure 19 illustrates the statement of this observation.)

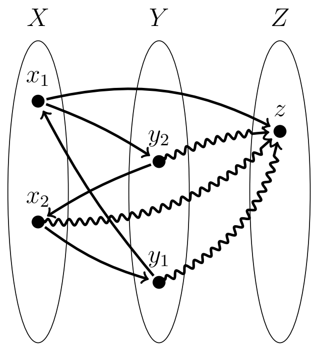

Figure 19: If x_1 dominates z then x_2, y_1, y_2 also dominate z.

□

Note that in Observation 32, as $\beta(D) = 1$, the inneighbourhood of the vertices x_1, x_2, y_1, y_2 is also the same, so these vertices split to out- and inneighbourhood in the same way every partite class $Z \neq X, Y$.

Proof of Lemma 30. We know from Lemma 29 that there is a partite class K which is a dominating set. (Figure 20 shows the main steps of the proof.)

Let k be an element of K for which $|N_+(k)|$ is maximal. If k itself dominates all the vertices not in K then we are done. (In that case we do not even need an exceptional class L.) Otherwise, there is a vertex l_1 in a partite class $L \neq K$ for which the edge between l_1 and k is oriented towards k. As $L \subseteq N_+(K)$, there must be a vertex $k_1 \in K$ which sends an edge to l_1.

Using Observation 31 for the vertices k, k_1 and l_1, we obtain that k_1 sends an edge not just to l_1 but to every vertex in $N_+(k) \setminus L$. By the choice of k this implies the existence of a vertex $l_2 \in L$ for which $(k, l_2), (l_2, k_1) \in E(D)$. Thus the vertices k, l_2, k_1, l_1 form a cyclic quadrangle. Applying Observation 32 this implies that these four vertices have the same outneighbourhood in $V(D) \setminus (K \cup L)$.

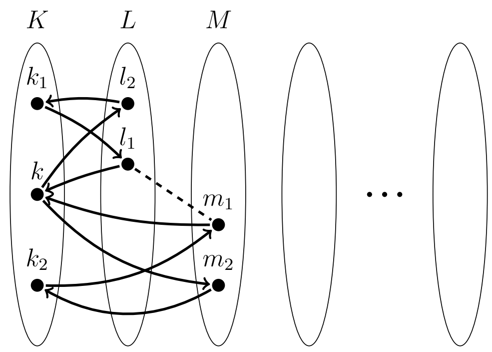

Figure 20: Two cyclic quadrangles give a contradiction.

We claim that $N_+(k)$ contains all vertices of $D \setminus (K \cup L)$. Assume on the contrary, that there is a vertex m_1 in a partite class $M \neq K, L$ which is not dominated by k. We can argue similarly as we did for l_1. Namely, since $M \subseteq N_+(K)$ there is some $k_2 \in K$ (perhaps identical to k_1) dominating m_1. Applying Observation 31 to the vertices k, m_1 and k_2, we obtain $(N_+(k) \setminus M) \subseteq N_+(k_2)$. Then by the choice of k we must have a vertex $m_2 \in M$ for which $(k, m_2), (m_2, k_2) \in E(D)$. So vertices k, m_2, k_2, m_1 also form a cyclic quadrangle, and Observation 32 gives us that $Z \cap N_+(k) = Z \cap N_+(m_2) = Z \cap N_+(k_2) = Z \cap N_+(m_1)$ for all partite classes $Z \neq K, M$.

The contradiction will be that the edge between l_1 and m_1 should be oriented both ways. Indeed, since $(l_1, k) \in E(D)$ and in L the inneighbours of k and m_1 are the same, we must have $(l_1, m_1) \in E(D)$. However, $(m_1, k) \in E(D)$ and the fact that k and l_1 split M in the same way implies $(m_1, l_1) \in E(D)$. This contradiction completes the proof of the lemma. $\square$

Remark 5. It is easy to see that in the proof of this lemma if there is a vertex $l_1 \in L \neq K$ which is not dominated by $k \in K$ then we can change the roles of the dominating vertex and the exceptional partite class, namely it is also true that $V(D) \setminus (L \cup K) \subseteq N_+(l_1)$. $\Diamond$

Now we are ready to prove the second statement of Theorem 23.

Proof of Theorem 23. We have already proven the first statement of the theorem. To prove the second part let D be a multipartite digraph without cyclic triangles and $\beta(D) = 2$. We use induction on the number of vertices. The base case is obvious. Let p be a vertex of D and consider the subdigraph $\hat{D} = D \setminus \{p\}$. (One can follow the proof on Figure 21.)

By induction $k(\hat{D}) \leq 4$. Let K, L, M and N be four partite classes of $\hat{D}$ that form a dominating set in $\hat{D}$. If $p \in \hat{N}_+(K \cup L \cup M \cup N)$ then we are done, the same four sets also dominate D. If $p \notin \hat{N}_+(K \cup L \cup M \cup N)$ then we will choose four other partite classes that will dominate D. First we choose P, the class of p. We partition every other partite class into three parts according to how it is connected to p. For any class Z, let Z_1 denote the set of vertices in Z dominated by p, let Z_2 be the set of vertices in Z nonadjacent to p, and let Z_3 denote the set of remaining vertices of Z, i.e., those which send an edge to p. We will refer to Z_i as the i-th part of the partite class Z, where $i = 1, 2, 3$. Note that K_3, L_3, M_3, N_3 are all empty, otherwise we would have $p \in \hat{N}_+(K \cup L \cup M \cup N)$.

Let D_2 be the subdigraph of D induced by the vertices in the second part of the partite classes of $D \setminus P$ in their partition above. This graph is also a multipartite digraph with no cyclic triangle and $\beta(D_2) = 1$. The latter follows from the fact that the vertices of D_2 are all nonadjacent to p and $\beta(D) = 2$. Thus by Lemma 29 the vertices of D_2 can be dominated by one partite class Q_2, the second part of some partite class Q of D. We choose Q to be the second partite class in our dominating set. Observe that all vertices of D not dominated so far, i.e., those not in $\hat{N}_+(P \cup Q)$ should belong to the third part of their partite classes. Let u be such a vertex. (If there is none, then we are done.) We know $u \notin K \cup L \cup M \cup N$ as none of these four classes has a third part. Since $K \cup L \cup M \cup N$ is a dominating set in $\hat{D}$ there is a vertex k in one of these four classes for which (k, u) is an edge of D. No vertex in the first part of a class can send an edge to a vertex lying in the third part of some other class, otherwise the latter two vertices would form a cyclic triangle with p. Thus, since K, L, M, N has no third parts, k must be in the second part of one of them.

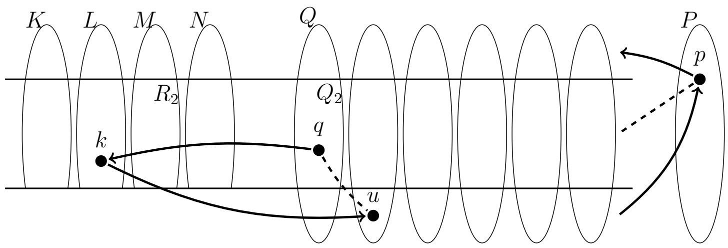

Figure 21: Domination of a multipartite digraph D with $\beta(D) = 2$.

Lemma 30 implies that there is a vertex $q \in Q_2$ with $V(D_2) \cap \hat{N}_+(q)$ containing $V(D_2)$ except one exceptional class R_2. We choose R, the partite class of R_2, to be the third partite class in our dominating set. If $u \notin \hat{N}_+(R)$, then $k \notin R$ and so q dominates k, i.e. k is an outneighbour of q. Observe that (u, q) cannot be an edge of D, otherwise q, k and u would form a cyclic triangle. But (q, u) cannot be an edge either, as $u \notin N_+(Q)$. Thus u and every so far undominated vertex is nonadjacent to q. Thus the set U of undominated vertices induces a subgraph $D[U]$ with $\beta(D[U]) = 1$, otherwise adding q we would get $\beta(D) \geq 3$. But then by Lemma 29 all vertices in U can be dominated by one additional, fourth class. □

Remark 6. It is not difficult to show that we only need the partite class R for the domination if it coincides with K, L, M or N. (Otherwise k cannot be an element of R hence q surely sends an edge to k and is nonadjacent to every $u \notin \hat{N}_+(P \cup Q)$.) Also, obviously if in D_2 we do not need the exceptional partite class, that is the vertex q dominates every other partite class except for Q_2, then we can dominate D with three partite classes.
Moreover, from the remark after the proof of Lemma 30 it follows that in the proof of this theorem if $R \in \{K, L, M, N\}$ but $Q \notin \{K, L, M, N\}$ then P, R and one additional partite class for the undominated vertices are enough for domination. Thus we only need four partite classes in the dominating set if both Q and R are equal to one of the dominating partite classes of $D \setminus \{p\}$. This observation may be useful in deciding whether there is a multipartite digraph D with no cyclic triangle for which $\beta(D) = 2$ and $k(D) = 4$. $\Diamond$

4.3.2 General case

Surprisingly, our proof of Theorem 22 is not a direct generalization of the argument proving Theorem 23 in the previous subsection. In fact, in a way it is conceptually simpler.

Proof of Theorem 22. We have seen that $h(1) = 1$ (and $h(2) = 4$) is an upper bound for $k(D)$ if $\beta(D) = 1$ (and if $\beta(D) = 2$). Now we prove that $h(\beta) = 3\beta + (2\beta + 1)h(\beta - 1)$ is an upper bound on $k(D)$ if $\beta(D) = \beta \geq 2$. Let D be a multipartite digraph without cyclic triangles and $\beta(D) = \beta$. (See Figure 22.) Let $k_1, k_2, \ldots, k_{2\beta}$ be vertices of D, each from a different partite class, such that $|\hat{N}_+(\cup_{i=1}^{2\beta}\{k_i\})|$ is maximal. Let the partite class of k_i be K_i for all i and let $\mathcal{K}$ denote $\cup_{i=1}^{2\beta}\{k_i\}$. First we declare the 2β partite classes of these vertices k_i to be part of our dominating set. Next we partition every other partite class into $2\beta + 2$ parts. For an arbitrary partite class $Z \neq K_i$ $(i = 1, \ldots, 2\beta)$ we denote by Z_0 the set $Z \cap N_+(\mathcal{K})$. For $i = 1, 2, \ldots, 2\beta$ let Z_i be the set of vertices in $Z \setminus Z_0$ that are not sending an edge to k_i, but are sending an edge to k_j for all $j < i$. Finally, we denote by $Z_{2\beta+1}$, the remaining part of Z, that is the set of those vertices of Z that send an edge to all vertices $k_1, k_2, \ldots, k_{2\beta}$. (As in the proof of Theorem 23 we will refer to the set Z_i as the i-th part of Z.) The subgraph D_i of D induced by the i-th parts of the partite classes of $D \setminus (\cup_{i=1}^{2\beta} K_i)$ is also a multipartite digraph with no cyclic triangle. For $1 \leq i \leq 2\beta$ it satisfies $\beta(D_i) \leq \beta - 1$, since adding k_i to any transversal independent set of D_i we get a larger transversal independent set. So by induction on β, each of these 2β digraphs D_i can be dominated by at most $h(\beta - 1)$ partite classes. We add the appropriate $2\beta h(\beta - 1)$ partite classes to our dominating set.

