Asymptotic values of graph parameters

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2009.

Asymptotic values of graph parameters

General problem:
p is a graph parameter
$*$ is a graph product

$$p(G),\, p(G*G),\, p(G*G*G),\, \dots\, p(G^{*k}) \longrightarrow q(G) = \lim_{k\to\infty} p(G^{*k})$$

$$q(G) = \lim_{k\to\infty} \sqrt[k]{p(G^{*k})}$$

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example: Shannon-capacity, Witsenhausen rate

Categorical product (direct product)

Definition. For two graphs F and G, their *categorical product* $F \times G$ is defined as follows

$V(F \times G) = V(F) \times V(G)$,

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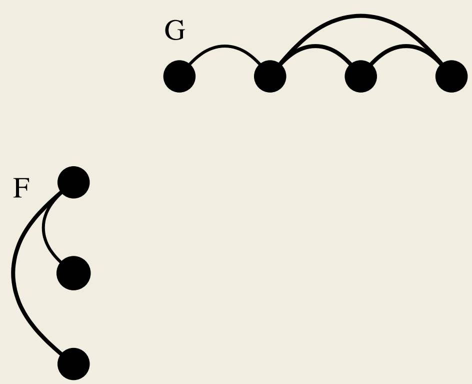

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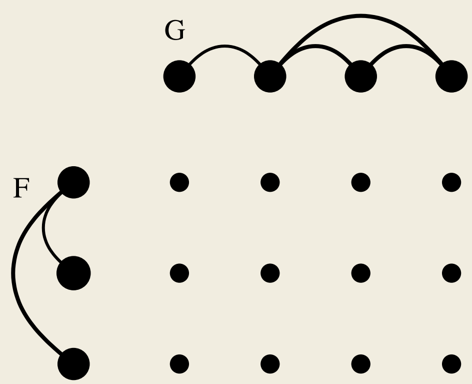

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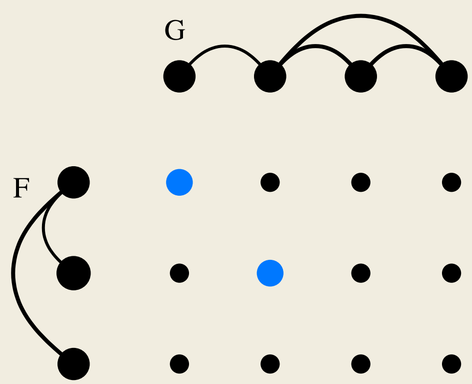

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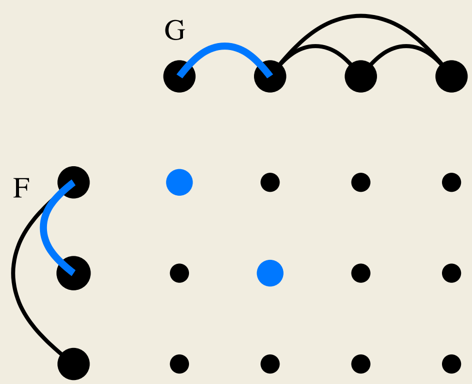

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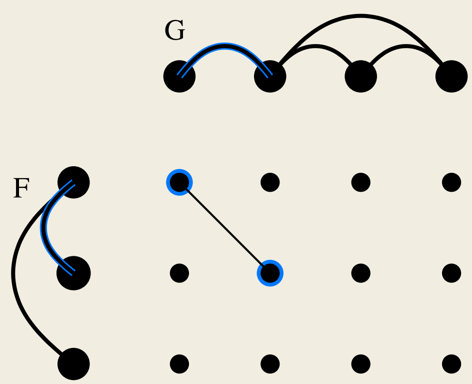

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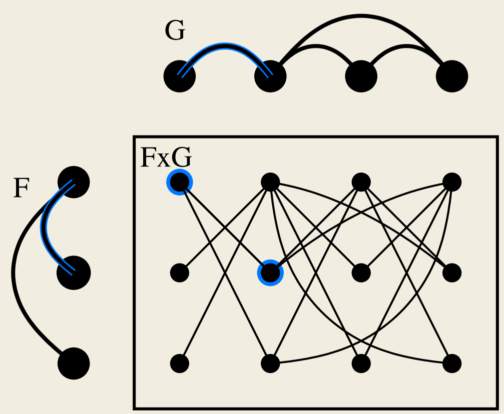

The nth categorical power $G^{\times n}$ is the n-fold categorical product of G.

Ultimate categorical independence ratio

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Definition. [BNR] The *ultimate categorical independence ratio* of a graph G is defined as

$$A(G) = \lim_{k\to\infty} i(G^{\times k}).$$

([BNR] J. I. Brown, R. J. Nowakowski, D. Rall: The Ultimate Categorical Independence Ratio of a Graph, SIAM J. Discrete Math. 9 (1996))

Self-universal graphs

$$i(G) \leq i(G^{\times 2}) \leq i(G^{\times 3}) \leq \ldots \leq i(G^{\times k-1}) \leq i(G^{\times k}) \leq \ldots$$

Theorem. [BNR] If $i(G) > \frac{1}{2}$ then $A(G) = 1$.

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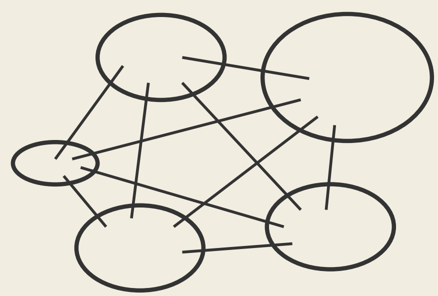
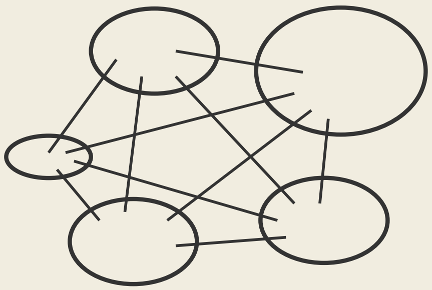
What about complete multipartite graphs?



Ultimate categorical independence ratio of complete multipartite graphs



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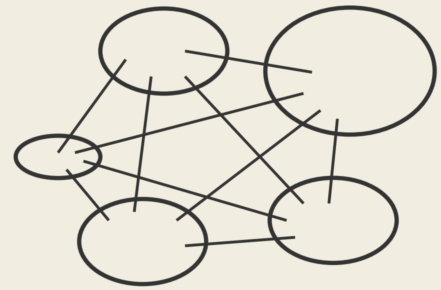

Theorem. [1] Let G be a graph for which
(1) $i(G) \leq \frac{1}{2}$
(2) $\forall v \in V(G):\ d(v) \geq |V(G)| - \alpha(G)$.
Then $i(G^{\times k}) = i(G)$ holds for every integer $k \geq 1$.

Corollary. [1] Let $G = K_{\ell_1,\ell_2,\ldots\ell_m}$ be a complete multipartite graph.
Let $n = \sum_{i=1}^{m} \ell_i$ be the number of vertices and let $\ell = \max_{1\leq i\leq m} \ell_i$ be the size of the largest partite class.
If $\ell \leq \frac{n}{2}$ then $A(G) = i(G) = \frac{\ell}{n}$, so G is self-universal, otherwise $A(G) = 1$.

[1] Á. Tóth, The ultimate categorical independence ratio of complete multipartite graphs, submitted to SIAM J. Discrete Math.

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$\exists J$ independent set in $G^{\times t}$: $\frac{|J|}{|V(G^{\times t})|} > i(G) \Rightarrow$

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Lemma. [1] Let G be a graph for which
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If J is an independent set in $G^{\times t}$ for which $\frac{|J|}{|J \cup N(J)|} > i(G)$ holds then there is an independent set K in $G^{\times(t-1)}$ for which $\frac{|K|}{|K \cup N(K)|} > i(G)$ holds.

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Lemma. [1] Let G be a graph for which (1) and (2) holds. If I is an independent set of G then $\frac{|I|}{|I \cup N(I)|} \leq \frac{\alpha(G)}{|V(G)|} = i(G)$.

