The ultimate categorical independence ratio of complete multipartite graphs

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Abstract

The independence ratio $i(G)$ of a graph G is the ratio of its independence number and the number of vertices. The ultimate categorical independence ratio of a graph G is defined as $\lim_{k\to\infty} i(G^{\times k})$, where $G^{\times k}$ denotes the kth categorical power of G. This parameter was introduced by Brown, Nowakowski and Rall, who asked about its value for complete multipartite graphs. In this paper we determine the ultimate categorical independence ratio of complete multipartite graphs.

1 Introduction

The *independence ratio* of a graph G is defined as $i(G) = \frac{\alpha(G)}{|V(G)|}$, that is, as the ratio of the independence number and the number of vertices.

Its asymptotic value with respect to what is called Cartesian graph exponentiation is the ultimate independence ratio which was introduced by Hell, Yu and Zhou [5] and futher investigated by Hahn, Hell and Poljak [4] and by Zhu [6]. Motivated by this concept Brown, Nowakowski and Rall [3] considered the analogous, but significantly different parameter, the ultimate categorical independence ratio which is defined with respect to the categorical power of graphs.

For two graphs F and G, their *categorical product* $F \times G$ is defined on the vertex set $V(F \times G) = V(F) \times V(G)$ with edge set $E(F \times G) = \{\{(u_1, v_1), (u_2, v_2)\} : \{u_1, u_2\} \in E(F) \text{ and } \{v_1, v_2\} \in E(G)\}$. The kth categorical power $G^{\times k}$ is the k-fold categorical product of G.

Definition. ([3]) The *ultimate categorical independence ratio* of a graph G is defined as

$$A(G) = \lim_{k\to\infty} i(G^{\times k}).$$

This parameter was also investigated by Alon and Lubetzky [2] and the characterization of maximum-size independent sets in categorical graph powers were considered by Alon, Dinur, Friedgut and Sudakov [1].

The authors of [3] investigated graphs for which $A(G) = i(G)$ holds and they called such graphs *self-universal*. In that article it is proven that some interesting graph families, for example Cayley graphs of Abelian groups, have this property.

The paper [3] mentions complete multipartite graphs as one of those families of graphs for which the determination of the ultimate categorical independence ratio remained an open problem. It follows from a result in [3] that if the largest partite class contains more than half of the vertices then the ultimate categorical independence ratio equals to one. In this paper we prove that in all other cases, i.e., when none of the parts of the complete multipartite graph has size greater than half the number of vertices then the graph is self-universal.

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2 The ultimate categorical independence ratio of complete multipartite graphs

We will use the following theorem of [3].

Theorem 1. ([3]) *If $i(G) > \frac{1}{2}$ then $A(G) = 1$.*

The main result of this paper is the following theorem. We get the result for complete multipartite graphs as a simple corollary. The first condition of Theorem 2, i.e., the one about the lower bound on the degrees is essential. The second condition is trivial; it must hold by Theorem 1. We denote by $d(v)$ the degree of the vertex v.

Theorem 2. *Let G be a graph for which $d(v) \geq |V(G)| - \alpha(G)$ holds for all vertices v of G and $i(G) \leq \frac{1}{2}$ holds. Then $i(G^{\times k}) = i(G)$ holds for every integer $k \geq 1$.*

Corollary 3. *Let $G = K_{\ell_1,\ell_2,\ldots\ell_m}$ be a complete multipartite graph. Let $n = \sum_{i=1}^{m} \ell_i$ be the number of vertices and let $\ell = \max_{1\leq i\leq m} \ell_i$ be the size of the largest partite class. If $\ell \leq \frac{n}{2}$ then $A(G) = i(G) = \frac{\ell}{n}$, so G is self-universal, otherwise $A(G) = 1$.*

Proof of Corollary 3 from Theorem 2. Since G is a complete multipartite graph $\alpha(G) = \ell$. If $\ell > \frac{n}{2}$ then $i(G) = \frac{\ell}{n} > \frac{1}{2}$, thus $A(G) = 1$ follows from Theorem 1.
If $\ell \leq \frac{n}{2}$ then $i(G) = \frac{\ell}{n} \leq \frac{1}{2}$. As G is a complete multipartite graph, the degree of its vertices is at least $|V(G)| - \alpha(G)$. Thus from Theorem 2 we get $A(G) = i(G) = \frac{\ell}{n}$. □

We remark that there are graphs which satisfy the conditions of Theorem 2 other than complete multipartite graphs. An example is given by the graph consisting of a 5-length cycle and three additional points joint to every vertex of the cycle.