If $\beta(D_{2\beta+1}) \leq \beta - 1$ also holds then the whole graph can be dominated by choosing $h(\beta - 1)$ additional partite classes. Otherwise let $\mathcal{L} = \{l_1, l_2, \ldots, l_\beta\}$ be an independent set of size β

with all its vertices in $V(D_{2\beta+1})$ belonging to distinct partite classes (of D), that are denoted by $L_1, L_2, \dots, L_\beta$, respectively. We claim that in the remaining part of $D_{2\beta+1}$, i.e., in $D_{2\beta+1} \setminus (\cup_{i=1}^{\beta} L_i)$ there is no other independent set of size β with all elements belonging to different partite classes. Assume on the contrary that $m_1 \in M_1, m_2 \in M_2, \dots, m_\beta \in M_\beta$ form such an independent set $\mathcal{M}$. As $\mathcal{L}$ is a maximal transversal independent set, every element of a partite class different from $L_1, \dots, L_\beta$ is connected to at least one of the l_i's. And since every element of $\mathcal{L}$ sends an edge to all the vertices $k_1, \dots, k_{2\beta}$, we must have $N_+(\mathcal{K}) \setminus (\cup_{i=1}^{\beta} L_i) \subseteq N_+(\mathcal{L})$ otherwise a cyclic triangle would appear. (The latter is because if k_i ($i \in \{1, 2, \dots, 2\beta\}$) sends an edge to v, and l_j ($j \in \{1, 2, \dots, \beta\}$) sends an edge to k_i, moreover l_j is connected with v then the edge between l_j and v must be oriented towards v.)

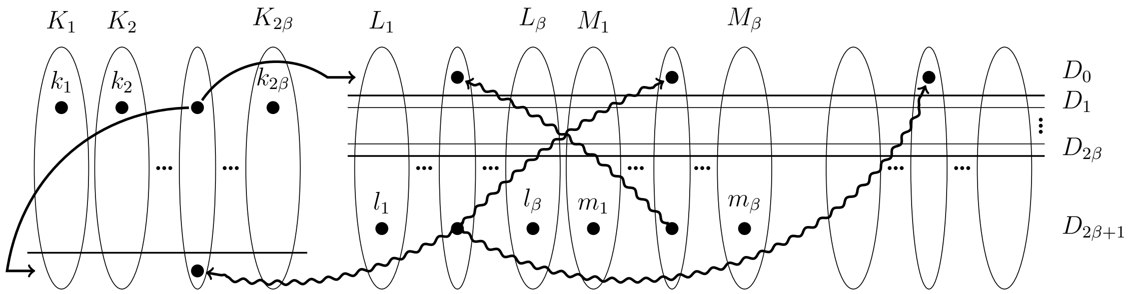

Figure 22: Domination of a multipartite digraph in the general case.

Similarly, we have $N_+(\mathcal{K}) \setminus (\cup_{i=1}^{\beta} M_i) \subseteq N_+(\mathcal{M})$. Thus if such an $\mathcal{M}$ exists then $\hat{N}_+(\mathcal{K}) \subseteq N_+(\mathcal{L} \cup \mathcal{M})$ while $\hat{N}_+(\mathcal{L} \cup \mathcal{M})$ also contains the additional vertices belonging to $\mathcal{L} \cup \mathcal{M}$. This contradicts the choice of $\mathcal{K}$. (Note that $\mathcal{L} \cup \mathcal{M}$ dominates also the vertices in $(K_1 \cup \dots \cup K_{2\beta}) \cap (N_+(k_1) \cup \dots \cup N_+(k_{2\beta}))$.) Thus if we add the classes $L_1, \dots, L_\beta$ to our dominating set, the still not dominated part of D can be dominated by $h(\beta - 1)$ further classes. So we constructed a dominating set of D containing at most $2\beta + 2\beta h(\beta - 1) + \beta + h(\beta - 1) = 3\beta + (2\beta + 1)h(\beta - 1)$ partite classes. This proves the statement. $\square$

Note that we have proved a little bit more than stated in Theorem 22. Namely, we showed that there is a set of at most $h_1(\beta)$ vertices of D which dominates the whole graph except perhaps their own partite classes and at most $h_2(\beta)$ other exceptional classes. From the proof we obtain the recursion formula $h_1(\beta) \leq 2\beta + (2\beta+1)h_1(\beta-1)$ and $h_2(\beta) \leq \beta + (2\beta+1)h_2(\beta-1)$.

4.3.3 Clique-acyclic digraphs

For the proof of Theorem 24 we will use the following theorem due to Chvátal and Lovász [19].

Theorem 33 (Chvátal, Lovász [19])**.** *Every directed graph D contains a semi-kernel, that is an independent set U satisfying that for every vertex $v \in D$ there is an $u \in U$ such that one can reach v from u via a directed path of at most two edges.*

Proof of Theorem 24. The statement is trivial for $\alpha(D) = 1$, since a transitive tournament is dominated by its unique vertex of indegree 0. We use induction on $\alpha = \alpha(D)$. Assume the theorem is already proven for $\alpha - 1$. Consider D with $\alpha(D) = \alpha$ and a semi-kernel U in D that exists by Theorem 33. (Figure 23 illustrates the proof.)

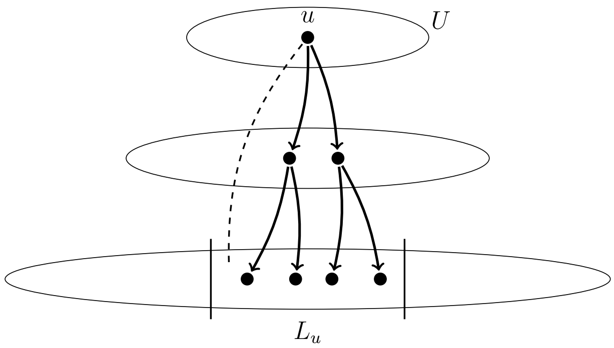

Figure 23: Domination of a clique-acyclic digraph.

We define a set S with $|S| \leq f(\alpha)$ elements dominating each vertex. Let $U \subseteq S$. Then S already dominates the outneighbourhood of U. Denote by T the second outneighbourhood of U (i.e., the set of all vertices not in U and not yet dominated). Observe that for every vertex $w \in T$ there is a vertex $u \in U$ such that neither (u, w) nor (w, u) is an edge. Indeed, let u be the vertex of U from which w can be reached by traversing two directed edges. Then $(w, u) \notin E(D)$ otherwise we would have a cyclic triangle. But $(u, w) \notin E(D)$ is immediate from knowing that w is not in the first outneighbourhood of U. Partition T into $|U| \leq \alpha$ classes L_u indexed by the elements of U where $w \in L_u$ means that u and w are nonadjacent. Thus all vertices in each class L_u are independent from the same vertex in U implying that the induced subgraph $D[L_u]$ has independence number at most $\alpha - 1$. Thus $D[L_u]$ can be dominated by at most $f(\alpha - 1)$ vertices. Add these to S for every $u \in U$. So all vertices can be dominated by at most $\alpha + \alpha f(\alpha - 1) = f(\alpha)$ vertices completing the proof. $\square$

For $\alpha(D) = 2$ the above theorem gives $\gamma(D) \leq f(2) = 4$. Compared to this the improvement of Theorem 25 is only 1, but as already mentioned, the cyclically oriented five-cycle shows that $\gamma(D) \leq 3$ is the best possible upper bound.

The proof of Theorem 25 goes along similar lines as the proof we had for the second statement of Theorem 23.

Proof of Theorem 25. We use induction on the number of vertices in D. Let p be a vertex of D, and partition the remaining vertices of D into three parts. (See Figure 24.) Let V_1 be the set of vertices that are dominated by p, V_2 the set of vertices nonadjacent to p, and let V_3 be the set of vertices which send an edge to p. We assume by induction that $D \setminus \{p\}$ can be dominated by three vertices. (The base case is obvious.) If at least one of these is located in V_3 then p is also dominated by them and we are done. Otherwise we create a new dominating set.

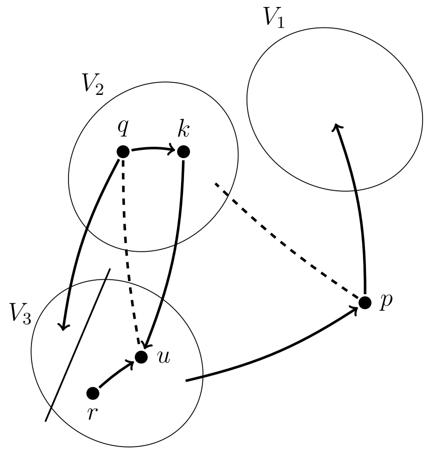

Figure 24: Domination of a clique-acyclic digraph D with $\alpha(D) = 2$.

First we choose p, and by p we dominate all the vertices in V_1. Observe that any two vertices in V_2 must be connected, because two nonadjacent vertices of V_2 and p would form an independent set of size 3. Thus $D[V_2]$ is a transitive tournament and so it can be dominated by just one vertex, let it be $q \in V_2$. Let U be the set of remaining undominated vertices. That is, $U = V_3 \setminus N_+(q)$. Consider an arbitrary element $u \in U$. We know that u is dominated by a vertex of the dominating set of $D \setminus \{p\}$. Let this vertex be k, it does not belong to V_3 as we assumed above. We also have $k \notin V_1$, otherwise there is a cyclic triangle on the vertices p, k and u. So $k \in V_2$, and thus q sends an edge to k. Since u is undominated, (q, u) is not an edge of D. With the edge (u, q), we would get a cyclic triangle on u, q and k. So u and all the vertices in U are nonadjacent to q, therefore $\alpha(D[U]) = 1$ and thus U can be dominated by one vertex r. Thus all vertices of D are dominated by the 3-element set $\{p, q, r\}$. This completes the proof. □

To prove Proposition 26 we formulate the following simple observation. Let $\chi(F)$ denote the chromatic number of graph F.