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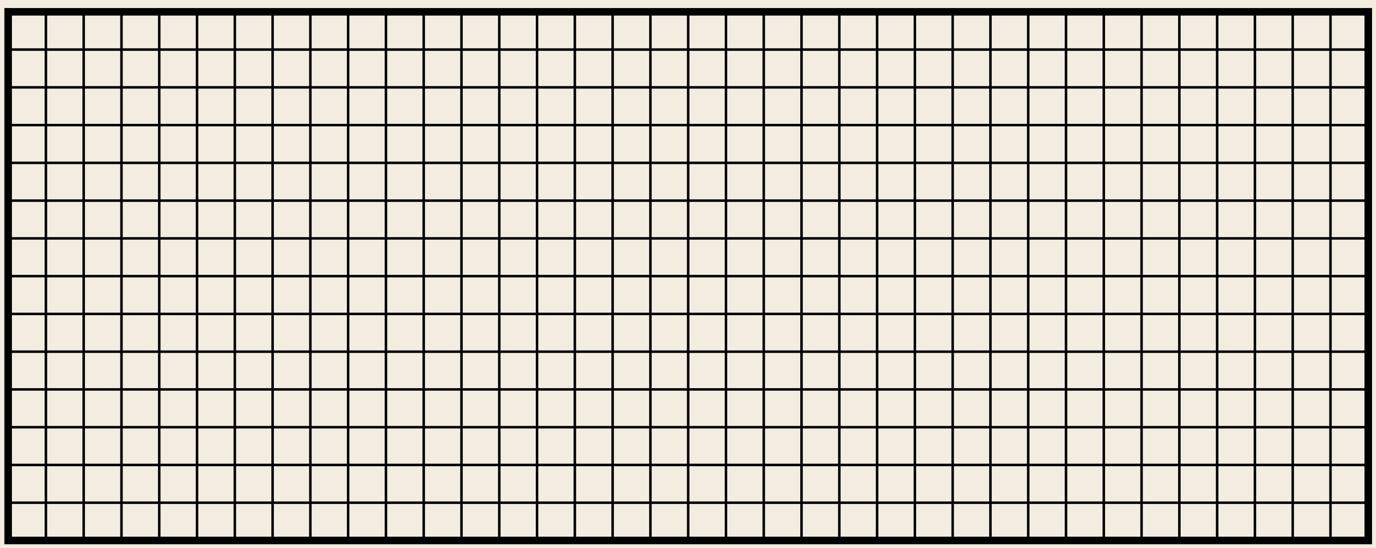

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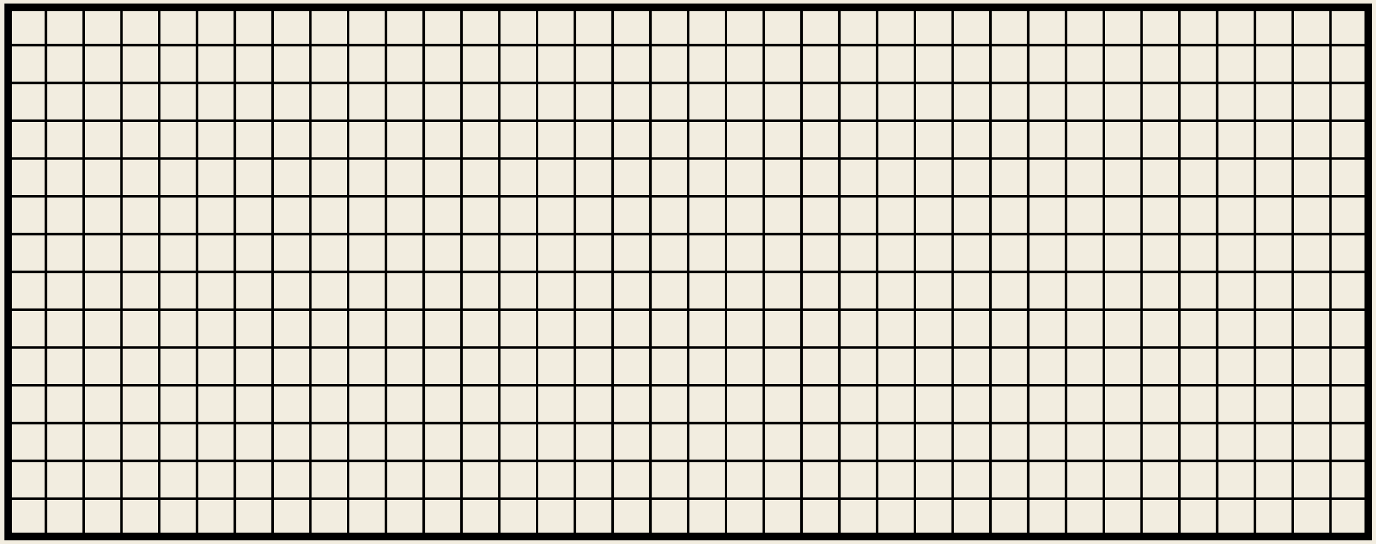

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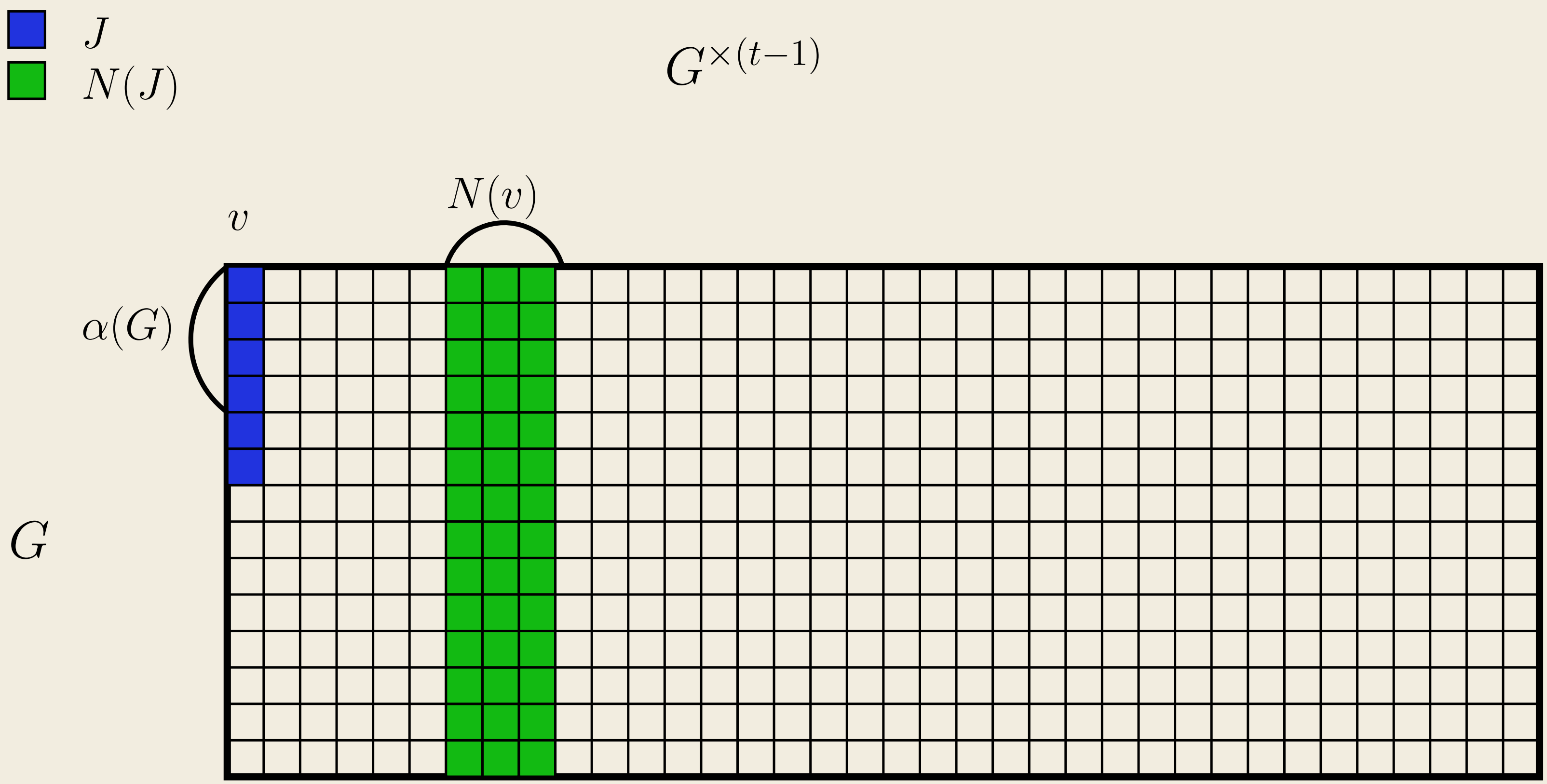

