On the ultimate lexicographic Hall-ratio

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October 30, 2008

Abstract

The Hall-ratio $\rho(G)$ of a graph G is the ratio of the number of vertices and the independence number maximized over all subgraphs of G. The ultimate lexicographic Hall-ratio of a graph G is defined as $\lim_{n\to\infty} \sqrt[n]{\rho(G^{\circ n})}$, where $G^{\circ n}$ denotes the nth lexicographic power of G (that is, n times repeated substitution of G into itself). Here we prove the conjecture of Simonyi stating that the ultimate lexicographic Hall-ratio equals to the fractional chromatic number for all graphs.

1 Introduction

The *Hall-ratio* of a graph G was investigated in [1, 2] where it is defined as

$$\rho(G) = \max\left\{\frac{|V(H)|}{\alpha(H)} \,:\, H \subseteq G\right\},$$

that is, as the ratio of the number of vertices and the independence number maximized over all subgraphs of G. (See also [3] and some of the references therein for an earlier appearance of the same notion on a different name.)

The asymptotic values of the Hall-ratio for different graph powers were investigated by Simonyi [8]. Among others, he considered the (appropriately normalized) asymptotic values of the Hall-ratio for the three exponentiations called normal, co-normal and lexicographic, respectively. In this paper we deal mainly with the asymptotic value of the Hall-ratio with respect to the lexicographic power. (Other questions related to the Hall-ratio, the fractional chromatic number and the lexicographic power discussed in [5].)

For two graphs F and G, their *lexicographic product* $F \circ G$ is defined on the vertex set $V(F \circ G) = V(F) \times V(G)$ with edge set $E(F \circ G) = \{\{u_1v_1, u_2v_2\} : \{u_1, u_2\} \in E(F),\text{ or } u_1 = u_2 \text{ and } \{v_1, v_2\} \in E(G)\}$. The nth lexicographic power $G^{\circ n}$ is the n-fold lexicographic product of G. The lexicographic product $F \circ G$ also known as the substitution of G into all vertices of F, the name we use follows the book [4].

Definition. ([8]) The *ultimate lexicographic Hall-ratio* of graph G is

$$h_\circ(G) = \lim_{n\to\infty} \sqrt[n]{\rho(G^{\circ n})}.$$

The normal and co-normal products of two graphs F and G are also defined on $V(F) \times V(G)$ as vertex sets and their edge sets are such that $E(F \odot G) \subseteq E(F \circ G) \subseteq E(F \cdot G)$ holds, where $F \odot G$ denotes the normal, $F \cdot G$ the co-normal product of F and G.
(In particular, $\{u_1v_1, u_2v_2\} \in E(F \odot G)$ if $\{u_1, u_2\} \in E(F)$ and $\{v_1, v_2\} \in E(G)$, or $\{u_1, u_2\} \in E(F)$ and $v_1 = v_2$, or $u_1 = u_2$ and $\{v_1, v_2\} \in E(G)$, while $\{u_1v_1, u_2v_2\} \in E(F \cdot G)$ if $\{u_1, u_2\} \in E(F)$ or $\{v_1, v_2\} \in E(G)$.)

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Denoting by $h_{\odot}(G)$ and $h(G)$ the normalized asymptotic values analogous to $h_\circ(G)$ for the normal and co-normal power, respectively, Simonyi [8] proved that $h(G) = \chi_f(G)$, where $\chi_f(G)$ is the fractional chromatic number of graph G, while $h_{\odot}(G) = R(G)$, where $R(G)$ denotes the so-called Witsenhausen rate. The latter is the normalized asymptotic value of the chromatic number with respect to the normal power and is introduced by Witsenhausen in [9] where its information theoretic relevance is also explained. The fractional chromatic number is the well-known graph invariant one obtains from the fractional relaxation of the integer program defining the chromatic number, see [7] for more details.