Observation 34. *Let D be a directed graph and $\bar{D}$ the complementary graph of the undirected graph underlying D. If D is clique-acyclic, then $\gamma(D) \leq \chi(\bar{D})$.*

Proof. It follows from the definition of $\chi(\bar{D})$ that the vertex set of D can be covered by $\chi(\bar{D})$ complete subgraphs of D. Since D is clique-acyclic, all these complete subgraphs can be dominated by one of their vertices. Thus all vertices are dominated by these $\chi(\bar{D})$ chosen vertices. □

Proof of Proposition 26. If the orientation of D is acyclic, then consider those vertices that have indegree zero. Let these form the set U_0. Delete these vertices and all vertices they dominate. Let set U_1 contain the indegree zero vertices of the remaining graph, and delete the vertices in $U_1 \cup N_+(U_1)$. Proceed this way to form the sets $U_2, \ldots, U_s$, where finally there are no remaining vertices after U_s and its neighbours are deleted. It follows from the construction that $U_0 \cup U_1 \cup \cdots \cup U_s$ is an independent set and dominates all vertices not contained in it.

The second statement immediately follows from Observation 34 and the fact that $\chi(\bar{D}) = \alpha(D)$ if D is perfect, an immediate consequence of the Perfect Graph Theorem [45]. □

4.3.4 On the exceptional classes

As already mentioned after the proof of Theorem 22, the statement of Theorem 22 could be formulated in a somewhat stronger form. Namely, we do not only dominate our multipartite digraph D by $h(\beta)$ partite classes, we actually dominate almost all of D by $h_1(\beta)$ vertices, where "almost" means that there is only a bounded number $h_2(\beta)$ of partite classes not dominated this way. The first appearance of this phenomenon is in Lemma 30 where we showed that if $\beta(D) = 1$ then a single vertex dominates the whole graph except at most one class. To complement this statement we show below that this exceptional class is indeed needed, we cannot expect to dominate the whole graph by a constant number of vertices. In other words, if we want to dominate with a constant number of singletons (and not by simply taking a vertex from each partite class), then we do need exceptional classes already in the $\beta(D) = 1$ case.

For a bipartite digraph D with partite classes A and B let $\gamma_A(D)$ denote the minimum number of vertices in A that dominate B and similarly let $\gamma_B(D)$ denote the minimum number of vertices in B dominating A. Let $\gamma_0(D) = \min\{\gamma_A(D), \gamma_B(D)\}$.

Theorem 35. *There exists a sequence of oriented complete bipartite graphs $\{D_k\}_{k=1}^{\infty}$ satisfying $\gamma_0(D_k) > k$.*

We note that the existence of D_k with n vertices in each partite class and satisfying $\gamma_0(D_k) > k$ follows by a standard probabilistic argument provided that $2\binom{n}{k}(1-2^{-k})^n < 1$. Our proof below is constructive, however.

Proof of Theorem 35. We give a simple recursive construction for D_k in which we blow up the vertices of a cyclically oriented cycle C_{2k+2} and connect the blown up versions of originally nonadjacent vertices that are an odd distance away from each other by copies of the already constructed digraph D_{k-1}.

Let D_1 be a cyclic 4-cycle, i.e., a cyclically oriented $K_{2,2}$. It is clear that neither partite class in this digraph can be dominated by a single element of the other partite class. Thus $\gamma_0(D_1) > 1$ holds.

Assume we have already constructed D_{k-1} satisfying $\gamma_0(D_{k-1}) > k-1$. Let the two partite classes of D_{k-1} be $A_{k-1} = \{a_1, \ldots, a_m\}$ and $B_{k-1} = \{b_1, \ldots, b_m\}$. Now we construct D_k as follows. (The construction of D_2 is shown on Figure 25.) Let the vertex set of D_k be $V(D_k) = A_k \cup B_k$, where

$$A_k := \{(j, a_i) : 1 \le j \le k+1, 1 \le i \le m\},$$
$$B_k := \{(j, b_i) : 1 \le j \le k+1, 1 \le i \le m\}.$$

There will be an oriented edge from vertex (j, a_i) to (r, b_s) if either $j = r$, or $j \not\equiv r+1 \pmod{k+1}$ and $(a_i, b_s) \in E(D_{k-1})$. All other edges between A_k and B_k are oriented towards A_k, i.e., this latter set of edges can be described as

$$\{((r, b_s), (j, a_i)) \, : \, j \equiv r+1 \pmod{k+1} \quad \text{or} \quad ((b_s, a_i) \in E(D_{k-1}) \text{ and } j \neq r)\}.$$

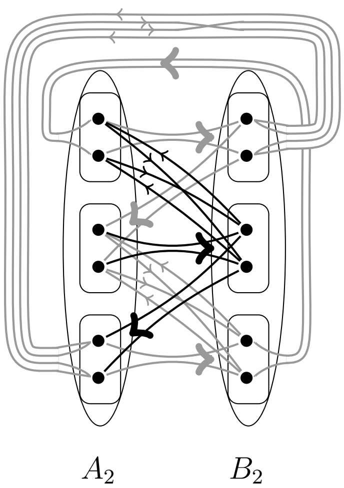

Figure 25: The construction of D_2.

It is only left to prove that $\gamma_0(D_k) > k$. Let us use the notation $A_k(j) = \{(j, a_i) : 1 \le i \le m\}$, $B_k(j) = \{(j, b_i) : 1 \le i \le m\}$. Consider a set K of k vertices of A_k, we show it cannot dominate B_k. There must be an $r \in \{1, \dots, k+1\}$ by pigeon-hole for which $K \cap A_k(r) = \emptyset$ and $K \cap A_k(r+1) \ne \emptyset$. (Addition here is meant modulo $(k+1)$.) Fix this r. We claim that some vertex in $B_k(r)$ will not be dominated by K. Indeed, the vertex $(r+1, a_i) \in K \cap A_k(r+1)$ does not send any edge into $B_k(r)$, so we have only at most $k-1$ vertices in K that can dominate vertices in $B_k(r)$ and all these vertices are in $A_k \setminus A_k(r)$. Notice that the induced subgraph of D_k on $B_k(r) \cup A_k \setminus A_k(r)$ admits a digraph homomorphism (that is an edge-preserving map) into D_{k-1}. Indeed, the projection of each vertex to its second coordinate gives such a map by the definition of D_k. So if the above mentioned $k-1$ vertices would dominate the entire set $B_k(r)$, then their homomorphic images would dominate the homomorphic image of $B_k(r)$ in D_{k-1}. The latter image is the entire set B_{k-1} and by our induction hypothesis it cannot be dominated by $k-1$ vertices of A_{k-1}. Thus we indeed have $\gamma_{A_k}(D_k) > k$.

The proof of $\gamma_{B_k}(D_k) > k$ is similar by symmetry. Thus we obtain $\gamma_0(D_k) > k$ as stated. □

4.4 Monochromatic partitions of Gallai-coloured graphs

In this section we extend the result about monochromatic covering of Gallai coloured graphs (Theorem 28) to partitioning. We say that the vertex set of an edge-coloured graph G can be partitioned into ℓ monochromatic connected parts, if there is a partition $\{V_1, \dots, V_\ell\}$ of $V(G)$ such that every $G[V_i]$ $(1 \le i \le \ell)$ is connected in some colour, where $G[S]$ denotes the induced subgraph by the subset of the vertex set S in G. (Note that, arbitrary subsets of the monochromatic connected components may not be used as parts of our partition because they can be disconnected in the corresponding colour.)

Let $\hat{g}(1) = 1$ and for $\alpha \ge 2$, let $\hat{g}(\alpha) = \max\{h(\alpha)(\alpha^2 + \alpha - 1), 2h(\alpha)\hat{g}(\alpha - 1) + h(\alpha) + 1\}$ where h is the function given by Theorem 22.

Theorem 36. *Suppose that the edges of a graph G are coloured so that no triangle is coloured with three distinct colours. Then, the vertex set of G can be partitioned into at most $\hat{g}(\alpha(G))$ monochromatic connected parts.*

To prepare the proof of Theorem 36 we need the following lemma about trees. (Another proof for this statement can be found in [6].)

Lemma 37. *Let $t \ge 1$ be an integer, and T be a tree of order at least t. Then there exist two sets $R \subseteq C \subseteq V(T)$ such that $|R| = t$, $|C| \le 2t$ and both $T[C]$ and $T[V(T) \setminus R]$ are connected.*

Proof. We describe an algorithm which gives the desired R and C. Initially they are empty sets. Let r be a vertex of T, this will be the root of the tree. Let x_0 be one of the farthest vertices from r in the tree, we will start our algorithm from this vertex. We add x_0 to C and R, and move up to its parent. (The vertex p is the parent of a vertex c if p is the first vertex on the unique path from c to the root. In this case c is a child of p.) During the algorithm we will move up and down in the tree. If we stepped up to the parent p of a vertex then first we check whether p is in C, and if not then we add it to C. If there is a child of p which is not yet in C then we step down to this child, otherwise we add p also to R and step up to its parent. If we stepped down to a child c then first we add it to C. If it has a child then we step down to that, otherwise we add c to R and step up to its parent. We stop the algorithm when the size of R reaches t.

So, for example, in the case shown on Figure 26 the algorithm runs as follows. We start from x_0 and add it to both C and R, next we step up to x_1 and add it to C, we step down to x_2 and add it to C then also to R, we step back to x_1 then down to x_3 and add it to C and to R, we step back to x_1 and add it to R. Afterwards we step up to x_4 and add it to C, step down to x_5 and add it to C, step down to x_6 add it to C and R, step up to x_5, step down to x_7 and add it to C and R, step up to x_5. The figure shows R and C at this point.

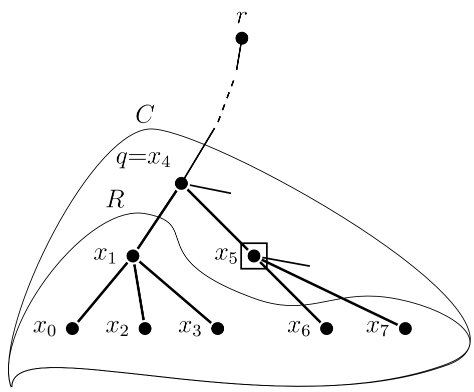

Figure 26: A state of the algorithm in the example.