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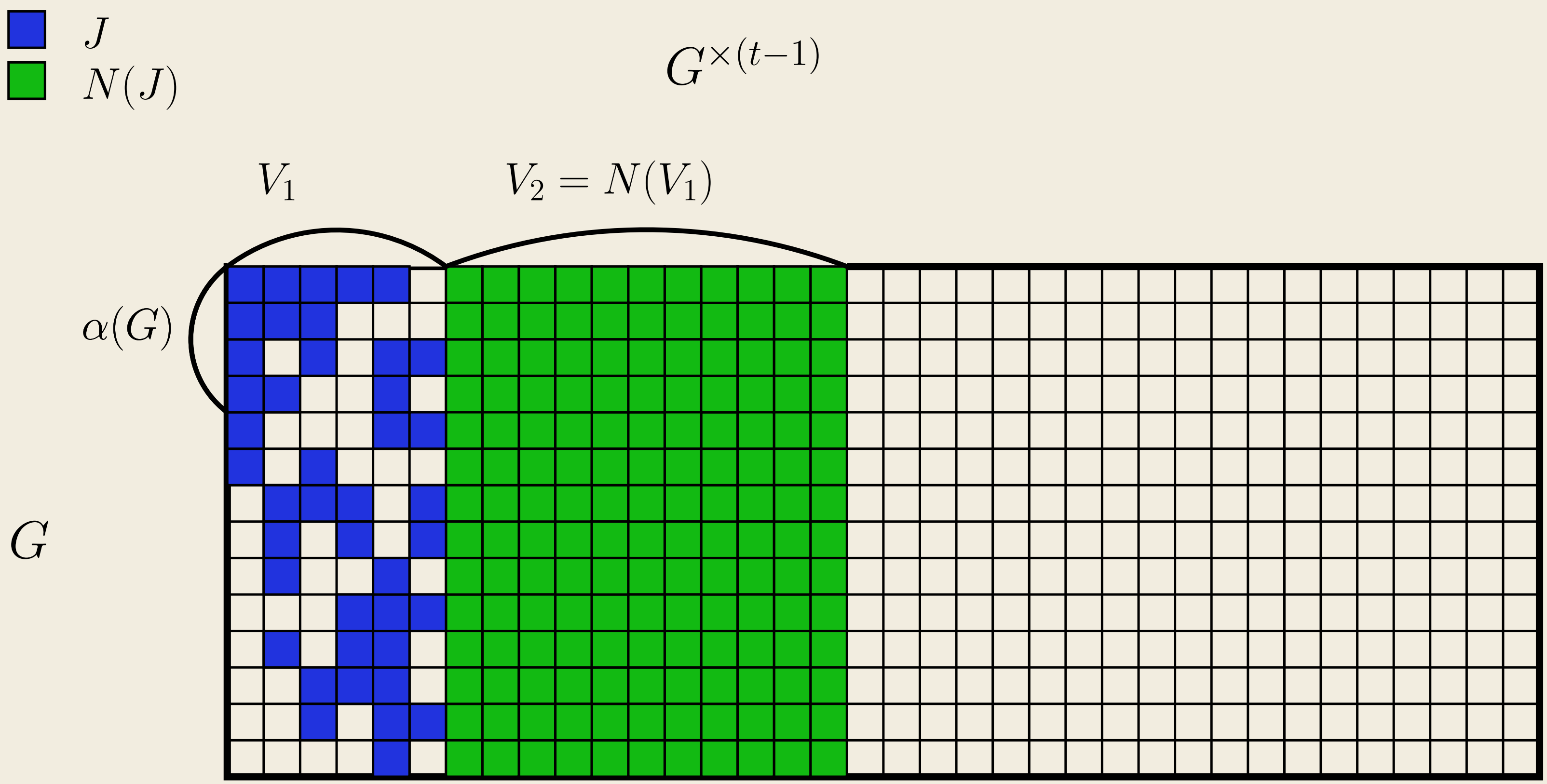

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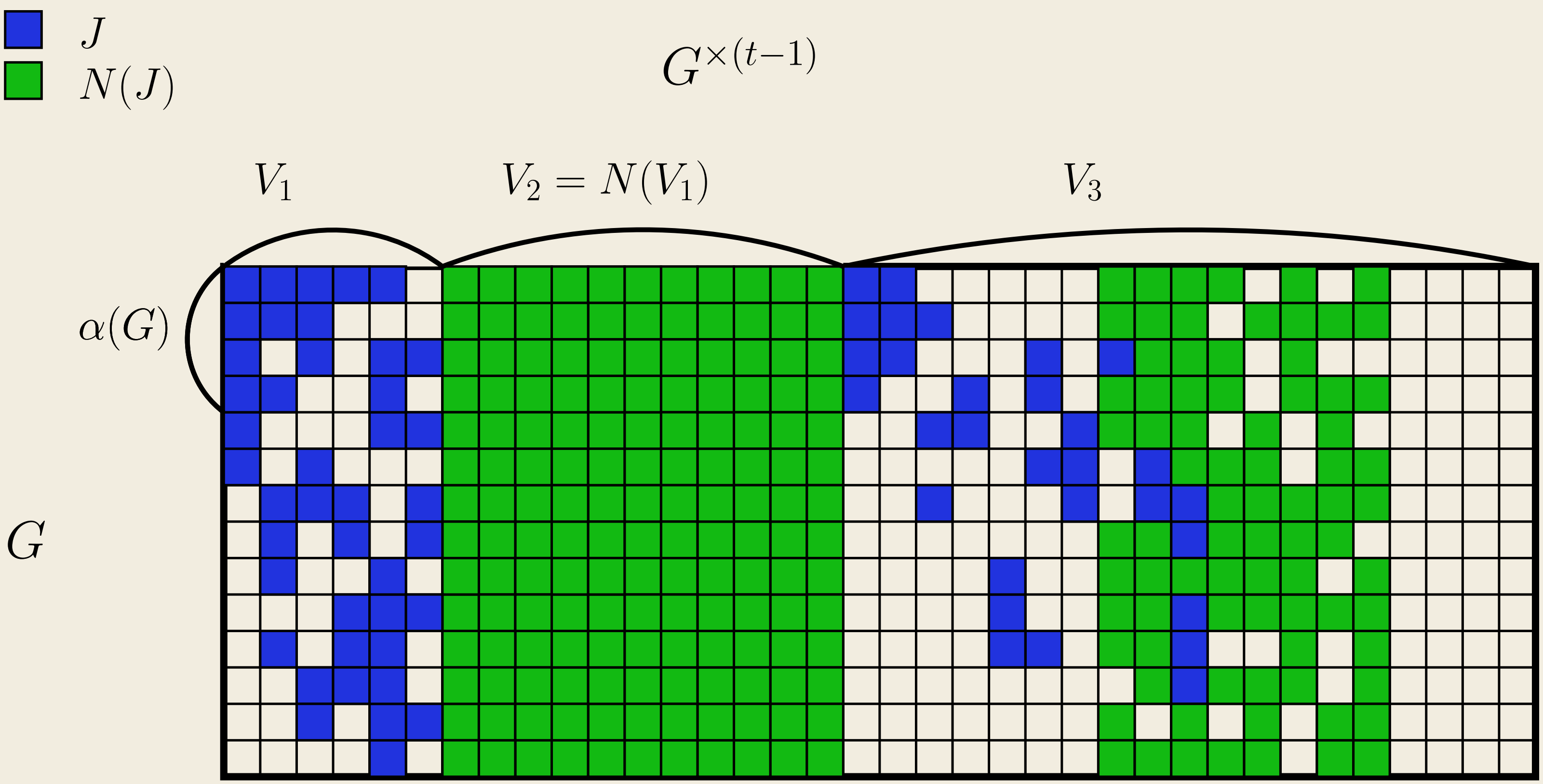

