Further remarks on long monochromatic cycles in edge-colored complete graphs

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Abstract

In [Discrete Math., **311** (2011), 688–689], Fujita defined $f(r,n)$ to be the maximum integer k such that every r-edge-coloring of K_n contains a monochromatic cycle of length at least k. In this paper we investigate the values of $f(r,n)$ when n is linear in r. We determine the value of $f(r,2r+2)$ for all $r \geq 1$ and show that $f(r,sr+c) = s+1$ if r is sufficiently large compared with posivite integers s and c.

1 Introduction

The *circumference* $c(G)$ of a graph G is the length of a longest cycle in G. In [4] Faudree et al. showed that for every graph G of order $n \geq 6$ we have $\max\{c(G), c(\overline{G})\} \geq \lceil 2n/3 \rceil$, where $\overline{G}$ denotes the *complement* of G. Furthermore, this bound is sharp.

Fujita [5] introduced the following concept and notation. Let $f(r,n)$ be the maximum integer k such that every r-edge-coloring of K_n contains a monochromatic cycle of length at least k. (For $i \in \{1,2\}$, we regard K_i as a cycle of length i.) Thus, Faudree et al. proved that $f(2,n) = \lceil 2n/3 \rceil$ for $n \geq 6$. Furthermore, they showed that $f(r,n) \leq \lceil n/(r-1) \rceil$ for infinitely many r and, for each such r, infinitely many n and conjectured that $f(r,n) \geq \lceil n/(r-1) \rceil$ for $r \geq 3$. However, Fujita [5] showed that this conjecture is not true for small n and r and then established the following lower bound for $f(r,n)$.

Theorem 1 ([5]). *For $1 \leq r \leq n$ we have $f(r,n) \geq \lceil n/r \rceil$.*

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He also showed that if $1 < n \leq 2r$ then $f(r,n) = 2$, while if $n = 2r+1$ then $f(r,n) = 3$ for $r \geq 1$. Motivated by his results we investigate the values of $f(r,n)$ when n is linear in r. In Section 2 we will consider the values of $f(r, 2r+2)$ for $r \geq 1$. In Section 3 we will show that $f(r, sr+c) = s+1$ if r is sufficiently large with respect to s and c. For terminology and notation not defined here we refer the reader to [2].

2 The value of $f(r, 2r+2)$

In this section we determine the exact value of $f(r, 2r+2)$ for all $r \geq 1$. By Theorem 1 we have that $f(r, 2r+2) \geq 3$. To show the reverse inequality for $r \geq 4$ we will use the following result of Ray-Chaudhury and Wilson (see [6]) regarding Kirkman Triple Systems. We handle the cases $r = 1, 2, 3$ separately.

Theorem 2 ([6])**.** *For any $t \geq 1$, the edge set of K_{6t+3} can be partitioned into $3t+1$ parts, where each part forms a graph isomorphic to $2t+1$ disjoint triangles.*

Theorem 3. *For $r \geq 3$, we have $f(r, 2r+2) = 3$. For $r = 1, 2$, we have $f(r, 2r+2) = 4$.*

Proof. Firstly, we consider the case $r \geq 4$, and proceed according to the residue of r modulo 3.

Claim 4. *$f(r, 2r+2) \leq 3$ for $r = 4, 7, 10, 13, \ldots$, that is, $r = 3k+1, k \geq 1$.*

Proof. For $r = 3k+1$ we have $n = 6k+4$. We start with a coloring of the edges of K_{6k+3} on the vertices $v_1, v_2, \ldots, v_{6k+3}$ with colors $c_1, c_2, \ldots, c_{3k+1}$ according to Theorem 2. It remains to color the edges incident with vertex v_{6k+4}. Without loss of generality we may assume that color c_{3k+1} contains the triangles on the vertices $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$, $\ldots$,$\{v_{6k+1}, v_{6k+2}, v_{6k+3}\}$. We color the edges from v_{6k+4} to the vertices $v_3, v_6, \ldots, v_{6k+3}$ with c_{3k+1}. The edges from v_{6k+4} to v_{3i-1} and v_{3i-2} will be colored with c_i for $i = 1, 2, \ldots, 2k+1 (\leq 3k)$. As the edges from v_{6k+4} colored with c_i ($i = 1, 2, \ldots, 2k+1$ or $i = 3k+1$) go to different c_i-colored triangles on the vertices $v_1, v_2, \ldots, v_{6k+3}$, the coloring so obtained does not contain a monochromatic cycle of length more than three. $\square$

Claim 5. *$f(r, 2r+2) \leq 3$ for $r = 5, 8, 11, 14, \ldots$, that is, $r = 3k+2, k \geq 1$.*