We show that at the end of the algorithm the conditions for R and C are satisfied. We only add a vertex to R if it is already added to C, therefore $R \subseteq C$ throughout the whole process. We add a vertex to C if a child or the parent of it is already in C, so $T[C]$ is a subtree of T. Similarly we add a vertex to R if all its children are already in R, which means that R is removable from T leaving the tree connected. Finally, we have to check the size of C. Almost every vertex in C is also a member of R except for the vertices of a path from the top vertex of C (i.e., the nearest one from the root), say q, to the parent of the vertex added last to R. But the vertices on the path from q to x_0 are all in R except for maybe q. By the choice of x_0 there is no longer path from q to a vertex of $T[C]$ than the one going to x_0, therefore we get $|C \setminus R| \leq |R|$ during the algorithm. At the end $|R| = t$ and this implies $|C| \leq 2t$. □

Now we are ready to prove Theorem 36.

Proof of Theorem 36. We proceed by induction on $\alpha = \alpha(G)$. If $\alpha = 1$, i.e. G is complete then there is a connected monochromatic spanning subgraph of G, as desired. Suppose $\alpha \geq 2$. We shall prove that the vertex set of G can be partitioned into at most $\hat{g}(\alpha) = \max\{h(\alpha)(\alpha^2 + \alpha - 1), 2h(\alpha)\hat{g}(\alpha - 1) + h(\alpha) + 1\}$ monochromatic connected parts. We may assume that $|V(G)| \geq \hat{g}(\alpha)$. Let T_0 be a maximum monochromatic subtree of G. Assume that the colour of the edges of T_0 is colour 0. By Theorem 27, T_0 has at least $|V(G)|(\alpha^2 + \alpha - 1)^{-1}$ vertices. Therefore using the definition of $\hat{g}(\alpha)$, we have $|V(T_0)| \geq h(\alpha(G))$. And so, by Lemma 37, there exist two sets R and C with $R \subseteq C \subseteq V(T_0)$ such that $|R| = h(\alpha)$, $|C| \leq 2h(\alpha)$ and both $T_0[C]$, $T_0[V(T_0) \setminus R]$ are connected. Write $R = \{u_1, \ldots, u_{h(\alpha)}\}$ and $C = \{u_1, \ldots, u_m\}$, where $h(\alpha) \leq m \leq 2h(\alpha)$. For every i with $1 \leq i \leq m$, let U_i be the set of vertices in $V(G) \setminus V(T_0)$ that are not adjacent to u_i, but adjacent to u_j for all $j < i$. For every i with $1 \leq i \leq m$, we have $\alpha(G[U_i]) \leq \alpha - 1$ because u_i can be added to any independent set of $G[U_i]$. By the inductive assumption, for every i with $1 \leq i \leq m$, there exists a partition $\mathcal{P}_i$ of U_i such that $|\mathcal{P}_i| \leq \hat{g}(\alpha - 1)$ and, for every $U \in \mathcal{P}_i$, the induced subgraph $G[U]$ has a connected spanning monochromatic subgraph.

Let $U_0 = V(G) \setminus \left(V(T_0) \cup \left(\bigcup_{1 \leq i \leq m} U_i\right)\right)$. Recall that T_0 was a non-extendable monochromatic tree of G and the triangles in G are coloured with at most two colours. Hence, for every $v \in U_0$, since v is adjacent to every vertex of $T_0[C]$, all the edges between v and C are coloured with the same colour, say $c_v (\neq 0)$. Let ℓ be the number of colours used on edges of C and U_0, assume that $1, \ldots, \ell$ are these colours. For each i with $1 \leq i \leq \ell$, let $A_i = \{v \in U_0 \mid c_v = i\}$. Note that $\{A_1, \ldots, A_\ell\}$ is a partition of U_0. Since there is no 3-coloured triangle in G, each edge between A_i and A_j is coloured with either colour i or j for i, j with $1 \leq i, j \leq \ell$ and $i \neq j$.

We construct a multipartite digraph D on U_0 as follows. Let $A_1, \ldots, A_\ell$ be the partition classes of D. For i, j with $1 \leq i, j \leq \ell$, $i \neq j$ and $v \in A_i$, $w \in A_j$, let $(v, w) \in E(D)$ if and only if the edge $\{v, w\}$ is in G and its colour is i. Note that $\beta(D) \leq \alpha(G)$ and D has no cyclic triangle. By Theorem 22, there exist $p \leq h(\alpha)$ partite classes which dominates $V(D)$, say $B_1, \ldots, B_p$. Set $B_{p+1} = \cdots = B_{h(\alpha)} = \emptyset$. For every i with $1 \leq i \leq h(\alpha)$, let B_i' be the set of vertices in $U_0 \setminus \left(\bigcup_{1 \leq i \leq h(\alpha)} B_i\right)$ that are dominated by B_i, but not dominated by B_j for all $j < i$, and let $B_i'' = \{u_i\} \cup B_i \cup B_i'$. For each i with $1 \leq i \leq h(\alpha)$, note that $G[B_i'']$ has a connected

monochromatic spanning subgraph in colour i.
Therefore $\mathcal{P} = \left(\bigcup_{1\leq i\leq m} \mathcal{P}_i\right) \cup \{B''_1, \dots, B''_{h(\alpha)}\} \cup \{V(T_0) \setminus R\}$ is a partition of $V(G)$ satisfying that $G[U]$ has a connected spanning monochromatic subgraph for every $U \in \mathcal{P}$. Furthermore,

$$\begin{aligned} |\mathcal{P}| &\leq \sum_{1\leq i\leq m} |\mathcal{P}_i| + h(\alpha) + 1 \leq \sum_{1\leq i\leq m} \hat{g}(\alpha - 1) + h(\alpha) + 1 = \\ &= m\hat{g}(\alpha - 1) + h(\alpha) + 1 \leq 2h(\alpha)\hat{g}(\alpha - 1) + h(\alpha) + 1 \leq \hat{g}(\alpha). \end{aligned}$$

This completes the proof of Theorem 36. $\square$

Chapter 5

Monochromatic covering of complete bipartite graphs

A special case of a conjecture generally attributed to Ryser (appeared in his student, Henderson's thesis, [38]) states that intersecting r-partite hypergraphs have a transversal of at most $r-1$ vertices. This conjecture is open for $r \geq 6$. It is trivially true for $r = 2$, the cases $r = 3, 4$ are solved in [30] and in [24], and for the case $r = 5$, see [24] and [51]. The following equivalent formulation is from [30],[25]. In the sequel let $r \geq 2$.

Conjecture 4 ([38], [30], [25])**.** *In every r-colouring of the edges of a complete graph, the vertex set can be covered by the vertices of at most $r-1$ monochromatic components.*

Gyárfás and Lehel proposed a bipartite version of this conjecture [30], [44]. A complete bipartite graph G with nonempty vertex classes X and Y is referred to here as a *biclique* $[X, Y]$, and X and Y will be called the *blocks* of this biclique.

Conjecture 5 (Gyárfás [30], Lehel [44])**.** *In every r-colouring of the edges of a biclique, the vertex set can be covered by the vertices of at most $2r-2$ monochromatic components.*

First we will see here that Conjecture 5, if true, is best possible. Let $G^* = [A, B]$ be a biclique with $|A| = r-1, |B| = r!$, and label the vertices of A with $\{1, 2, \ldots, r-1\}$ and those of B with the $(r-1)$-length permutations of the elements of $\{1, 2, \ldots, r\}$. For $k \in A$ and $\pi = j_1 j_2 \ldots j_{r-1} \in B$, let the colour of the edge $\{k, \pi\}$ be j_k.
Since each vertex in B is incident to $r-1$ edges of distinct colour, every monochromatic component of G^* is a star with $(r-1)!$ leaves centered at A. Furthermore, G^* has a vertex cover with $2r-2$ monochromatic components, just take the r monochromatic stars centered at vertex $r-1$, and add one edge from each vertex $k = 1, 2, \ldots, r-2$ of A.

Proposition 38 (Gyárfás [30]). *The vertex set of G^* cannot be covered with less than $2r-2$ monochromatic components.*

Proof. Let $\mathcal{C}$ be a cover of $V(G^*) = A \cup B$ by monochromatic stars centered in A. Let a_k denote the number of monochromatic stars of $\mathcal{C}$ on vertex $k \in A$. We may assume that $a_1 \leq a_2 \leq \cdots \leq a_{r-1}$.

We show first that $a_i \geq i+1$ holds for some $1 \leq i \leq r-1$. Suppose on the contrary that $a_{r-1} < r, a_{r-2} < r-1, \ldots, a_1 < 2$. Thus we can select a colour $j_{r-1} \in \{1, \ldots, r\}$ different from the a_{r-1} colours of all stars of $\mathcal{C}$ centered at $r-1$. Then we can select a new colour $j_{r-2} \in \{1, \ldots, r\} \setminus \{j_{r-1}\}$ different from the a_{r-2} colours of all stars of $\mathcal{C}$ centered at $r-2$, etc. Thus we end up by selecting $r-1$ distinct colours $j_1, \ldots, j_{r-1}$. This is a contradiction since the $(r-1)$-permutation $j_1 j_2 \ldots j_{r-1} \in B$ is uncovered by $\mathcal{C}$.

Now let $a_i \geq i+1$, for some $1 \leq i \leq r-1$, then for the number of stars in $\mathcal{C}$ we have

$$\sum_{k=1}^{r-1} a_k = \sum_{k=1}^{i-1} a_k + \sum_{k=i}^{r-1} a_k \geq (i-1) + (i+1)(r-i).$$

Since

$$(i-1) + (i+1)(r-i) = -i^2 + ri + r - 1 \geq 2r - 2$$

holds for every $1 \leq i \leq r-1$, the proposition follows. □

It is worth noting that Conjecture 5 (similarly to Conjecture 4) becomes obviously true if the number of monochromatic components is just one larger than stated in the conjecture.

Proposition 39 (Gyárfás [30]). *In every r-colouring of the edges of a biclique, the vertex set can be covered by the vertices of at most $2r-1$ monochromatic components.*

Proof. For an edge $\{u, v\}$ of the biclique G, consider the monochromatic component (double star) formed by the edges in the colour of $\{u, v\}$ incident to u or v. In all other colours consider the (at most $r-1$) monochromatic stars centered at u and at v. This gives $2r-1$ monochromatic components covering the vertices of G. □

In Section 5.1 we show that Conjecture 5 can be reduced to design-like conjectures: for example, one can assume that all components of all colour classes are complete bipartite graphs. It is worth noting that similar reduction is not known for Conjecture 4.