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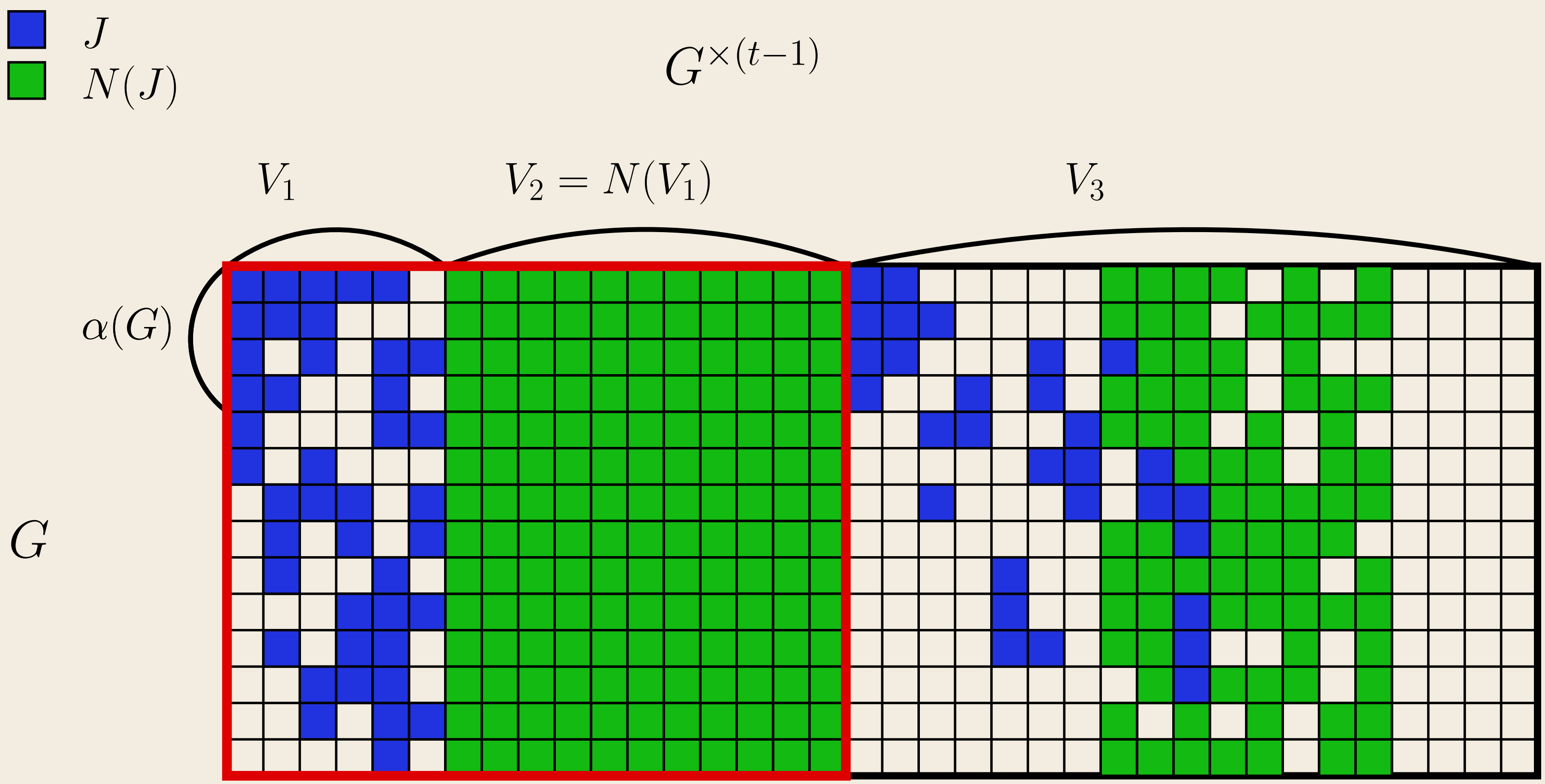

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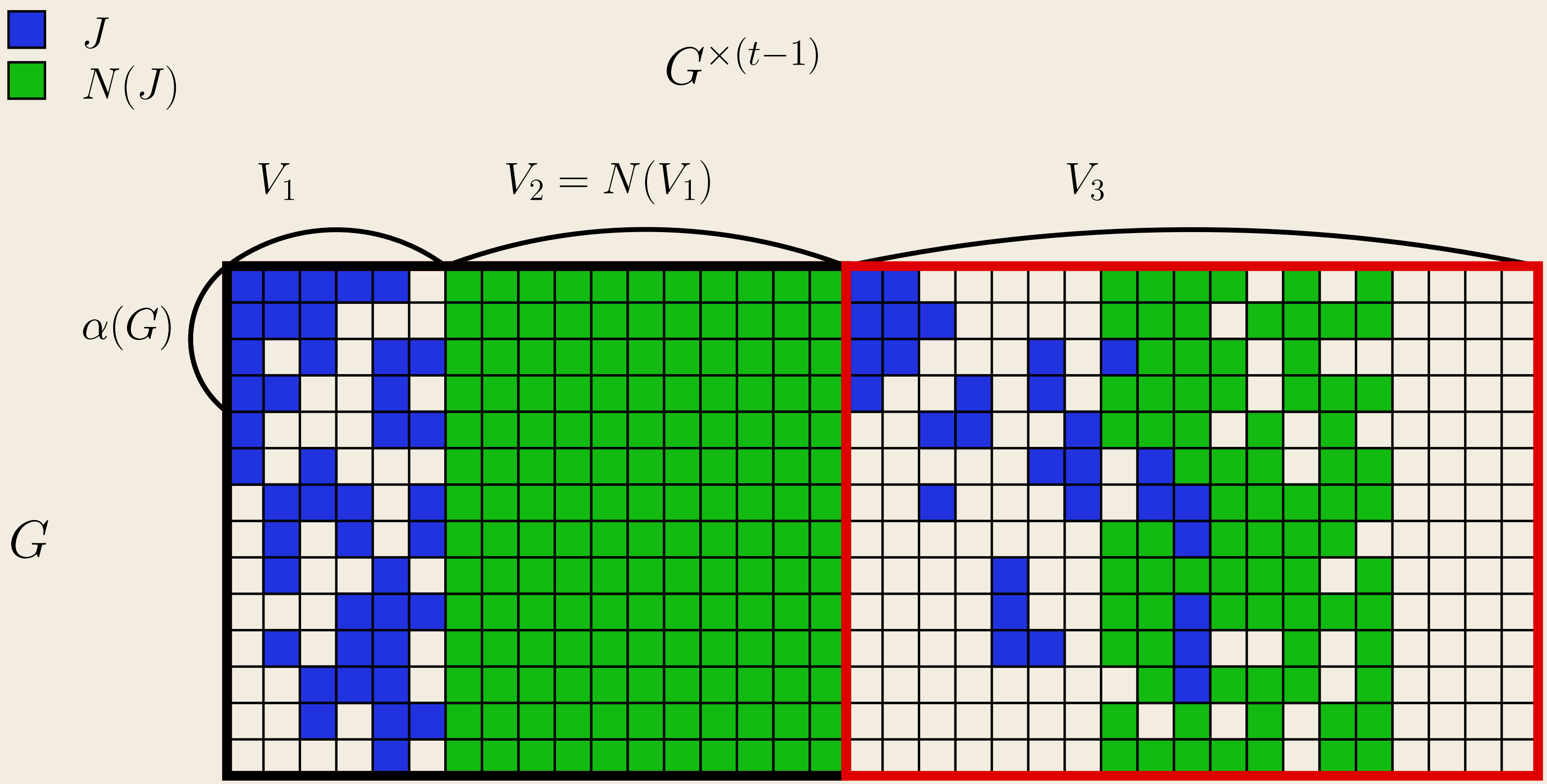