Proof. For $r = 3k+2$ we have $n = 6k+6$. As in the previous case we start with a coloring of the edges of K_{6k+3} on the vertices $v_1, v_2, \ldots, v_{6k+3}$ with colors $c_1, c_2, \ldots, c_{3k+1}$ according to Theorem 2. Now it remains to color the edges incident with three vertices, v_{6k+4}, v_{6k+5}, v_{6k+6}, and we have one unused color, c_{3k+2}. Without loss of generality we may assume that color c_{3k+1} contains the triangles on the vertices $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$,$\ldots$,$\{v_{6k+1}, v_{6k+2}, v_{6k+3}\}$. We color the edges from v_{6k+6} to $v_3, v_6, \ldots, v_{6k+3}$ with c_{3k+1}. We give color c_i for $i = 1, 2, \ldots, 2k+1 (\leq 3k)$ to the edges from v_{6k+4} to v_{3i} and to v_{3i-1}, from v_{6k+5} to v_{3i-1} and to v_{3i-2}, from v_{6k+6} to v_{3i-2}. (See Figure 1.)

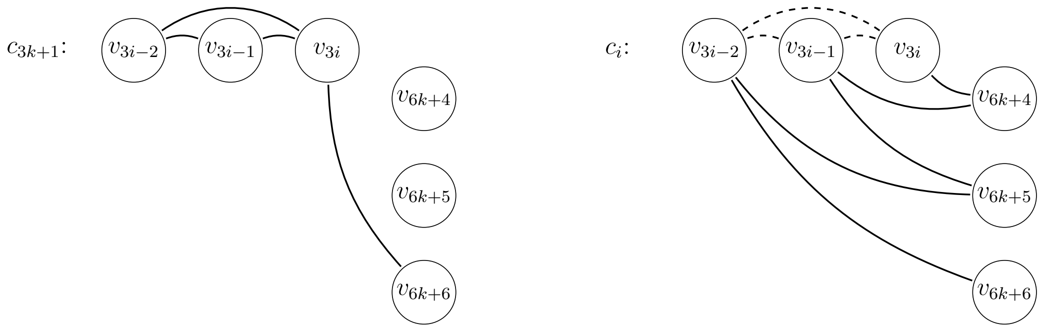

Figure 1: The edge between $v_{3i-2}, v_{3i-1}, v_{3i}$ and $v_{6k+4}, v_{6k+5}, v_{6k+6}$, in color c_{3k+1} and in color c_i ($i \in \{1, 2, \dots, 2k+1\}$), respectively. The dashed edges are missing.

We left one edge from each of the vertices $v_1, v_2, \dots, v_{6k+3}$ (from v_{3i-2} to v_{6k+4}, from v_{3i-1} to v_{6k+6}, from v_{3i} to v_{6k+5}, for $i = 1, 2, \dots, 2k+1$) and the 3 edges between $v_{6k+4}, v_{6k+5}, v_{6k+6}$. We color these edges with color c_{3k+2}. It is easy to check that in this coloring every monochromatic cycle is a triangle. $\square$

In the third case we prove the following stronger statement.

Claim 6. *$f(r, 2r+3) \leq 3$ for $r = 6, 9, 12, 15, \dots$, that is, $r = 3k, k \geq 2$.*

As $f(r, n_1) \leq f(r, n_2)$ if $n_1 \leq n_2$ this implies $f(r, 2r+2) = 3$ for $r = 6, 9, 12, 15, \dots$, that is, $r = 3k, k \geq 2$.

Proof. For $r = 3k$ we have $n = 6k+3$. We start with a coloring of the edges of K_{6k+3} on the vertices $v_1, v_2, \dots, v_{6k+3}$ with colors $c_1, c_2, \dots, c_{3k}$ and c_{3k+1} according to Theorem 2. In contrast with the previous cases now we have to get rid of one color. We may assume that color c_{3k+1} contains the $2k+1$ triangles on the vertices $\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}, \dots, \{v_{6k+1}, v_{6k+2}, v_{6k+3}\}$. We recolor the edges of the ith triangle to color c_i for $i = 1, 2, \dots, 2k+1 (\leq 3k)$ and obtain the desired coloring of the edges of K_{6k+3}. $\square$

It remains to deal with the small values of r.

Claim 7. *$f(r, 2r+2) = 4$ for $r = 1, 2$.*

Proof. $f(1, 4) = 4$ is trivial. (In general, $f(1, n) = n$.)

We get $f(2, 6) \geq 4$ from the fact that a graph of order 6 without a cycle of length at least four can have at most 7 edges (see [3] for the general result) while K_6 has 15 edges. The reverse inequality follows from the construction $E(K_6) = E(K_{2,4}) \cup E(\overline{K_{2,4}})$. $\square$

Claim 8. *$f(3, 8) \leq 3$.*

Proof. In order to show the claim we give the following list of edges in color $1, 2, 3$ on the K_8. In what follows we let $V(K_8) = \{1, 2, \dots, 8\}$ and let E_i be the edge set of color i. (See also Figure 2.)