We shall prove Conjecture 5 for $r = 2, 3, 4, 5$ in Sections 5.2.1 and 5.2.2, in fact in stronger forms defined in Section 5.1 (Propositions 41, 42 and Theorems 43, 44).

5.1 Equivalent formulations, notations

We shall see that Conjecture 5 is equivalent to further design-like conjectures, thus an r-colouring will also be called a partition of the edge set into r subgraphs. Let a biclique $[X, Y]$ be partitioned into graphs $G_1, G_2, \ldots, G_r$, then we will say that i is the colour of the edges in G_i $(i = 1, \ldots, r)$.

5.1.1 Spanning partition

Let us call a graph partition $G_1, \ldots, G_r$ of biclique G a *spanning partition* if each vertex $v \in V(G)$ is included in every $V(G_i)$, $i = 1, \ldots, r$. Notice that it is enough to prove Conjecture 5 for spanning partitions. Indeed, assuming that $v \notin V(G_1)$ and $\{v, w\} \in E(G_2)$, just take the at most $r-2$ monochromatic components from $G_3, \ldots, G_r$ that contain v and add the at most r monochromatic components from $G_1, G_2, \ldots, G_r$ that contain w, together they form a cover of all vertices of G with at most $2r-2$ monochromatic components. Thus we have the following equivalent form of Conjecture 5.

Conjecture 6. *If a biclique has a spanning partition into r graphs, then its vertex set can be covered by at most $2r-2$ monochromatic components.*

5.1.2 Partition into bi-equivalence graphs

A *bi-equivalence graph* is a bipartite graph whose connected components are bicliques; the *width* of a bi-equivalence graph is the number of its components. (A graph whose connected components are complete graphs, i.e. cliques, is usually called equivalence graph, that is the reason of this name.)

Conjecture 7. *If a biclique has a spanning partition into r bi-equivalence graphs, then its vertex set can be covered by at most $2r-2$ biclique components.*

Conjecture 7 and 6 are equivalent. On the one hand, it is clear that validity of Conjecture 6 implies that Conjecture 7 is also true.
On the other hand, suppose we have an r-colouring of a biclique $G = [X, Y]$ such that some monochromatic component C, say in colour 1, is not a biclique. Let $x \in X$, $y \in Y$ be non-adjacent vertices in C, w.l.o.g. $\{x, y\}$ has colour 2. Observe that the $2(r-2)$ monochromatic stars in colours $3, \ldots, r$ centered at x and at y, plus the component C, and the component in colour 2 containing $\{x, y\}$ cover $V(G)$, leading to a cover with at most $2r-2$ monochromatic components. Thus Conjecture 6 follows from Conjecture 7.

Let a biclique $[X, Y]$ be partitioned into the bi-equivalence graphs $G_1, G_2, \ldots, G_r$. Any connected component of G_i is a biclique, its vertex classes will be called *blocks in colour i*.

Denote by $B_i[u_1, \ldots, u_k]$ the connected component of G_i which contains the vertices $u_1, \ldots, u_k$, if they are in the same component of G_i, and in this case let $X_i[u_1, \ldots, u_k] = X \cap V(B_i[u_1, \ldots, u_k])$ and $Y_i[u_1, \ldots, u_k] = Y \cap V(B_i[u_1, \ldots, u_k])$ be the corresponding blocks. Otherwise we set $B_i[u_1, \ldots, u_k] = \emptyset$, $X_i[u_1, \ldots, u_k] = Y_i[u_1, \ldots, u_k] = \emptyset$.
Note that $B_i[u] \neq \emptyset$ for any $u \in V(G)$ in a spanning partition. In the sequel we will also use the fact that for any colour $i \in \{1, 2, \ldots, r\}$ and any vertices $u, v \in V(G)$, the blocks $X_i[u]$ and $X_i[v]$ are either disjoint or equal.

5.1.3 Antichain partition

Let us call a spanning bi-equivalence graph partition $G_1, \ldots, G_r$ of biclique G an *antichain partition* if no blocks properly contain each other, that is if no colours $i, j \in \{1, \ldots, r\}$ and vertices $u, v \in V(G)$ exist such that $X_i[u] \subsetneq X_j[v]$ or $Y_i[u] \subsetneq Y_j[v]$.
If $v \in X$ and $|X_i[v]| = 1$ (or $v \in Y$ and $|Y_i[v]| = 1$) then we call vertex v a *singleton* block in colour i. Note that if a colouring has the antichain property (i.e., we have an antichain partition), then a singleton block in some colour is a singleton in every colour, in this case we just say that v is a singleton.

It turns out that it is enough to prove Conjecture 7 for antichain partitions. Indeed, assume that in a spanning partition there are two blocks properly containing each other, that is $X_1[y] \subsetneq X_2[x]$, for some biclique components $B_1[y]$, $B_2[x]$ and vertices $x \in X$, $y \in Y$, $x \notin X_1[y]$. The colour of the edge $\{x, y\}$ is neither 1 nor 2, w.l.o.g. it is 3. Since $B_3[y] = B_3[x]$ and $X_1[y] \subseteq X_2[x]$, the collection

$$\{B_i[x] \ : \ i \in \{1, 2, \ldots, r\}\} \cup \{B_i[y] \ : \ i \in \{1, 2, \ldots, r\} \setminus \{1, 3\}\}$$

is a cover with at most $2r - 2$ monochromatic components. Thus we obtain the following equivalent form of Conjecture 5.

Conjecture 8. *If a biclique has an antichain partition into r bi-equivalence graphs, then its vertex set can be covered by at most $2r - 2$ biclique components.*

Our example in Proposition 38 showing that Conjecture 5 is sharp is not an antichain partition (not even a spanning partition). It is possible that for antichain partitions (or even for spanning partitions) a stronger result holds.

Question 9. *Suppose that a biclique has an antichain partition into r bi-equivalence graphs. Can one cover its vertex set by at most r biclique components?*

For $r = 2, 3, 4$ the answer to Question 9 is affirmative (see Section 5.2.1). Note that one-factorizations of $K_{r,r}$ show that one cannot expect a cover with less than r biclique components.

5.1.4 Reduced colouring

Finally we note an important reduction used extensively in the proofs later. We call a pair $u, v \in X$ or $u, v \in Y$ *equivalent* if in every bi-equivalence graph of the bi-equivalence graph partition of the biclique G, u and v belong to the same block. We may assume w.l.o.g. that there is no pair of equivalent vertices, and in this case we say that *the colouring is reduced.* Indeed, if there were two vertices $u, v \in X$ and for every $w \in Y$, the edges $\{u, w\}$ and $\{v, w\}$ have the same colour, then v could be added to any monochromatic component of $G \setminus \{v\}$ containing u. Hence if Conjecture 8 holds for $G \setminus \{v\}$ then it also holds for G.

In a reduced r-colouring of a biclique, the number of vertices is bounded by a function of r. In fact, one can easily see the following.

Proposition 40. *Suppose a biclique $[X, Y]$ has a partition into r bi-equivalence graphs and no two vertices of X are equivalent. Then $|X| \leq r!$, and equality is possible.*

Proof. It is easy to check that the partition of the graph G^* defined before Proposition 38 is a reduced one, hence the second statement follows.

To see the first statement, the case $r = 1$ is obvious. Assuming it is true for some $r \geq 1$, suppose indirectly that $|X| \geq (r + 1)! + 1$ in some partition into $r + 1$ bi-equivalence graphs. Then for any fixed $v \in Y$ there are $r! + 1$ edges of the same colour from v, say in colour $r + 1$, to $A \subseteq X$. Let B be the set of vertices in Y that send edges in at least two different colours to A. By the assumption $B \neq \emptyset$ and since the colour class $r + 1$ is a bi-equivalence graph, $[A, B]$ has no edge of colour $r+1$. This means no two vertices of A are equivalent in the induced r-partition on $[A, B]$, and thus $|A| > r!$ contradicts the inductive hypothesis. $\square$

5.2 Bi-equivalence partitions for small r values

In the present section we prove Conjecture 5 for $r \leq 5$.

5.2.1 The case of $r = 2, 3$ and 4

First we show that Conjecture 5 (in its original form) is true for $r = 2$ and $r = 3$.

Proposition 41.

(i) If the edges of a biclique G are coloured with 2 colours, then the vertex set can be covered by the vertices of at most 2 monochromatic components.

(ii) If the edges of a biclique G are coloured with 3 colours, then the vertex set can be covered by the vertices of at most 4 monochromatic components.

Proof. To show (i) let C_1 be a monochromatic component in colour 1. Set $X_1 = V(C_1) \cap X$, $Y_1 = V(C_1) \cap Y$. Suppose $X_1 \neq X$ and $Y_1 \neq Y$. Then the edges between X_1 and $Y \setminus Y_1$ as well as the edges between Y_1 and $X \setminus X_1$ are coloured with colour 2, and they form monochromatic components C_2 and C_3 on $X_1 \cup (Y \setminus Y_1)$ and $Y_1 \cup (X \setminus X_1)$, respectively. (These components could coincide if there is an edge in colour 2 between X_1 and Y_1, or between $X \setminus X_1$ and $Y \setminus Y_1$.) The components C_2 and C_3 cover the vertex set of G, as desired. If one (or both) of $X \setminus X_1$, $Y \setminus Y_1$ is empty then the corresponding monochromatic component does not exists, in this case it can be substituted by C_1 in the cover.

The proof of (ii) is similar. Let C_1 be a monochromatic component in colour 1. Set $X_1 = V(C_1) \cap X$, $Y_1 = V(C_1) \cap Y$. Suppose $X_1 \neq X$ and $Y_1 \neq Y$. Then the edges between X_1 and $Y \setminus Y_1$ as well as the edges between Y_1 and $X \setminus X_1$ are coloured with colour 2 and 3, and so they form two bicliques whose edges are coloured with 2 colours. Using (i) we have that their vertex set can be covered by the vertices of at most 2 monochromatic components. Hence the at most 4 components together form a cover of G. If one (or both) of $X \setminus X_1$, $Y \setminus Y_1$ is empty then the corresponding biclique, and so the monochromatic components does not exists but they can be substituted by C_1 in the cover. $\square$

Notice that with the argument used above we would get that if the edges of a biclique G are coloured with 3 colours, then the vertex set can be covered by the vertices of at most 8 monochromatic components. (While the conjecture states that 6 components are enough.)