To prove Theorem 2 we need the following lemma. We denote by $N(U)$ the neighborhood of the set U in a graph G, that is, $N(U)$ is the set of all vertices v of G for which there is a vertex u in U so that u and v are adjacent in G. (Notice that if U is not independent then $N(U)$ and U will not be disjoint.)

Lemma 4. *Let G be a graph for which $d(v) \geq |V(G)| - \alpha(G)$ holds for all vertices v of G and $i(G) \leq \frac{1}{2}$ holds. If J is an independent set in $G^{\times k}$ for which $\frac{|J|}{|J\cup N(J)|} > i(G)$ holds then there is an independent set K in $G^{\times(k-1)}$ for which $\frac{|K|}{|K\cup N(K)|} > i(G)$ holds.*

Proof. Let G be a graph for which

$$\text{for all vertices } v \text{ of } G: \; d(v) \geq |V(G)| - \alpha(G) \tag{1}$$

and

$$i(G) \leq \frac{1}{2} \tag{2}$$

holds. Let J be an independent set in $G^{\times k}$ for which $\frac{|J|}{|J\cup N(J)|} > i(G)$.

Consider $G^{\times k}$ in the form $G \times G^{\times(k-1)}$. We denote the vertex set of $G^{\times(k-1)}$ by U and the vertex set of $G^{\times k}$ by V. Let U_1, U_2 and U_3 be the following subsets of U:

$$\begin{aligned} &U_1 = \{v \in U \,:\, |J \cap (V(G) \times \{v\})| > \alpha(G)\}, \\ &U_2 = N(U_1), \\ &U_3 = U \setminus (U_1 \cup U_2). \end{aligned}$$

For $i = 1, 2, 3$ let $V_i = V(G) \times U_i$ and denote by J_i the corresponding subsets of J: $J_i = J \cap V_i$.

It follows from (1) that for all subset P of $V(G)$ for which $|P| > \alpha(G)$ stands, it holds that $N(P) = V(G)$; because for every vertex v in $V(G)$ there are at most $\alpha(G)$ vertices which are non-adjacent to v, so there

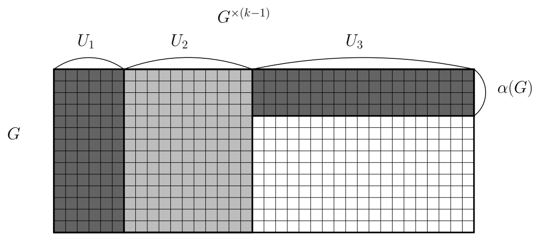

Figure 1: Partition of $G^{\times k}$ (the size of J is at most the size of the dark grey area and the size of $N(J)$ is at least the size of the bright grey area)

must be a vertex in P which is adjacent to v. This fact and the definition of the categorical product imply that every vertex of $V(G) \times N(U_1)$ is a neighbor of a vertex in J_1, i.e., $N(J_1) = V(G) \times N(U_1) = V_2$, thus U_1 is an independent set of $G^{\times(k-1)}$ and J_2 is empty. It also follows that $U_1 \cap U_2 = \emptyset$, $V_1 \cap V_2 = \emptyset$, so $U = U_1 \cup U_2 \cup U_3$ is a partition of U, $V = V_1 \cup V_2 \cup V_3$ is a partition of V and $J = J_1 \cup J_3$ is a partition of J.