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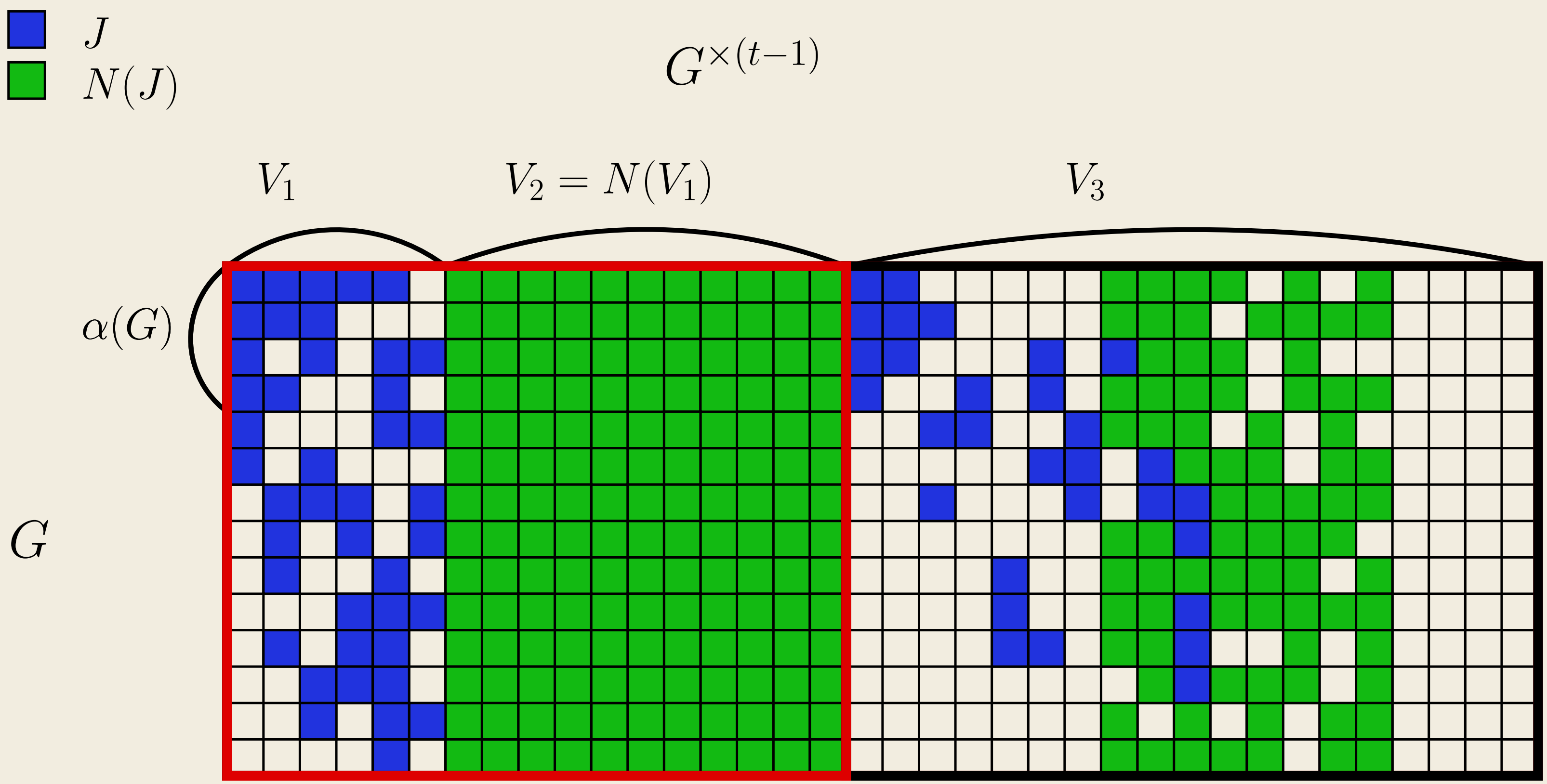

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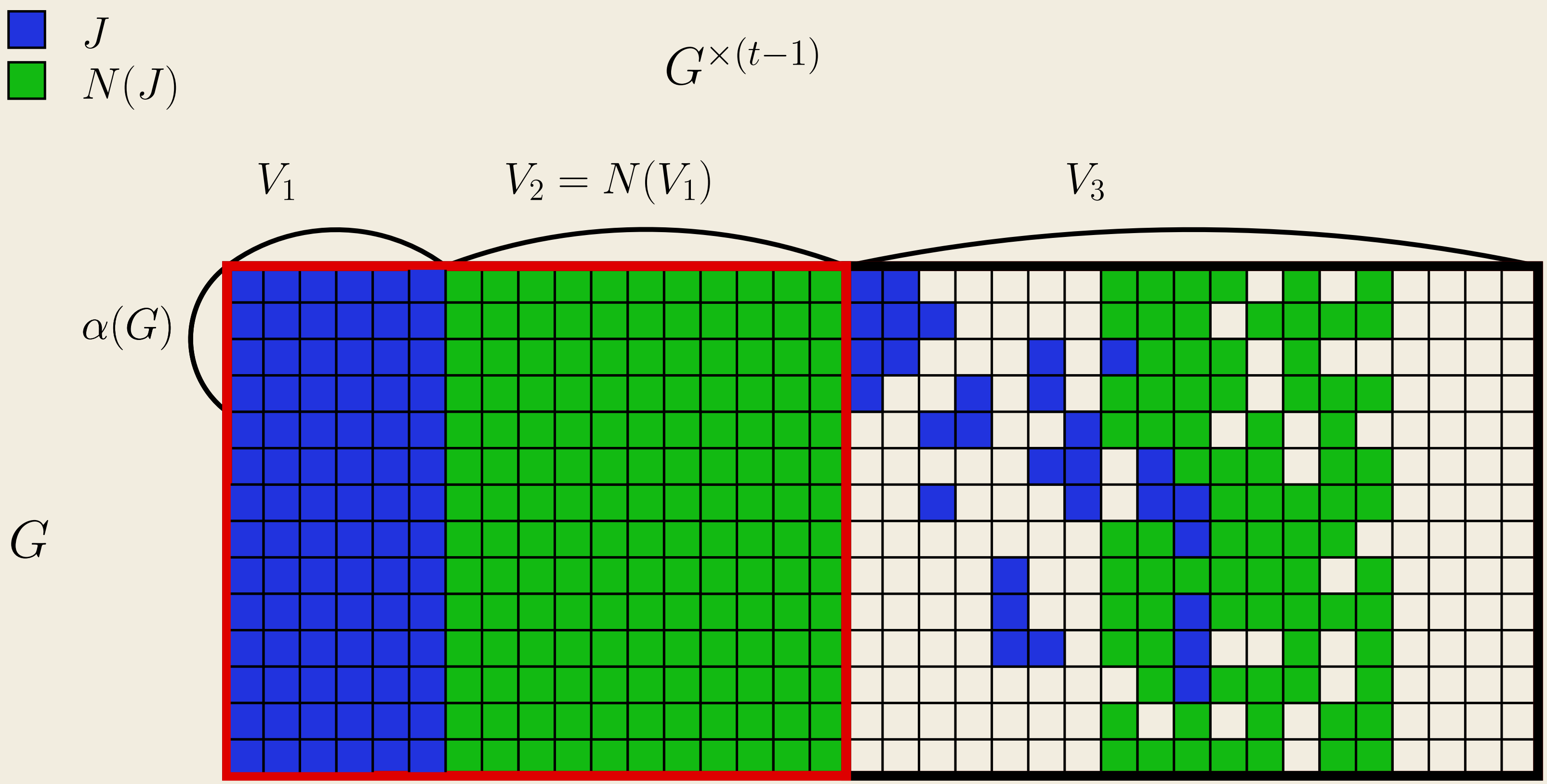

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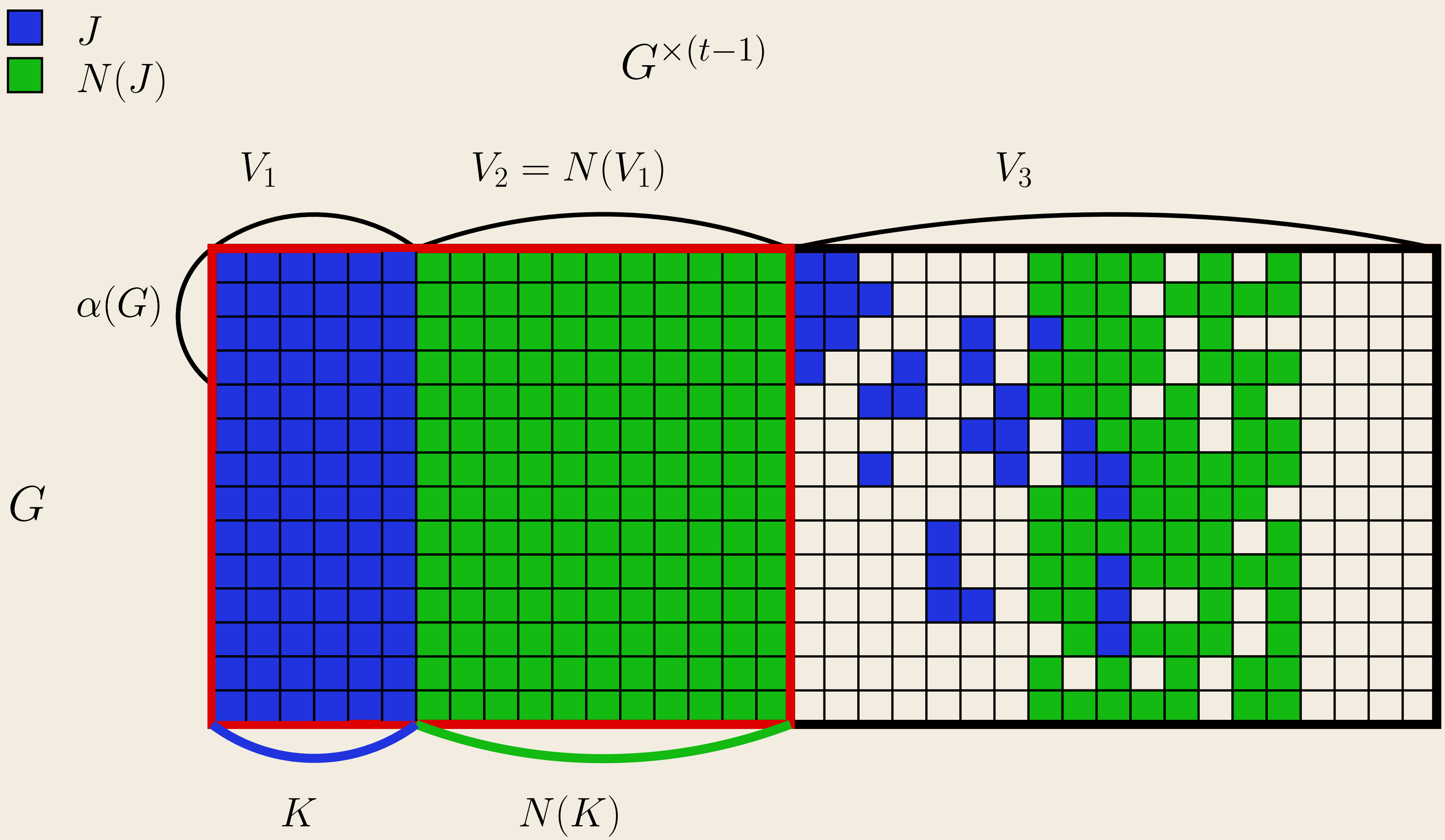