Next, we show that the answer for Question 9 is positive for $r = 3$.

Proposition 42. *If a biclique has an antichain partition into* 3 *bi-equivalence graphs, then one of them has at most* 3 *connected components.*

Proof. Let G_i, $i = 1, 2, 3$, be the bi-equivalence graphs in a reduced antichain partition of a biclique $[X, Y]$. We may assume that $|X| > 3$, since otherwise the width of each bi-equivalence graph is at most 3. Let $y \in Y$, we have $X = X_1[y] \cup X_2[y] \cup X_3[y]$. We may assume that $|X_1[y]| \geq 2$, let $x_1, x_2 \in X_1[y]$. From $Y_1[x_1] = Y_1[x_2] (= Y_1[y])$ it follows that $Y_2[x_1] \cup Y_3[x_1] = Y_2[x_2] \cup Y_3[x_2] (= Y \setminus Y_1[y])$. The vertices x_1 and x_2 are not equivalent hence we conclude that $Y_2[x_2] = Y_3[x_1]$, $Y_3[x_2] = Y_2[x_1]$, because any two blocks in the same colour are disjoint or coincide. Therefore $Y_2[y] \subseteq Y \setminus (Y_2[x_1] \cup Y_2[x_2]) = Y_1[y]$, and so $Y_2[y] = Y_1[y]$. This yields $Y = Y_2[y] \cup Y_2[x_1] \cup Y_2[x_2]$, thus the width of G_2 is at most 3. $\square$

One can prove the following, in some sense stronger, statement for $r = 3$, see its proof in [7]. Let a biclique $[X, Y]$ be partitioned into 3 bi-equivalence graphs. If one of those has more than three nontrivial components, then some of the other two is spanning and has two connected components.

Now we turn to the case $r = 4$, and answer Question 9 affirmatively (hence Conjecture 8 is also verified in this case).

Theorem 43. *If a biclique has an antichain partition into* 4 *bi-equivalence graphs, then its vertex set can be covered by at most* 4 *monochromatic components of the same colour, or equivalently, one of the bi-equivalence graphs has width at most* 4.

Proof. Let G_i, $i = 1, 2, 3, 4$, be the bi-equivalence graphs in a reduced antichain partition of a biclique $[X, Y]$.

First we show that if $|X_i[u]| \leq 2$ for every colour i and vertex u, then the statement holds. To see this, let $y \in Y$, we have $X = X_1[y] \cup X_2[y] \cup X_3[y] \cup X_4[y]$. Let s be the number of blocks of G_1 in $X \setminus X_1[y]$, their union is equal to the union of the three blocks $X_2[y], X_3[y], X_4[y]$. Recall that in an antichain partition singleton blocks are singletons in every colour. From this and from the assumption that every other block has size 2, it follows that $s = 3$. Thus the width of G_1 is 4.

Thus we may assume that there are three distinct vertices, $x_1, x_2, x_3 \in X$ in some block of G_1. Let

$$Y(c_1, c_2, c_3) = \{y \in Y \mid \{y, x_i\} \text{ is coloured with } c_i,\ i = 1, 2, 3\}.$$

The three-tuple (c_1, c_2, c_3) will be called the type of the subset $Y(c_1, c_2, c_3)$. In terms of this notation $Y(1, 1, 1) \neq \emptyset$. When the wildcard character $*$ is used for a colour, then the colour of the corresponding edge between $\{x_1, x_2, x_3\}$ and the set of that type is undetermined (e.g. $Y(3, 3, 4) \subseteq Y(3, *, 4)$ is true).
In a bi-equivalence graph partition certain types cannot coexist as is expressed in the next rule. If a, b are distinct colours, then at least one of the sets $Y(a, a, *)$ and $Y(a, b, *)$ must be empty. Indeed, if $y_1 \in Y(a, a, *)$ and $y_2 \in Y(a, b, *)$, then (y_2, x_1, y_1, x_2) is a path belonging to some biclique of G_a, hence the edge $\{x_2, y_2\}$ must have colour a, and not b. This rule remains valid also when relabelling the vertices x_1, x_2, x_3, that is when the colours in the types are moved to different positions. Thus, for instance, types $(2, *, 2)$ and $(2, *, 3)$ cannot coexist.

We claim that there is no (nonempty) “three of a kind” type in $Y \setminus Y(1, 1, 1)$. Assume on the contrary that $Y(2, 2, 2) \neq \emptyset$. Since x_1 and x_2 are not equivalent, we have $Y(3, 4, *) \neq \emptyset$, $Y(4, 3, *) \neq \emptyset$, and therefore, $Y(3, 3, *) = \emptyset$, $Y(4, 4, *) = \emptyset$. Moreover, this must hold for the pair x_1, x_3 and the pair x_2, x_3, which is clearly impossible.

At least one of $Y(2, 2, 3)$ and $Y(2, 2, 4)$ is empty. To see this, assume $Y(2, 2, 3) \neq \emptyset$ and $Y(2, 2, 4) \neq \emptyset$. Then again $Y(3, 4, *) \neq \emptyset$, $Y(4, 3, *) \neq \emptyset$. Moreover, $Y(3, 4, *) = Y(3, 4, 2)$,

$Y(4,3,*) = Y(4,3,2)$ and there is no other nonempty type. This yields

$$Y = Y(1,1,1) \cup Y(2,2,3) \cup Y(2,2,4) \cup Y(3,4,2) \cup Y(4,3,2),$$

in particular $Y_3[x_3] \cup Y_4[x_3] = Y_2[x_1]$, violating the antichain property.

Now w.l.o.g. assume that either $Y(2,2,3) \neq \emptyset$ or no (nonempty) pair type exists in $Y \setminus Y(1,1,1)$. In both cases every (nonempty) type in $Y \setminus Y(1,1,1)$ has a colour 3. Then the components $B_3[x_i], i = 1,2,3$, form a cover provided $Y_3[z] \cap (Y \setminus Y(1,1,1)) \neq \emptyset$, for all $z \in X$. If some z does not satisfy this, then by the antichain property, $Y(1,1,1) = Y_3[z]$, and $B_3[x_i], i = 1,2,3$, and $B_3[z]$ together form a cover. The width of G_3 is at most 4. □

5.2.2 The case of $r = 5$

In this section we shall verify Conjecture 8 for $r = 5$, in a stronger form. Actually we will show that under the appropriate conditions there is a cover with at most $2r - 2 = 8$ monochromatic components in the same colour, or equivalently, one of the bi-equivalence graphs of the partition has width at most 8.

Theorem 44. *If a biclique has an antichain partition into 5 bi-equivalence graphs, then its vertex set can be covered by at most 8 monochromatic components of the same colour.*

The proof of Theorem 44 is organized as follows. Let $G_i, i = 1,2,3,4,5$, be the bi-equivalence graphs in a reduced antichain partition of the biclique $[X,Y]$. First we prove two lemmas, and from them we conclude that in some colour there exists a block of size at least 9, otherwise we are done. Then we define the notation of type (similarly as we did in the proof of Theorem 43), and state some rules on them. We finish the proof by case analysis.

5.2.3 Existence of a large block

We need the following two technical lemmas.

Lemma 45. *If each $G_i, i = 1,\dots,5$, has width at least 6, then $[X,Y]$ contains at most two singletons in both vertex class.*

Proof. Suppose on the contrary that one class has three singletons, say $x_1, x_2, x_3 \in X$ with $|X_i[x_j]| = 1$, for every $1 \leq i \leq 5$, and $1 \leq j \leq 3$. Then taking any $y \in Y$, we may assume that $\{y, x_1\} \in E(G_1)$, $\{y, x_2\} \in E(G_2)$ and $\{y, x_3\} \in E(G_3)$. In particular, we obtain that $X = \{x_1, x_2, x_3\} \cup X_4[y] \cup X_5[y]$.

For any $z \in X_4[y]$, we have $X_5[z] \cap X_5[y] = \emptyset$, hence by the antichain property, $X_5[z] = X_4[y]$. Therefore G_5 has five components: $B_5[x_1]$, $B_5[x_2]$, $B_5[x_3]$, $B_5[z]$, $B_5[y]$, a contradiction. □

Lemma 46. *Let each $G_i, i = 1, \dots, 5$, have width at least 9. If $[X, Y]$ contains at most two singletons in both of its vertex classes, then there is a colour i and a vertex u for which $|X_i[u]| \geq 9$ or $|Y_i[u]| \geq 9$.*

Proof. Assume that for every colour i and vertex u we have $|X_i[u]| \leq t$ and $|Y_i[u]| \leq t$. Let G_1 be the graph with the maximum number of edges among $G_i, i = 1, \dots, 5$. The trivial inequality $|E(G)| \leq 5|E(G_1)|$ will give us a lower bound on t.

For a vertex $u \in X$ we have $Y = Y_1[u] \cup Y_2[u] \cup Y_3[u] \cup Y_4[u] \cup Y_5[u]$. From $|Y_i[u]| \leq t$ we get $|Y| \leq 5t$. Similarly it follows that $|X| \leq 5t$. Since G contains at most two singletons, and the width of G_1 is at least 9 we have $5t \geq |Y| \geq 2 \cdot 1 + 7 \cdot 2 = 16$, therefore $t \geq 4$.

Let $\underline{x}$ and $\underline{y}$ be vectors which contain the sizes of the components of G_1 in X and in Y, respectively. Our assumptions on G_1 mean that the length of $\underline{x}$ and $\underline{y}$ is at least 9, they have at most two elements equal to 1, and all their elements are at most t. Using this notation $|E(G_1)| = \underline{x} \cdot \underline{y}$, and $|E(G)| = |X||Y| = (\underline{x} \cdot \underline{1})(\underline{y} \cdot \underline{1})$, where $\underline{1}$ is the constant 1 vector with appropriate length. We are going to investigate the function

$$\mathrm{diff}(\underline{x}, \underline{y}) = |E(G)| - 5|E(G_1)| = (\underline{x} \cdot \underline{1})(\underline{y} \cdot \underline{1}) - 5(\underline{x} \cdot \underline{y}),$$

and determine its minimum over all possible values of $\underline{x}$ and $\underline{y}$. If for a given value of t this function is positive for any $\underline{x}$, $\underline{y}$, then there is no partition of G into graphs with the above conditions.