It follows from the above discussion that the value of $h_\circ(G)$ falls into the interval $[R(G), \chi_f(G)]$. We remark that the lower bound $R(G)$ is sometimes better but sometimes worse than the easy lower bound $\rho(G)$, cf. [8]. Thus we know that

$$\max\{\rho(G), R(G)\} \leq h_\circ(G) \leq \chi_f(G).$$

For some types of graphs the upper and lower bounds are equal, so this formula gives the exact value of the ultimate lexicographic Hall-ratio. For instance, if G is a perfect graph, then $\chi_f(G) = \chi(G) = \omega(G) \leq \rho(G)$. If G is a vertex-transitive graph, then $\chi_f(G) = \frac{|V(G)|}{\alpha(G)} \leq \rho(G)$. (The proof of the fact that $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$ holds for vertex-transitive graphs, can be found for example in [7].)

The length of the interval $[\max\{\rho(G), R(G)\}, \chi_f(G)]$ is positive in general. An example is the 5-wheel, W_5 constisting of a 5-length cycle and an additional point joint to every vertex of the cycle. It is clear that $\rho(W_5) = 3$. To get an upper bound for $R(W_5)$, one can find a coloring of $C_5^{\odot 2}$ with 5 colors (see [9]) which can be completed to a coloring of $W_5^{\odot 2}$ with 12 colors, so $\chi(W_5^{\odot 2}) \leq 12$, since $\chi(G^{\odot n}) \leq (\chi(G))^n$ (see, e.g., [4] for the easy proof) and by the definition of $R(G)$ we get $R(W_5) \leq \sqrt{12}$. Furthermore, $\chi_f(W_5) = \chi_f(C_5) + 1 = \frac{7}{2} > \max\{3, \sqrt{12}\}$.

It was conjectured in [8], that in fact, $h_\circ(G)$ always coincides with the larger end of the above interval. The main goal of this paper is to prove this conjecture.

2 The ultimate lexicographic Hall-ratio

In this section we prove our main result.

Theorem 1.

$$h_\circ(G) = \chi_f(G)$$

We know $h_\circ(G) \leq \chi_f(G)$ thus it is enough to prove the reverse inequality.

Preparing for the proof we introduce some notations. Let n be a positive integer and let α be a positive real number. Denote by $p_G(n, \alpha)$ the number of vertices maximized over all subgraphs of $G^{\circ n}$ with independence number at most α, that is

$$p_G(n, \alpha) = \max\{|V(H)| \,:\, H \subseteq G^{\circ n}, \alpha(H) \leq \alpha\}$$

and let

$$q_G(n, \alpha) = \frac{p_G(n, \alpha)}{\alpha}.$$

Clearly, $p_G(n, \alpha) = p_G(n, \lfloor\alpha\rfloor)$ and $q_G(n, \alpha) \leq q_G(n, \lfloor\alpha\rfloor)$. In spite of this fact it will be useful that $p_G(n, \alpha)$ is defined also for non-integral α values.

Now we are going to prove some technical lemmas.

Lemma 2. *The ultimate lexicographic Hall-ratio can be expressed by the values of $q_G(n, \alpha)$ as follows.*

$$h_\circ(G) = \lim_{n\to\infty} \max\left\{ \sqrt[n]{q_G(n, \alpha)} \,:\, \alpha \in \mathbb{R}_+ \right\} \tag{1}$$

Proof. The Hall-ratio of the nth lexicographic power of G can be calculated by the above terms the following simple way:

$$\rho(G^{\circ n}) = \sup\{q_G(n,\alpha) \, : \, \alpha \in \mathbb{R}_+\}.$$

Since $p_G(n,\alpha)$ is a bounded, monotone increasing function and $q_G(n,\alpha)$ is the fraction of this and the strictly monotone increasing identity function, the above supremum is always reached. Since $q_G(n,\alpha) \leq q_G(n,\lfloor\alpha\rfloor)$, it is reached at some integer value of α, so the maximum value belongs to one of the subgraphs of $G^{\circ n}$. Thus we get $h_\circ(G) = \lim\limits_{n\to\infty} \sqrt[n]{\rho(G^{\circ n})} = \lim\limits_{n\to\infty} \max\left\{ \sqrt[n]{q_G(n,\alpha)} \, : \, \alpha \in \mathbb{R}_+ \right\}.$ □