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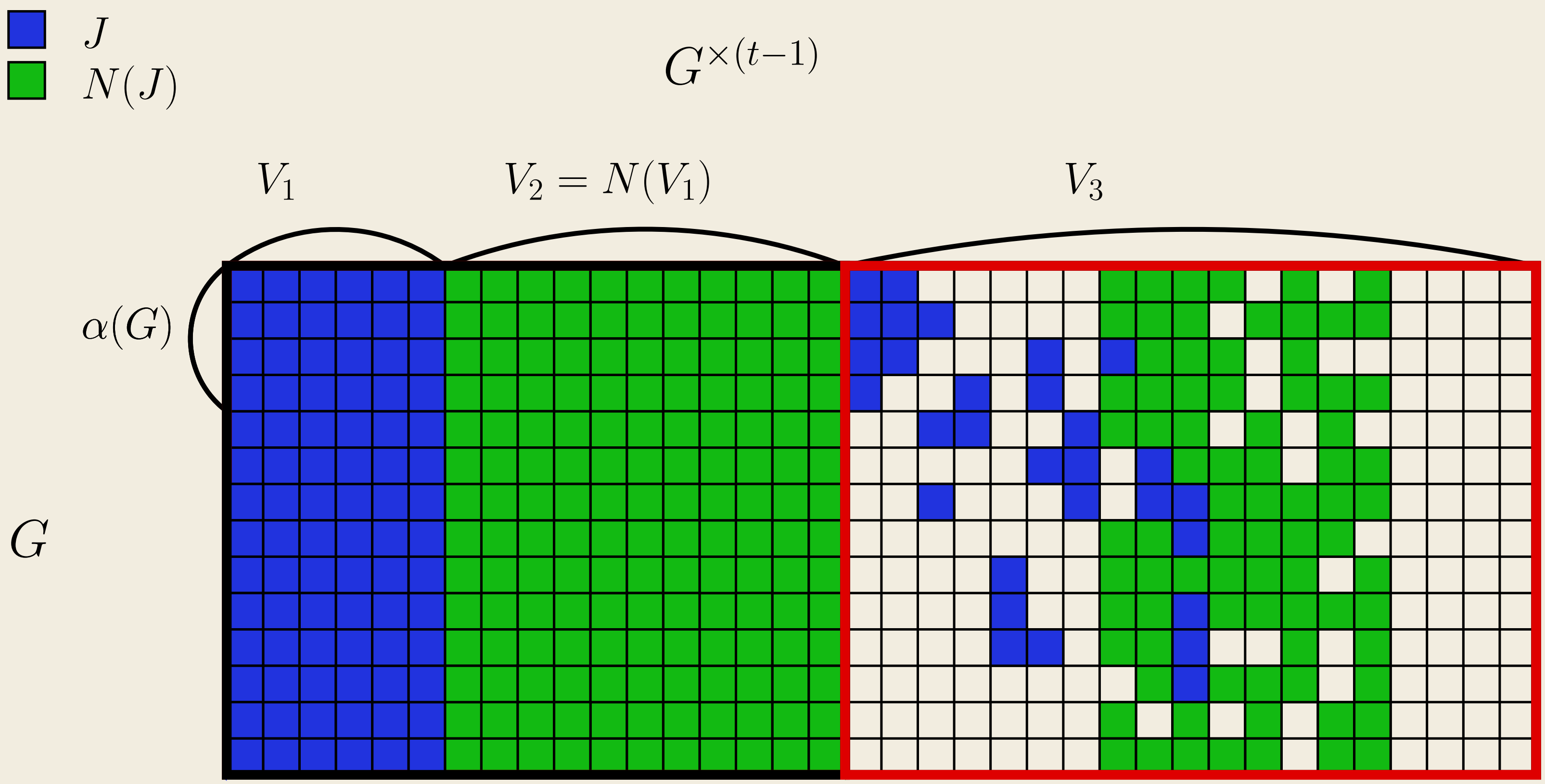

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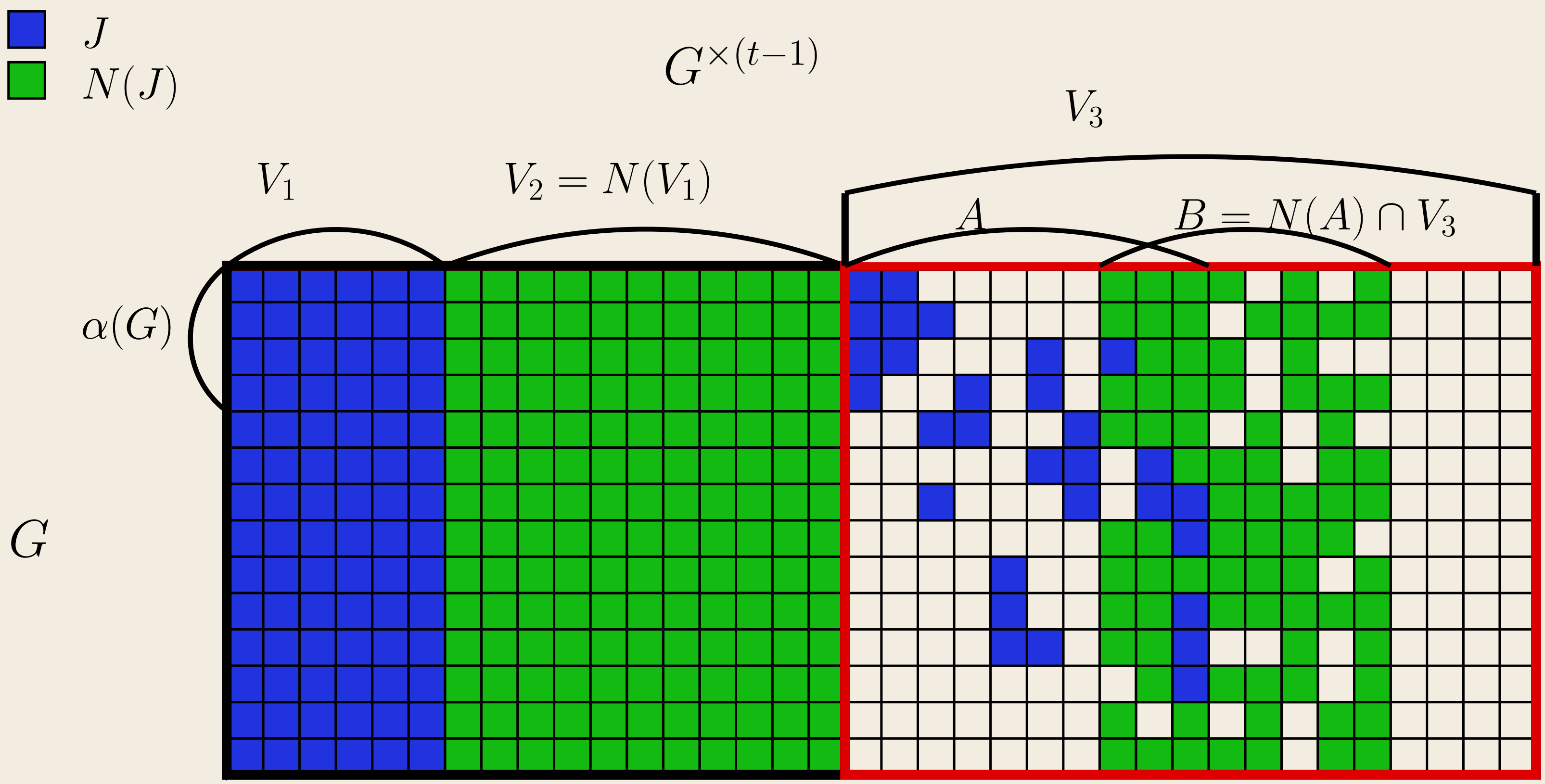