In the first steps we minimize $\mathrm{diff}(\underline{x}, \underline{y})$, for any fixed $|X|$ and $|Y|$, that is we maximize $|E(G_1)| = \underline{x} \cdot \underline{y}$.

Step 1: We may assume that the length of $\underline{x}$ is equal to 9, and so the length of y is also 9. This is because otherwise we could join two components of G_1 and increase the number of edges. So we have $\underline{x} = (x_1, \dots, x_9)$ and $\underline{y} = (y_1, \dots, y_9)$.

Step 2: We can reorder the components of G_1 such that $\underline{y}$ is ordered non-increasingly. After that we may assume that the elements of $\underline{x}$ are also ordered non-increasingly. Indeed, otherwise we could swap two elements with $x_i < x_j$ for $1 \leq i < j \leq 9$ and this operation would not decrease the value of $\underline{x} \cdot \underline{y}$. (The increment is $(x_j - x_i)(y_i - y_j) \geq 0$.) Hence we get $y_1 \geq y_2 \geq \cdots \geq y_9$ and $x_1 \geq x_2 \geq \cdots \geq x_9$.

Step 3: If we increase an element x_i of $\underline{x}$ by some constant c and decrease x_j for $j > i$ by the same constant, we cannot decrease the number of edges of G_1. (The increment is $c(y_i - y_j) \geq 0$.) By repeated use of this operation (observing the condition that each element of $\underline{x}$ and $\underline{y}$ is at most t, and these vectors contain at most two elements equal to 1) we obtain that $x_1 = \cdots = x_p = t$, $t > x_{p+1} \geq 2$, $x_{p+2} = \cdots = x_7 = 2$, $x_8 = x_9 = 1$ and similarly $y_1 = \cdots = y_q = t$, $t > y_{q+1} \geq 2$,

$y_{q+2} = \cdots = y_7 = 2$, $y_8 = y_9 = 1$. From $|X| \leq 5t$ it follows that $p < 5$, and similarly we get $q < 5$.

Thus for a given $|X|$ and $|Y|$, the maximum value $|E(G_1)| = \underline{x} \cdot \underline{y}$ is determined by the vectors $\underline{x}, \underline{y}$ standardized as above. In the next steps we further minimize $\text{diff}(\underline{x}, \underline{y})$ by changing also $|X|$ and $|Y|$.

Step 4: If $x_{p+1} \neq 2$ then let $\underline{x}^-$ and $\underline{x}^+$ be vectors almost the same as $\underline{x}$, but at the $(p+1)$-th position they have $x_{p+1} - 1 \geq 2$ and $x_{p+1} + 1 \leq t$, respectively. We claim that $\text{diff}(\underline{x}^-, \underline{y})$ or $\text{diff}(\underline{x}^+, \underline{y})$ is not greater than $\text{diff}(\underline{x}, \underline{y})$. Indeed, $\text{diff}(\underline{x}, \underline{y}) - \text{diff}(\underline{x}^-, \underline{y}) = \text{diff}(\underline{x}^+, \underline{y}) - \text{diff}(\underline{x}, \underline{y}) = |Y| - 5y_{p+1}$ which means that $\text{diff}(\underline{x}, \underline{y})$ is a middle element of an arithmetic progression between $\text{diff}(\underline{x}^-, \underline{y})$ and $\text{diff}(\underline{x}^+, \underline{y})$. Thus we may assume that $x_{p+1} = 2$ and similarly $y_{q+1} = 2$ (with appropriate p and q). Furthermore we assume $p \leq q$, and set $r = q - p \geq 0$.

Step 5: Now we can express $\text{diff}(\underline{x}, \underline{y})$ as a function of p and r in the following way.

$$\begin{aligned} \text{diff}(\underline{x}, \underline{y}) &= (\underline{x} \cdot \underline{1})(\underline{y} \cdot \underline{1}) - 5(\underline{x} \cdot \underline{y}) \\ &= (tp + 2(7-p) + 2)(t(p+r) + 2(7-p-r) + 2) \\ &\quad -5(t^2 p + 2tr + 4(7-p-r) + 2), \end{aligned}$$

In this expression the coefficient of r is $p(t-2)^2 + 6(t-2) > 0$, as $t \geq 4$. Therefore $\text{diff}(\underline{x}, \underline{y})$ is minimal if $r = 0$, that is $p = q$, and so $\underline{x} = \underline{y}$. In this case $\text{diff}(\underline{x}, \underline{x}) = p^2(t^2 - 4t + 4) + p(-5t^2 + 32t - 44) + 106$, thus the formula has an extremum if $\frac{\mathrm{d}}{\mathrm{d}p}\text{diff}(\underline{x}, \underline{x}) = 0$, that is $p = \frac{5t^2 - 32t + 44}{2(t^2 - 4t + 4)}$. (This extremum is a minimum since $\frac{\mathrm{d}^2}{\mathrm{d}p^2}\text{diff}(\underline{x}, \underline{x}) = 2(t^2 - 4t + 4) = 2(t-2)^2 > 0$, because $t \geq 4$.)

From the above formula we get $p = 1.5$, for $t = 8$, which gives that the minimum value of $\text{diff}(\underline{x}, \underline{y})$ for any $\underline{x}$, $\underline{y}$ is at least $25 > 0$. (Actually the minimum is 34 which is taken on the integer values $p = 1$ and $p = 2$.) Thus $|E(G)| \leq 5|E(G_1)|$ cannot hold for $t = 8$, it completes the proof. $\square$

Applying Lemma 45 and 46 we have the following corollary.

Corollary 47. *If each $G_i, i = 1, \ldots, 5$, has width at least 9, then there is a colour i and a vertex u for which $|X_i[u]| \geq 9$ or $|Y_i[u]| \geq 9$.*

5.2.4 Types and rules on them

Let $X_1[x_1, x_2, \ldots, x_9] \neq \emptyset$ be a block containing at least nine distinct vertices. Similarly to the proof of Theorem 43, for a sequence of given colours $c_1, \ldots, c_9$, let

$$Y(c_1, \ldots, c_9) = \{y \in Y \mid \{y, x_i\} \text{ is coloured with } c_i,\ i = 1, \ldots, 9\}.$$

The nine-tuple $(c_1, \ldots, c_9)$ will be called the type of the subset $Y(c_1, \ldots, c_9) \subseteq Y$. Again, when the wildcard character $*$ is used for the i-th colour position in a type, then the colour of the corresponding edges to x_i are undetermined.

In a bi-equivalence graph partition certain types cannot coexist as is expressed in the next rule.

Type rule. *If a, b are two distinct colours, then at least one of the sets $Y(a, a, *, \ldots, *)$ and $Y(a, b, *, \ldots, *)$ must be empty.*

Indeed, if $y_1 \in Y(a, a, *, \ldots, *)$ and $y_2 \in Y(a, b, *, \ldots, *)$, then (y_2, x_1, y_1, x_2) is a path belonging to G_a, hence the edge $\{x_2, y_2\}$ must have colour a, and not b.

Notice that the Type rule remains valid when permuting colours and/or when relabelling the vertices $x_1, x_2, \ldots, x_9$, that is when the colours in the types are moved to different positions. Thus, for instance, types $(*, 5, *, \ldots, *, 3)$ and $(*, 3, *, \ldots, *, 3)$ cannot coexist.

We will need a simple corollary of the antichain property as follows.

Starring rule. *If $Y_c[w] \subseteq Y(c_1, \ldots, c_9)$, for some $w \in X$, then equality must hold. In other words vertex w "stars" the set $Y(c_1, \ldots, c_9)$ in colour c.*

This is because $Y(c_1, \ldots, c_9) \subseteq Y(c_1, *, \ldots, *) = Y_{c_1}[x_1]$.

Distinguishing rule 1. *If $Y(2, 2, *, \ldots, *) \neq \emptyset$ and $Y(3, 3, *, \ldots, *) \neq \emptyset$, then $Y(4, 4, *, \ldots, *) = \emptyset$ and $Y(5, 5, *, \ldots, *) = \emptyset$, furthermore, $Y(4, 5, *, \ldots, *) \neq \emptyset$ and $Y(5, 4, *, \ldots, *) \neq \emptyset$.*

To see this recall that no equivalent vertices exist in the colouring, in particular x_1, x_2 must be distinguished by the components in colours 4 and 5. If $Y(4, 4, *, \ldots, *) \neq \emptyset$, then by the Type rule, $B_i[x_1] = B_i[x_2]$ for every $i = 1, 2, 3, 4$, implying $B_5[x_1] = B_5[x_2]$, hence x_1, x_2 would be equivalent.

An immediate corollary of Distinguishing rule 1 is stated for convenience as follows.

Distinguishing rule 2. *At least one of $Y(2, 2, 2, *, \ldots, *)$ and $Y(3, 3, 3, *, \ldots, *)$ must be empty.*

5.2.5 Case analysis

Now we are ready to prove Theorem 44.

Proof of Theorem 44. From Corollary 47 it follows that there is a block containing at least nine distinct vertices, $X_1[x_1, \ldots x_9] \neq \emptyset$. That is, $Y(1, \ldots, 1) \neq \emptyset$. We shall proceed with investigating the partition of $Y' = Y \setminus Y(1, \ldots, 1)$ into different types. Note that if $Y(c_1, \ldots, c_9) \subseteq Y'$, then we have $c_i \neq 1$, for every $i = 1, \ldots, 9$. Let $Y(c_1, \ldots, c_9) \subseteq Y'$. Since $c_i \in \{2, 3, 4, 5\}$ for $1 \leq i \leq 9$, some colour must repeat at least three times.

We shall consider the following three cases:

(1) there is a (nonempty) type in Y' such that a colour repeats at least five times;

(2) no (nonempty) type in Y' repeats a colour more than four times, and there is a (nonempty) type repeating a colour four times;

(3) no (nonempty) type in Y' repeats a colour more than three times.

In the sequel when we write "w.l.o.g. we assume", we mean "by appropriately permuting the colours and relabelling $x_1, x_2, \ldots, x_9$ we may assume".

Case 1: *there is a (nonempty) type in Y' such that a colour repeats at least five times, say $Y(2,2,2,2,2,*,\ldots,*) \neq \emptyset$.*

Observe that colour 2 cannot repeat seven times. Indeed, in every (nonempty) type in Y' different from $(2,2,2,2,2,2,2,*,*)$ colour 2 is not used on the first seven positions, by the Type rule. Hence one colour among $3,4$, and 5 must repeat at least three times contradicting Distinguishing rule 2. Thus w.l.o.g. we assume that $Y(2,2,2,2,2,*,3,*,*) \neq \emptyset$.