Thus our aim is to show that $\lim\limits_{n\to\infty} \max\left\{ \sqrt[n]{q_G(n,\alpha)} \, : \, \alpha \in \mathbb{R}_+ \right\} \geq \chi_f(G).$

Let $g : V(G) \to \mathbb{R}_{+,0}$ be an optimal fractional clique of G. That is, (denoting the set of independent sets in G by $S(G)$), it is a fractional clique:

$$\forall U \in S(G) : \sum_{v\in U} g(v) \leq 1, \tag{2}$$

and it is optimal:

$$\sum_{v\in V(G)} g(v) = \chi_f(G). \tag{3}$$

Lemma 3.

$$q_G(n,\alpha) \geq \sum_{v\in V(G)} g(v) q_G\left(n-1, g(v)\alpha\right)$$

Proof. Every subgraph of $G^{\circ n}$ can be imagined as if the vertices of G would be substituted by subgraphs of $G^{\circ(n-1)}$. Furthermore, every independent set of $G^{\circ n}$ can be thought of as having the vertices of an independent set of G substituted by independent sets of (the above subgraphs of) $G^{\circ(n-1)}$.
If we substitute every vertex v of G by a subgraph of $G^{\circ(n-1)}$ with independence number at most $g(v)\alpha$, then we get a subgraph of $G^{\circ n}$ with independence number at most $\max\limits_{U\in S(G)} \sum\limits_{v\in U} g(v)\alpha \leq \alpha \max\limits_{U\in S(G)} \sum\limits_{v\in U} g(v) \leq \alpha$, because of (2).
Thus we get

$$p_G(n,\alpha) \geq \sum_{v\in G} p_G\left(n-1, g(v)\alpha\right).$$

It follows from this inequality and the definition of $q_G(n,\alpha)$ that

$$q_G(n,\alpha) = \frac{p_G(n,\alpha)}{\alpha} \geq \frac{1}{\alpha}\sum_{v\in G} p_G\left(n-1, g(v)\alpha\right) = \sum_{v\in G} \frac{g(v)\alpha}{\alpha}\frac{p_G\left(n-1, g(v)\alpha\right)}{g(v)\alpha} = \sum_{v\in V(G)} g(v) q_G\left(n-1, g(v)\alpha\right).$$

□

Next we bound the $q_G(n,\alpha)$ function from below, it will be important for later calculations. Let us define function $r_G(n,\alpha)$ as follows.

$$r_G(1,\alpha) = \begin{cases} c_G, & \text{if } 1 \leq \alpha \leq m = |V(G)| \\ 0, & \text{otherwise} \end{cases}$$

where c_G is a positive constant, which bounds $q_G(1,\alpha)$ from below for all $1 \leq \alpha \leq m = |V(G)|$. Such c_G exists, for example $c_G = \frac{1}{m}$ is a good choice.
For $n \geq 2$ let

$$r_G(n,\alpha) = \sum_{v\in V(G)} g(v) r_G\left(n-1, g(v)\alpha\right).$$

By Lemma 3 and by the construction of $r_G(n,\alpha)$ it holds for all positive integer n and all positive real number α that

$$r_G(n,\alpha) \le q_G(n,\alpha). \tag{4}$$

Thus it is enough to show that $\limsup_{n\to\infty} \max\left\{ \sqrt[n]{r_G(n,\alpha)} \,:\, \alpha \in \mathbb{R}_+ \right\} \ge \chi_f(G)$.

To make the calculations simpler, we express α as m^β, that is $\beta = \log_m \alpha$ and introduce

$$s_G(n,\beta) = r_G(n,m^\beta),$$

where n is a positive integer, β is a real number. Since this transformation does not change the maximum value of the function (only its place), it holds that

$$\max\left\{ \sqrt[n]{r_G(n,\alpha)} \,:\, \alpha \in \mathbb{R}_+ \right\} = \max\left\{ \sqrt[n]{s_G(n,\beta)} \,:\, \beta \in \mathbb{R} \right\}. \tag{5}$$

Thus it is enough to prove that $\limsup_{n\to\infty} \max\left\{ \sqrt[n]{s_G(n,\beta)} \,:\, \beta \in \mathbb{R} \right\} \ge \chi_f(G)$.