It is easy to see that the ratio $\frac{|J|}{|J \cup N(J)|}$ is a convex linear combination of $\frac{|J_1|}{|J_1 \cup N(J_1)|}$, i.e., the corresponding fraction of J in $V_1 \cup V_2$, and $\frac{|J_3|}{|J_3 \cup (N(J_3) \cap V_3)|}$, i.e., the corresponding fraction of J in V_3. (Here we use the trivial fact that for all positive x_1, x_2, y_1, y_2 the equality $\frac{x_1+x_2}{y_1+y_2} = \alpha\frac{x_1}{y_1} + (1-\alpha)\frac{x_2}{y_2}$ holds for some $\alpha \in [0,1]$.) Thus $\frac{|J|}{|J \cup N(J)|} > i(G)$ implies that

$$\frac{|J_1|}{|J_1 \cup N(J_1)|} > i(G) \tag{3}$$

or

$$\frac{|J_3|}{|J_3 \cup (N(J_3) \cap V_3)|} > i(G). \tag{4}$$

In the first case, when (3) holds, it follows from $N(J_1) = V_2$ that $\frac{|U_1|}{|U_1 \cup N(U_1)|} = \frac{|V_1|}{|V_1 \cup V_2|} \geq \frac{|J_1|}{|J_1 \cup N(J_1)|} > i(G)$. Since U_1 is an independent set in $G^{\times(k-1)}$, we can choose U_1 to be the set K in the statement and we are done.

In the second case, when (4) holds we investigate the structure of J further. Let A and B be the following subsets in U_3:

$$A = \{v \in U_3 \,:\, J_3 \cap (V(G) \times \{v\}) \neq \emptyset\},$$
$$B = N(A) \cap U_3.$$

We prove that $|A| > |B|$. From (1) we get that for every vertex w in B the inequality $|N(J_3) \cap (V(G) \times \{w\})| \geq |V(G)| - \alpha(G)$ holds, so $|N(J_3) \cap V_3| \geq (|V(G)| - \alpha(G))|B|$. On the other hand, for every vertex v in U_3 the inequality $|J \cap (V(G) \times \{v\})| \leq \alpha(G)$ holds by the definition of U_1, U_2 and U_3, so $|J_3| \leq \alpha(G)|A|$. Thus $\frac{|J_3|}{|N(J_3) \cap V_3|} \leq \frac{\alpha(G)|A|}{(|V(G)| - \alpha(G))|B|}$. Furthermore, $|A| \leq |B|$ would imply that $\frac{|J_3|}{|N(J_3) \cap V_3|} \leq \frac{\alpha(G)}{(|V(G)| - \alpha(G))}$, so (using that $J_3 \cap N(J_3) = \emptyset$ as J_3 is independent) we have $\frac{|J_3|}{|J_3 \cup (N(J_3) \cap V_3)|} \leq \frac{\alpha(G)}{|V(G)|} = i(G)$, which contradicts (4). Hence

$$|A| > |B|. \tag{5}$$

We know that $\frac{|J|}{|J \cup N(J)|} > i(G) = \frac{\alpha(G)}{|V(G)|}$ which means that

$$\frac{|J|}{|N(J)|} > \frac{\alpha(G)}{|V(G)| - \alpha(G)}. \tag{6}$$

Let L be the following subset of $V(G^{\times k})$:

$$L = V(G) \times M, \text{ where } M = U_1 \cup (A \setminus B).$$

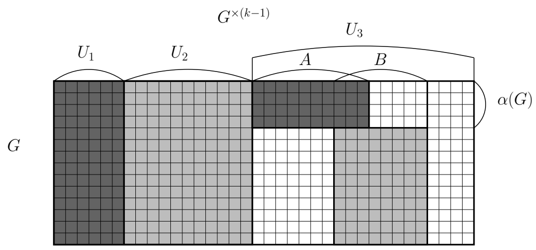

Figure 2: Partition of $G^{\times k}$, the structure of J and $N(J)$

We prove that $\frac{|L|}{|L \cup N(L)|} > i(G)$. From the structure of J (we have seen that $J = J_1 \cup J_3$, $J_1 \subseteq V(G) \times U_1$, $|J_3| \le \alpha(G)|A|$, $N(J_1) = V(G) \times U_2$ and $|N(J_3) \cap V_3| \ge (V(G) - \alpha(G))|B|$) it follows that

$$\frac{|J|}{|N(J)|} = \frac{|J_1| + |J_3|}{|N(J_1)| + |N(J_3) \cap V_3|} \le \frac{|V(G)||U_1| + \alpha(G)|A|}{|V(G)||U_2| + (|V(G)| - \alpha(G))|B|}. \tag{7}$$