A similar pigeon hole argument shows that in every (nonempty) type in Y' different from $(2,2,2,2,2,*,*,*,*)$ colour 3 must be used on the first five positions, otherwise Distinguishing rule 2 is violated. Therefore by the Type rule, $Y_3[x_7] = Y(*,\ldots,*,3,*,*) \subseteq Y(2,2,2,2,2,*,*,*,*)$, thus by the Starring rule, $Y_3[x_7] = Y(2,2,2,2,2,*,*,*,*)$ follows. Then we obtain that

$$Y' = \left(\bigcup\{Y_3[x_i] \mid 1 \leq i \leq 5\}\right) \cup Y_3[x_7].$$

If the six connected components $B_3[x_i], 1 \leq i \leq 5$ and $B_3[x_7]$ do not cover X, then there is an uncovered vertex $w \in X$ which stars $Y(1,\ldots,1)$ in colour 3, by the Starring rule. In this case $B_3[x_i], 1 \leq i \leq 5$, $B_3[x_7]$, and $B_3[w]$ cover Y (thus the whole vertex set of G).

Consequently, in either case G_3 has width at most 7.

Case 2: *no (nonempty) type in Y' repeats a colour more than four times, and there is a (nonempty) type repeating a colour four times, say $Y(2,2,2,2,c_5,\ldots,c_9) \neq \emptyset$, where $c_5,\ldots,c_9 \neq 2$.*

We also know that among the five colours, $c_5,\ldots,c_9$, there are two distinct colours, w.l.o.g. we assume that $c_5 = 3$ and $c_6 = 4$.

Assume now that in every (nonempty) type in Y' different from $(2,2,2,2,*,\ldots,*)$ colour 3 is used somewhere on the first four positions. Then a similar argument that we used in Case 1 shows that the width of G_3 is at most 6. By the same reason repeated for colour 4, it remains to consider the situation when, for each colour 3 and 4, there is a (nonempty) type in Y' different from $(2,2,2,2,*,\ldots,*)$ missing 3 and 4 on the first four positions, respectively.

Since a colour cannot repeat three times on the first four positions, w.l.o.g. we have that $Y(4,4,5,5,*,\ldots,*) \neq \emptyset$, moreover $Y(c_1,c_2,c_3,c_4,*,\ldots,*) \neq \emptyset$, where among c_1,c_2,c_3,c_4 both colours 3 and 5 repeat twice. By the Type rule, either $(c_1, c_2, c_3, c_4) = (5, 5, 3, 3)$ or $(c_1, c_2, c_3, c_4) = (3, 3, 5, 5)$. In each case Distinguishing rule 1 is violated.

Case 3: *no (nonempty) type in Y' repeats a colour more than three times.*

Then by the pigeon hole principle, each (nonempty) type in Y' has a colour repeated three times. Furthermore, if a type uses just three colours, then each of its three colours is repeated exactly three times.

Let $Y(c, c, c, *, \ldots, *) \neq \emptyset$, for some $c = 2, 3, 4$, or 5. If each (nonempty) type uses colour c at some position, then either the connected components $B_c[x_i], 3 \leq i \leq 9$ cover X, or some $w \in X$ stars $Y(1, \ldots, 1)$ in colour c, hence $B_c[x_i], 3 \leq i \leq 9$ and $B_c[w]$ cover Y (thus the whole vertex set of G). In each situation G_c has width at most 8. We claim that this must happen for some c.

Assume that colour 2 repeats three times in some (nonempty) type, and some other (nonempty) type misses colour 2. W.l.o.g. let $T_2 = (3, 3, 3, 4, 4, 4, 5, 5, 5)$ be a (nonempty) type. By repeating the same idea, we see that, for every $c = 3, 4, 5$, some (nonempty) type T_c misses c.
Thus T_3 has three triplets in colours $2, 4, 5$ at some positions. The last three positions of T_3 is not a triplet in 5 due to Distinguishing rule 2 and the Type rule. W.l.o.g. assume that $T_3 = (5, 5, *, 5, *, \ldots, *)$. Then again, by Distinguishing rule 2 and the Type rule, it follows that $T_3 = (5, 5, 4, 5, 2, 2, 4, 4, 2)$ (the last three positions can be permuted).
Finally, for the possible positions of the three 5's of T_4 with respect to T_2 and T_3, we conclude as before that $T_4 = (*, *, 5, *, 5, 5, *, *, *)$. This contradicts Distinguishing rule 1 on positions 5 and 6 (meaning that x_5 and x_6 would be equivalent). The proof of Theorem 44 is complete. $\square$

5.3 Homogeneous coverings

Chen asked (in 1998) whether a stronger version of Conjecture 7 can be true, i.e. whether $2r-2$ biclique components *of the same bi-equivalence graph G_i*, $1 \leq i \leq r$, can cover $[X, Y]$. Call such a cover a *homogeneous cover.*

Given r, let $g(r)$ be the smallest m such that in every biclique with a spanning partition into r bi-equivalence graphs $G_1, \ldots, G_r$, there is a partition class G_i with width at most m. Let $h(r)$ be the smallest m such that for every biclique with a spanning partition into r bi-equivalence graphs, the width of *every partition class* is at most m.
It is clear that $g(r) \leq h(r)$. It is proven in [7] about the functions g and h that $g(r) \geq cr^{3/2}$ for some positive constant c, and $h(r) = 2^{r-1}$. It is a challenging question how they separate.

Although the above lower bound for $g(r)$ means that there are no homogeneous covers with $2r-2$ bicliques in general for spanning partitions, they might exist for antichain partitions, in fact we proved this in Sections 5.2.1 and 5.2.2 for $r \leq 5$.

Question 10. *Suppose that a biclique has an antichain partition into r bi-equivalence graphs. Is it true that some of them has width at most $2r-2$?*

5.4 The dual form, transversals of r-partite intersecting hypergraphs

Conjectures 4 and 7 can be translated into dual forms as conjectures about transversals of r-partite r-uniform intersecting hypergraphs. The approach already turned out to be very useful, for example results of Füredi established in [26] can be applied. A survey on the subject is [31].

An r-uniform hypergraph H is defined by a finite set $V(H)$ called the vertex set of H, and by a set $E(H)$ of r-sets of $V(H)$ called edges of H. An r-uniform hypergraph H is called r-partite if there is a partition $V(H) = V_1 \cup \cdots \cup V_r$ such that $|e \cap V_i| = 1$, for all $i = 1, \ldots, r$ and $e \in E(H)$. A hypergraph H is called *intersecting* if $e \cap f \neq \emptyset$ for any $e, f \in E(H)$. A set $T \subseteq V(H)$ is called a transversal of H provided $e \cap T \neq \emptyset$, for all $e \in E(H)$; the minimum cardinality of a transversal of H is the transversal number of H denoted by $\tau(H)$.

To formulate the dual form of Conjecture 4, one should consider the monochromatic components (also the single ones) of an edge-coloured graph G as vertices of a hypergraph $\mathcal{H}$. The vertices are arranged into partite classes according to the colour of the monochromatic component. The hyperedges of $\mathcal{H}$ correspond to the vertices of G consisting of those monochromatic components of G which contain the given vertex. From an r-edge-coloured graph we obtain an r-partite r-uniform hypergraph. If G is complete then $\mathcal{H}$ is intersecting. The dual of Conjecture 4 is Ryser's conjecture for intersecting hypergraphs in its usual form as follows.

Conjecture 11. *If $\mathcal{H}$ is an intersecting r-partite hypergraph then $\tau(\mathcal{H}) \leq r-1$.*

There are infinitely many examples of intersecting r-partite hypergraphs with transversal number equal to $r-1$. Take a finite projective plane of order q, then truncate it by removing one point and the incident $q+1$ lines. The remaining lines taken as edges define an intersecting $(q+1)$-partite hypergraph with transversal number equal to q. (Note that the truncated projective plane is the dual of an affine plane.)

Ryser's conjecture for general hypergraphs states that if $\mathcal{H}$ is an r-partite hypergraph then $\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$, where $\nu(\mathcal{H})$ is the maximum number of pairwise disjoint edges in $\mathcal{H}$. It is the dual of the following statement: in every r-colouring of the edges of a graph G, the vertex set can be covered by the vertices of at most $(r-1)\alpha(G)$ monochromatic components, as it is formulated in [30], [25].

Concerning our biclique cover conjectures, the dual of a spanning partition of a complete bipartite graph into r graphs gives two r-partite hypergraphs, $\mathcal{H}_1, \mathcal{H}_2$ on the same vertex set (corresponding to the set of monochromatic components) with different edge sets (corresponding to the set of vertices in the two partite classes of the bipartite graph). And $h_1 \cap h_2 \neq \emptyset$ holds for every $h_1 \in E(\mathcal{H}_1), h_2 \in E(\mathcal{H}_2)$, moreover at each vertex there is at least one edge from both hypergraphs. We call such hypergraph pairs cross-intersecting. Then Conjecture 6 reads as follows:

Conjecture 12. *Let $\mathcal{H}_1, \mathcal{H}_2$ be a pair of cross-intersecting r-partite hypergraphs. Then we have $\tau(\mathcal{H}_1 \cup \mathcal{H}_2) \leq 2r - 2$.*

As we have seen we may assume that the biclique is partitioned into bi-equivalence graphs and we obtain an equivalent form of Conjecture 6. Translating this property to the dual problem this means that the hypergraph pair $\mathcal{H}_1$, $\mathcal{H}_2$ is 1-cross-intersecting, that is for every $h_1 \in E(\mathcal{H}_1)$, $h_2 \in E(\mathcal{H}_2)$, we have $|h_1 \cap h_2| = 1$.

In case of Ryser's conjecture for intersecting hypergraphs it is not known whether with a similar assumption we would obtain an equivalent form. Assuming that for every $e, f \in E(H)$ we have $|e \cap f| = 1$ (that is, the colour classes consist of disjoint complete graphs) seems a special case of the conjecture. It was conjectured by Lehel [44] that in this case Ryser's conjecture is true in a stronger form.

Conjecture 13. *Suppose that an intersecting r-partite hypergraph $\mathcal{H}$ has no isolated vertices and its edges pairwise intersect in precisely one vertex. Then some partite class of $\mathcal{H}$ contains at most $r - 1$ elements, in particular $\tau(\mathcal{H}) \leq r - 1$.*

One can easily prove that under the above conditions each partite class contains at most $\binom{2(r-1)}{r-1}$ vertices, see [7].

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