Observe that the following equalities hold:

$$s_G(1,\beta) = \begin{cases} c_G, & \text{if } 0 \le \beta \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$s_G(n,\beta) = \sum_{v\in V(G)} g(v) s_G\left(n-1, \log_m g(v) + \beta\right), \quad n \ge 2.$$

We get the formula for $s_G(1,\beta)$ from the definition of the function $s_G(n,\beta)$. The second equality follows by writing

$$s_G(n,\beta) = r_G(n,m^\beta) = \sum_{v\in V(G)} g(v) r_G\left(n-1, g(v)m^\beta\right) = \sum_{v\in V(G)} g(v) s_G\left(n-1, \log_m(g(v)m^\beta)\right) =$$
$$= \sum_{v\in V(G)} g(v) s_G\left(n-1, \log_m g(v) + \beta\right).$$

Lemma 4. *It holds for all graph G that*

$$\limsup_{n\to\infty} \max\left\{ \sqrt[n]{s_G(n,\beta)} \,:\, \beta \in \mathbb{R} \right\} \ge \chi_f(G). \tag{6}$$

Proof. Let us determine the integral of the function $s_G(n,\beta)$.

$$\int_{\beta=-\infty}^{\infty} s_G(1,\beta)\,\mathrm{d}\beta = c_G$$

$$\int_{\beta=-\infty}^{\infty} s_G(n,\beta)\,\mathrm{d}\beta = \int_{\beta=-\infty}^{\infty} \sum_{v\in V(G)} g(v) s_G\left(n-1, \log_m g(v) + \beta\right)\,\mathrm{d}\beta =$$
$$= \sum_{v\in V(G)} \left(g(v) \int_{\beta=-\infty}^{\infty} s_G(n-1, \log_m g(v) + \beta)\,\mathrm{d}\beta \right) =$$
$$= \sum_{v\in V(G)} \left(g(v) \int_{\beta=-\infty}^{\infty} s_G(n-1, \beta)\,\mathrm{d}\beta \right) =$$
$$= \left(\sum_{v\in V(G)} g(v) \right) \int_{\beta=-\infty}^{\infty} s_G(n-1,\beta)\,\mathrm{d}\beta = \chi_f(G) \int_{\beta=-\infty}^{\infty} s_G(n-1,\beta)\,\mathrm{d}\beta, \qquad n \ge 2,$$

where in the last equation we used (3). Hence,

$$\int_{\beta=-\infty}^{\infty} s_G(n,\beta)\,\mathrm{d}\beta = c_G(\chi_f(G))^{n-1}.$$

For a function $f(x)$ we call the support of $f(x)$, denoted by $T(f(x))$, the set of reals x for which $f(x) \neq 0$. Let us determine $T(s_G(n,\beta))$.
$T(s_G(1,\beta)) = [0,1]$. Let g_G be any real value satisfying $g_G \leq \log_m g(v) \leq 0$ for all $v \in V(G)$. Such g_G exists, for example $g_G = \min\{\log_m g(v) \,:\, v \in V(G)\}$ is a good choice. Thus $T(s_G(n,\beta)) \subseteq [0, 1-(n-1)g_G]$.