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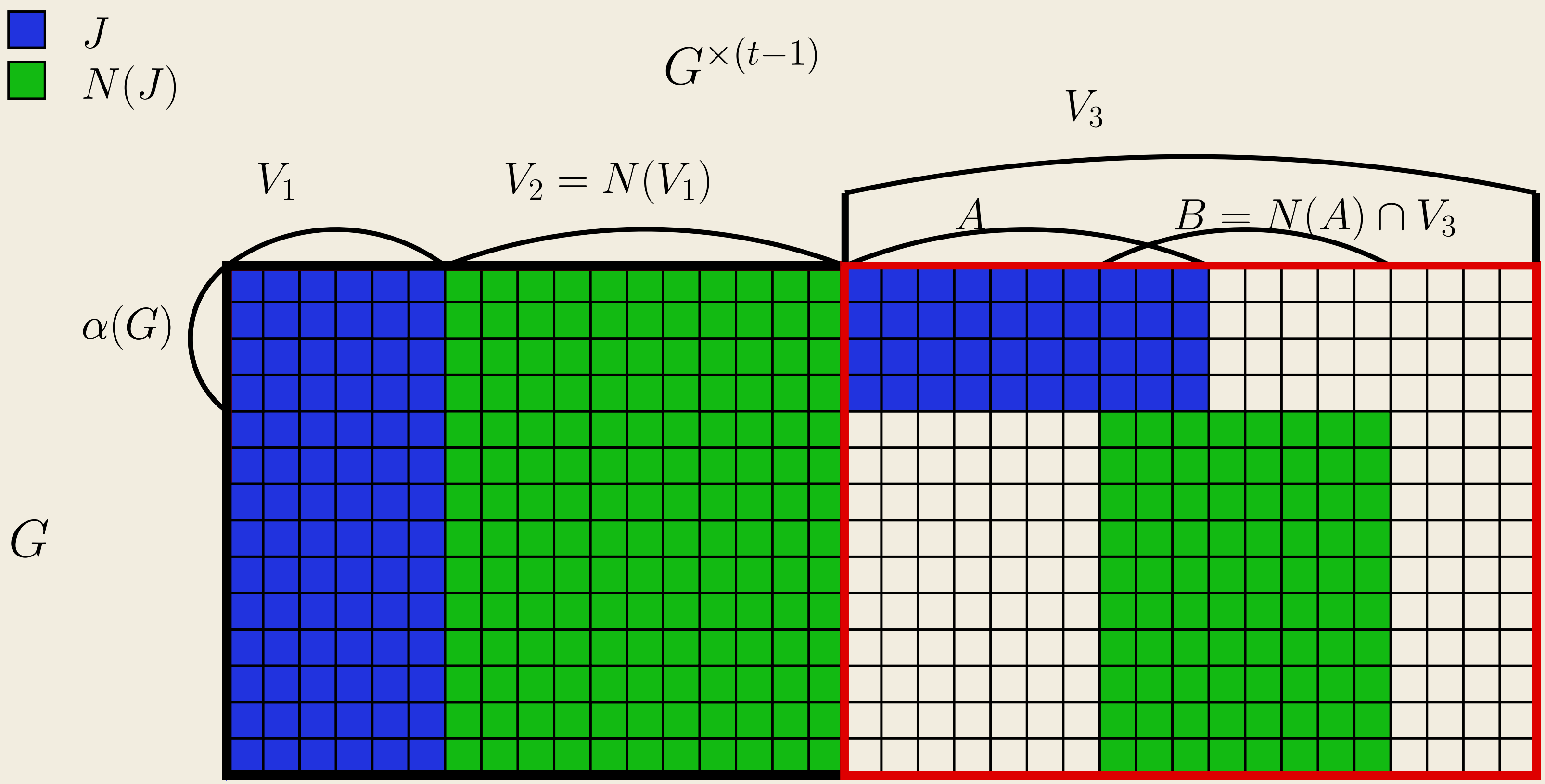

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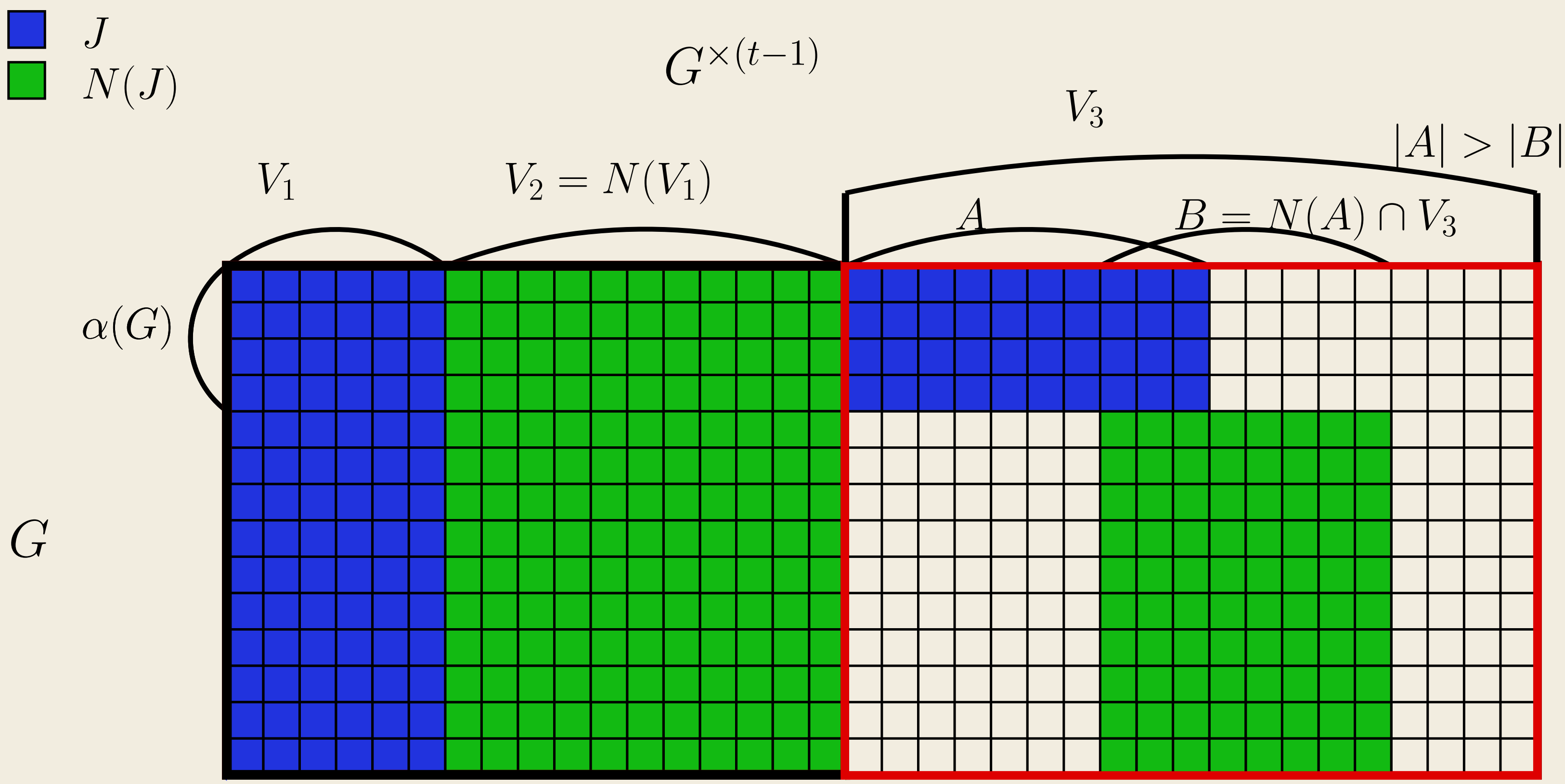