By the definition of B we get $N(A \setminus B) \subseteq (B \setminus A) \cup U_2$. This with the definition of U_2 imply that $N(M) \subseteq U_2 \cup (B \setminus A)$, hence

$$\frac{|L|}{|N(L)|} \ge \frac{|V(G) \times (U_1 \cup (A \setminus B))|}{|V(G) \times (U_2 \cup (B \setminus A))|} = \frac{|V(G)|(|U_1| + |A \setminus B|)}{|V(G)|(|U_2| + |B \setminus A|)}. \tag{8}$$

The difference between the right hand side of (8) and (7)

in the numerator:	$-\alpha(G)|A \cap B|$	$+$	$(|V(G)| - \alpha(G))|A \setminus B|$,
in the denominator:	$-(|V(G)| - \alpha(G))|A \cap B|$	$+$	$\alpha(G)|B \setminus A|$.

Note that the numerators belong to the independent sets and the denominators belong to their neighborhoods. Furthermore, analysing the two parts of these differences we have that

$$\frac{-\alpha(G)|A \cap B|}{-(|V(G)| - \alpha(G))|A \cap B|} = \frac{\alpha(G)}{|V(G)| - \alpha(G)} \tag{9}$$

and

$$\frac{(|V(G)| - \alpha(G))|A \setminus B|}{\alpha(G)|B \setminus A|} > 1 \ge \frac{\alpha(G)}{|V(G)| - \alpha(G)}, \tag{10}$$

because (5) implies that $|A \setminus B| > |B \setminus A|$ and from (2) it follows that $\alpha(G) \le |V(G)| - \alpha(G)$.
Since the numerator and the denominator of the right hand side of (8) are the sum of the numerators and the denominators of the right hand side of (7), the left hand side of (9) and the left hand side of (10), respectively, it follows from the bounds in (6), (7), (8), (9) and (10) that $\frac{|L|}{|N(L)|} > \frac{\alpha(G)}{|V(G)| - \alpha(G)}$. (We use the fact that for positive y_1 and y_2 the inequalities $\frac{x_1}{y_1} > K$ and $\frac{x_2}{y_2} > K$ imply that $\frac{x_1 + x_2}{y_1 + y_2} > K$ holds and the similar easy statement that if $y_1 > 0$ and $y_1 + y_2 > 0$ then from $\frac{x_1}{y_1} > K$ and $\frac{x_2}{y_2} = K$ it also follows $\frac{x_1 + x_2}{y_1 + y_2} > K$. We need the second one since the numerator and the denominator of the left hand side of (9) are negative.) This means that

$$\frac{|L|}{|L \cup N(L)|} > i(G). \tag{11}$$

From the structure of L and from (11) we get that $\frac{|M|}{|M \cup N(M)|} = \frac{|L|}{|L \cup N(L)|} > i(G)$. Since U_1 and $A \setminus B = A \setminus N(A)$ are independent and $N(A \setminus B) \cap U_1 = \emptyset$, we have that M is an independent set. Thus setting $K = M$ we found the independent set in $G^{\times(k-1)}$ the existence of which is claimed by the lemma. $\square$

Proof of Theorem 2. Suppose indirectly that there is a positive integer k for which $i(G^{\times k}) \neq i(G)$. As the sequence $\{i(G^{\times k})\}_{k=1}^{\infty}$ is nondecreasing we get that there is a maximal independent set I in $G^{\times k}$ for which $i(G) < \frac{|I|}{|V(G^{\times k})|} = \frac{|I|}{|I \cup N(I)|}$. By the iterative use of Lemma 4 we obtain that there is an independent set J in G for which $\frac{|J|}{|J \cup N(J)|} > i(G) = \frac{\alpha(G)}{|V(G)|}$. As $|J| \leq \alpha(G)$ and since the assumption $d(v) \geq |V(G)| - \alpha(G)$ for all vertices v of G implies that $|N(J)| \geq |V(G)| - \alpha(G)$, we get that $\frac{|J|}{|J \cup N(J)|} \leq i(G)$. Hence we got a contradiction proving the theorem. □

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