It is clear from the above discussion that $\int_{\beta=-\infty}^{\infty} s_G(n,\beta)\,\mathrm{d}\beta$ asymptotically equals to $(\chi_f(G))^n$, i.e., the limit of their fraction equals 1 as n goes to infinity. The length of the support of $s_G(n,\beta)$ can be bounded from above by a linear function of n, let this function be $d_G n$ where d_G is a constant. These facts imply that $\limsup\limits_{n\to\infty} \max\left\{ \sqrt[n]{s_G(n,\beta)} \,:\, \beta \in \mathbb{R}\right\} \geq \chi_f(G)$. Suppose indirectly that there is an $\varepsilon > 0$ and $N \in \mathbb{N}_+$, for which $\forall n > N$, $\forall \beta \in \mathbb{R}$: $s_G(n,\beta) < (\chi_f(G)-\varepsilon)^n$, then $\int_{\beta=-\infty}^{\infty} s_G(n,\beta)\,\mathrm{d}\beta < d_G n(\chi_f(G)-\varepsilon)^n$. Since $\lim\limits_{n\to\infty} \frac{d_G n(\chi_f(G)-\varepsilon)^n}{\chi_f(G)^n} = \lim\limits_{n\to\infty} (1-\frac{\varepsilon}{\chi_f(G)})^n = 0$, it is in contradiction with the statement at the begining of this paragraph. $\square$

By now we have essentially proved Theorem 1, it needs only to be summarized.
Proof of Theorem 1. The preceding lemmas imply that

$$h_\circ(G) = \lim_{n\to\infty} \max\left\{ \sqrt[n]{q_G(n,\alpha)} \,:\, \alpha \in \mathbb{R}_+\right\} \geq \limsup_{n\to\infty} \max\left\{ \sqrt[n]{r_G(n,\alpha)} \,:\, \alpha \in \mathbb{R}_+\right\} =$$
$$= \limsup_{n\to\infty} \max\left\{ \sqrt[n]{s_G(n,\beta)} \,:\, \beta \in \mathbb{R}\right\} \geq \chi_f(G),$$

where the stated relations follow from (1), (4), (5) and (6), respectively.

Thus we have proved

$$h_\circ(G) = \chi_f(G).$$

$\square$

Remark. There are graphs for which the sequence $\left\{ \sqrt[n]{\rho(G^{\circ n})} \right\}_{n=1}^{\infty}$ does not reach its limit $\chi_f(G)$ for any finite n. The 5-wheel is an example for which no t attains $\sqrt[t]{\rho(W_5^{\circ t})} = \chi_f(W_5) = \frac{7}{2}$. This is because if there was such a t then there must be a subgraph H of $W_5^{\circ t}$ for which $\frac{|V(H)|}{\alpha(H)} = \left(\frac{7}{2}\right)^t = \frac{7^t}{2^t}$, but this fraction is irreducible and $|V(H)| \leq |V(W_5^{\circ t})| = 6^t$.

Remark. It is known from the theorem of McEliece and Posner [6] (cf. also in [7]) that the normalized asymptotic value of the chromatic number with respect to the co-normal product is the fractional chromatic number. This theorem with the result proven here implies that the normalized asymptotic value of each of the Hall-ratio, the fractional chromatic number and the chromatic number with respect to both the co-normal and the lexicographic power equals to the fractional chromatic number. This is because $\rho(G) \leq \chi_f(G) \leq \chi(G)$ holds for every graph G and the lexicographic power of a graph is a subgraph of its co-normal power. These relations were already known except for the asymptotic value of the Hall-ratio for the lexicographic power. As we mentioned, it is proven in [8] that the normalized asymptotic value of the Hall-ratio for the co-normal power equals to the fractional chormatic number. The multiplicativity of the fractional chromatic number for the lexicographic product is a theorem in [4].

3 On the ultimate direct Hall-ratio

An analogous asymptotic value of the Hall-ratio can be defined also with respect to the direct power. For two graphs F and G, their *direct* or *categorical product* $F \times G$ is defined on the vertex set $V(F \times G) = V(F) \times V(G)$ with edge set $E(F \times G) = \{\{(u_1, v_1), (u_2, v_2)\} : \{u_1, u_2\} \in E(F) \text{ and } \{v_1, v_2\} \in E(G)\}$. The nth direct power $G^{\times n}$ is the n-fold direct product of G. The *ultimate direct Hall-ratio*, $h_\times(G) = \lim_{n\to\infty} \rho(G^{\times n})$ was defined in [8]. It is shown there that this graph parameter is bounded from above by the fractional chromatic number and conjectured that equality holds for all graphs.