$$E_1 = \{\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,5\}, \{3,5\}, \{4,6\}, \{4,7\}\}$$
$$E_2 = \{\{1,2\}, \{1,7\}, \{2,6\}, \{2,7\}, \{2,8\}, \{3,6\}, \{4,5\}, \{4,8\}, \{5,8\}, \{6,8\}\}$$
$$E_3 = \{\{1,8\}, \{2,3\}, \{2,4\}, \{3,4\}, \{3,7\}, \{3,8\}, \{5,6\}, \{5,7\}, \{6,7\}, \{7,8\}\}$$

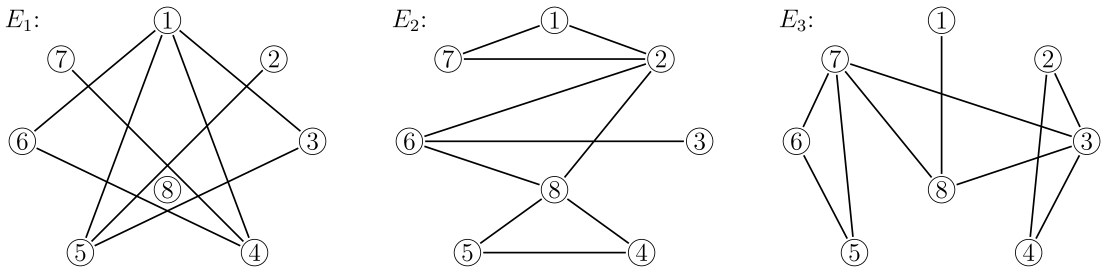

Figure 2: The graphs on $V(K_8)$ with edge sets E_1, E_2 and E_3, respectively.

One can easily check that the above coloring shows $f(3, 8) \leq 3$. □

This completes the proof of Theorem 3. □

3 On the value of $f(r, sr + c)$ for positive constants s and c

In the previous section we determined $f(r, 2r+2)$ for every $r \geq 1$. This suggests the more general problem: determine $f(r, sr + c)$ for positive constants s and c. Of course, $f(r, sr + c) \geq s + 1$ by Theorem 1. In Theorem 10 we show that $f(r, sr + c) = s + 1$ for r sufficiently large with respect to s and c. In order to do so, we will exhibit an r-edge-coloring of K_{sr+c} in which the longest monochromatic cycle has length $s + 1$. The edge-colorings used in the proof of Theorem 3 depended heavily on Theorem 2. The proof of Theorem 10 will, in an analogous manner, depend on Theorem 9. This is an immediate consequence of a result by Chang [1] on resolvable balanced incomplete block designs. For information on such designs, see [7].

Theorem 9 ([1])**.** *Let $q \geq 3$. Then for sufficiently large t (namely if $q(q-1)t + q > \exp\{\exp\{q^{12q^2}\}\}$ is satisfied), the edge set of $K_{q(q-1)t+q}$ can be partitioned into $qt + 1$ parts, where each part is isomorphic to $(q-1)t + 1$ disjoint copies of K_q.*

Observe that the case $q = 3$ in Theorem 9 is Theorem 2 (where t sufficiently large is simply $t \geq 1$).

Theorem 10. *For any pair of integers s, c with $s, c \geq 2$, there is an R such that $f(r, sr + c) = s + 1$ for all $r \geq R$.*

Proof. As $f(r, n)$ is monotone increasing in n we may assume that $sr + c = (s+1)st + (s+1)$ for some t. First we color the edges of $K_{(s+1)st+(s+1)}$ with $(s+1)t + 1 = r + \frac{c-1}{s}$ colors using Theorem 9 for $q = s + 1$. Then we reduce the number of colors by $\frac{c-1}{s}$ in the following way. Considering two colors c_1 and c_2 we want to recolor as many c_1-colored K_{s+1}'s to c_2 as we can (without creating a monochromatic cycle of length at least $s + 2$). Every color class consists of $st+1 = \frac{s}{s+1}r + \frac{c}{s+1}$ disjoint K_{s+1}'s and every c_1-colored K_{s+1} intersects $s+1$ copies of c_2-colored K_{s+1}'s. If we recolor such c_1-colored K_{s+1}'s which do not share intersecting c_2-colored K_{s+1}'s then we cannot create new monochromatic cycles. Hence recoloring a c_1-colored K_{s+1} can exclude at most $s(s+1)$ others. Therefore we can recolor at least $\frac{1}{s(s+1)+1}$th of the c_1-colored K_{s+1}'s with color c_2. At least $\frac{1}{s(s+1)+1}$th of the remaining c_1-colored K_{s+1}'s can be recolored with c_3, and so on. Finishing with the c_1 color class we continue with another one.

To remove one color class we need at most $\log_{\frac{s(s+1)+1}{s(s+1)}} \left(\frac{s}{s+1}r + \frac{c}{s+1}\right)$ other classes. Thus we can avoid $\frac{c-1}{s}$ color classes with the remaining r class if $\left(\frac{c-1}{s}\right) \log_{\frac{s(s+1)+1}{s(s+1)}} \left(\frac{s}{s+1}r + \frac{c}{s+1}\right) \leq r$, which is true for sufficiently large r compared with s and c. $\square$

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