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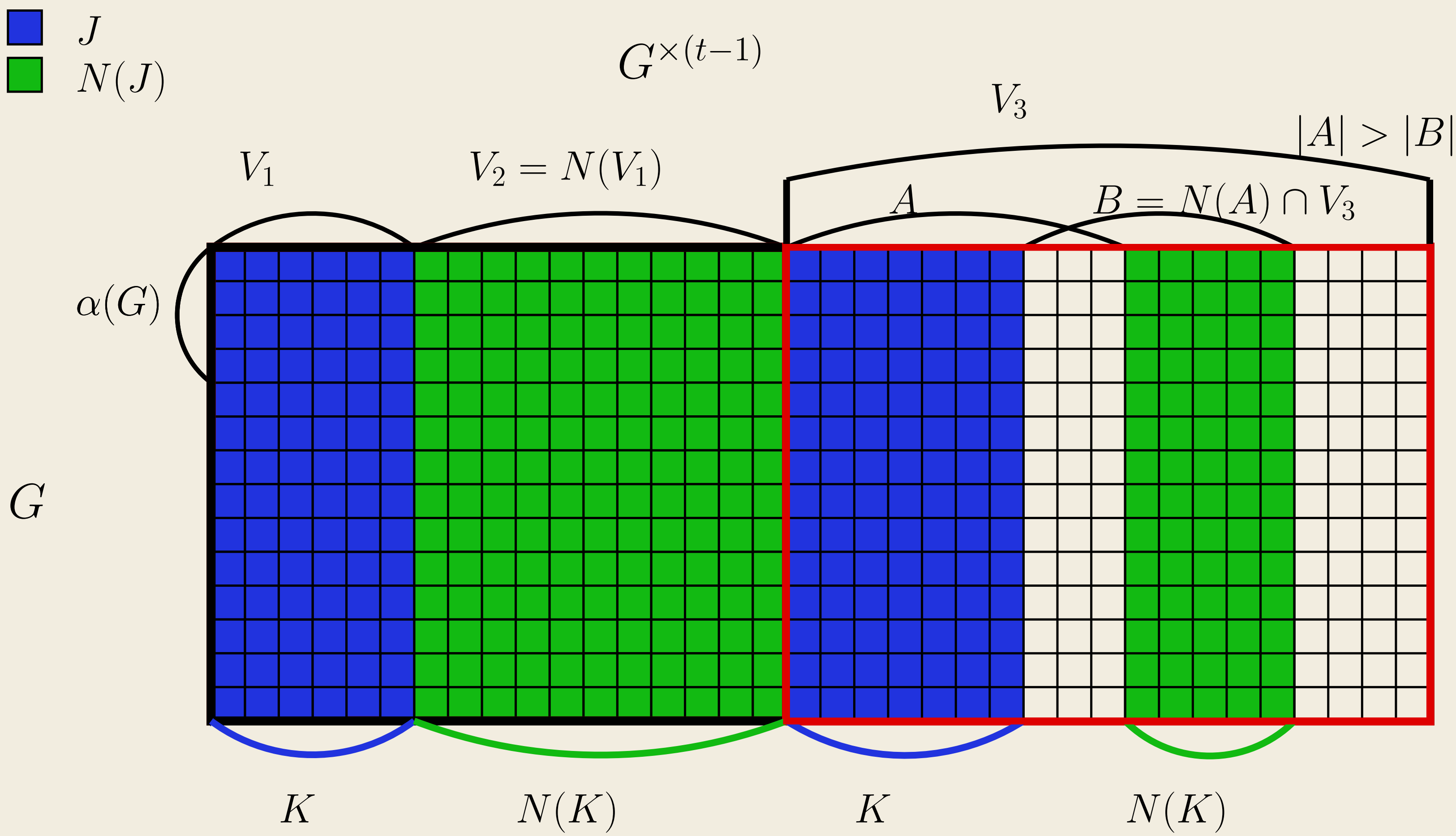

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Ultimate categorical independence ratio of complete multipartite graphs

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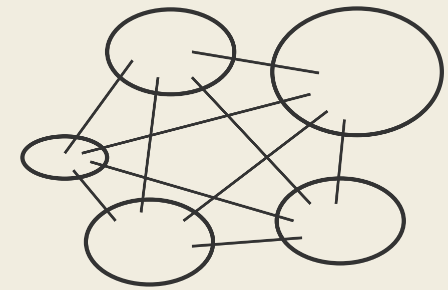

Corollary. [1] Let $G = K_{\ell_1,\ell_2,\ldots\ell_m}$ be a complete multipartite graph.
Let $n = \sum_{i=1}^{m} \ell_i$ be the number of vertices and let $\ell = \max_{1\leq i\leq m} \ell_i$ be the size of the largest partite class.
If $\ell \leq \frac{n}{2}$ then $A(G) = i(G) = \frac{\ell}{n}$, so G is self-universal, otherwise $A(G) = 1$.

[1] Á. Tóth, The ultimate categorical independence ratio of complete multipartite graphs, submitted to SIAM J. Discrete Math.

Hall-ratio

Definition. [CGyL] The *Hall-ratio* of a graph G is defined as

$$\rho(G) = \max\left\{\frac{|V(H)|}{\alpha(H)} \,:\, H \subseteq G\right\}.$$

[CGyL] M. Cropper, A. Gyárfás, J. Lehel: Hall-ratio of the Mycielski graphs, Discrete Math. 306 (2006)

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Definition. [CGyL] The *Hall-ratio* of a graph G is defined as

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Definition. [S] The *ultimate lexicographic Hall-ratio* of graph G is

$$h_\circ(G) = \lim_{n\to\infty} \sqrt[n]{\rho(G^{\circ n})}.$$

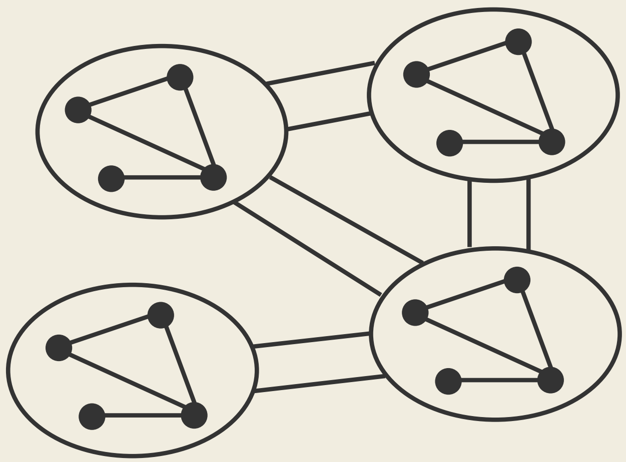
[CGyL] M. Cropper, A. Gyárfás, J. Lehel: Hall-ratio of the Mycielski graphs, Discrete Math. 306 (2006)
[S] G. Simonyi: Asymptotic values of the Hall-ratio for graph powers, Discrete Math. 306 (2006)

Lexicographic product

Definition. For two graphs F and G, their *lexicographic product* $F \circ G$ is defined as follows

$$V(F \circ G) = V(F) \times V(G),$$

$$E(F \circ G) = \{\{u_1v_1, u_2v_2\} : \{u_1, u_2\} \in E(F), \text{ or } u_1 = u_2 \text{ and } \{v_1, v_2\} \in E(G)\}.$$



The nth lexicographic power $G^{\circ n}$ is the n-fold lexicographic product of G.

The ultimate lexicographic Hall-ratio

Theorem. [S] For normal power the asymptotic value of the Hall-ratio is the Witsenhausen rate ($R(G)$).

Theorem. [S] For co-normal power the asymptotic value of the Hall-ratio is the fractional chromatic number ($\chi_f(G)$).

Corollary. [S]

$$\max\{\rho(G), R(G)\} \leq h_\circ(G) \leq \chi_f(G).$$

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Theorem. [2]

$$h_\circ(G) = \chi_f(G)$$

[2] Á. Tóth, On the ultimate lexicographic Hall-ratio, Discrete Math. 309 (2009) 3992-3997.

Thank you for your attention!