It is easy to see that this conjecture holds for perfect and vertex-transitive graphs. It is proved in [8] that it is also true for wheel graphs. By using a similar argument which was used in the proof of that result we prove the following generalization.

Proposition 5. *Let G be a graph for which $h_\times(G) = \chi_f(G)$ holds. Let $\hat{G}$ be the graph we obtain from G by connecting each of its vertices to an additional vertex. Then $h_\times(\hat{G}) = \chi_f(\hat{G})$ holds, too.*

Proof. $h_\times(G) = \lim_{n\to\infty} \rho(G^{\times n}) = \chi_f(G)$ means that

$$\forall \varepsilon > 0 : \exists n_0(\varepsilon) : \forall n \geq n_0 : \rho(G^{\times n}) \geq \chi_f(G) - \varepsilon \tag{7}$$

by definition of the limit and since $h_\times(G) \leq \chi_f(G)$.

Adding a new vertex w to G increases the fractional chromatic number by 1, as it does not lie in a common independent set with any other vertex of the graph. Therefore $\chi_f(\hat{G}) = \chi_f(G) + 1$.

Thus we have to show that $h_\times(\hat{G}) = \lim_{n\to\infty} \rho(\hat{G}^{\times n}) = \chi_f(\hat{G}) = \chi_f(G) + 1$, i.e.,

$$\forall \varepsilon > 0 : \exists \hat{n}_0(\varepsilon) : \forall n \geq \hat{n}_0 : \rho(\hat{G}^{\times n}) \geq \chi_f(G) + 1 - \varepsilon. \tag{8}$$

By the monoton increasing property of the sequence $\left\{\rho(G^{\times i})\right\}_{i=1}^{\infty}$ it is enough to find for all ε a suitable $\hat{n}_0$ for which $\rho(\hat{G}^{\times n_0}) \geq \chi_f(G) + 1 - \varepsilon$. It follows from (7) that for all $\varepsilon > 0$ there is an n_0 and $H \subseteq G^{\times n_0}$, for which $\frac{|V(H)|}{\alpha(H)} \geq \chi_f(G) - \varepsilon$ holds. Denote by k the number of vertices and by α the independence number of H. Let $v_1, v_2, \ldots v_k$ be the vertices of H and let $v_1, v_2, \ldots v_\alpha$ be the vertices of a maximum size independent set in H. Let $\hat{H}$ be the subgraph of $\hat{G}^{\times 2n_0}$ induced on the vertex set $P_1 \cup P_2 \cup Q$, where

$$\begin{aligned} P_1 &= \{(v_1, w^{n_0}), (v_2, w^{n_0}), \ldots (v_\alpha, w^{n_0})\}, \\ P_2 &= \{(w^{n_0}, v_1), (w^{n_0}, v_2), \ldots (w^{n_0}, v_\alpha)\} \text{ and} \\ Q &= \{(v_{\alpha+1}, v_{\alpha+1}), (v_{\alpha+2}, v_{\alpha+2}), \ldots (v_k, v_k)\}. \end{aligned}$$

The number of vertices of $\hat{H}$ is $k + \alpha$. Its independence number is less than or equal to α, because on the vertex set $P_1 \cup P_2$ we get a complete bipartite graph, thus every independent set of $\hat{H}$ can contain vertices only from P_1 or only from P_2, but on the set $P_1 \cup Q$ and $P_2 \cup Q$ the induced subgraph isomorph to H.
It follows that $\frac{|V(\hat{H})|}{\alpha(\hat{H})} \geq \frac{k+\alpha}{\alpha} = \frac{k}{\alpha} + 1 = \frac{|V(H)|}{\alpha(H)} + 1 \geq \chi_f(G) + 1 - \varepsilon$, thus $\hat{n}_0 = 2n_0$ is a good choice to satisfy (8).

Thus we have proved that $h_\times(\hat{G}) = \lim_{n\to\infty} \rho(\hat{G}^{\times n}) = \chi_f(\hat{G})$. □

Acknowledgement

I am grateful to Gábor Simonyi for helpful discussions throughout the whole time of this research.

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