DIPLOMA THESIS

Asymptotic values of graph parameters

Ágnes Tóth

Supervisor: Dr. Gábor Simonyi

Department of Computer Science and Information Theory
Faculty of Electrical Engineering and Informatics
Budapest University of Technology and Economics

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1 Introduction

Several graph parameters show an interesting behaviour when they are investigated for different powers of graphs. One of the most famous examples of such behaviour is that of the Shannon capacity of graphs (introduced by Shannon [17], see Körner, Orlitsky [12] for a survey of related topics) which is the theoretical upper limit of channel capacity for error-free coding in information theory. This graph parameter is defined as the normalized limit of the independence number under the so-called normal power and its exact value is not known even for small, simple graphs (for example odd cycles with length more then five).

The normalized asymptotic value of the chromatic number with respect to the normal power is the Witsenhausen rate. It is introduced by Witsenhausen in [19], where its information theoretic relevance is also explained.

If we investigate the chromatic number for the co-normal power we get the fractional chromatic number as the corresponding limit by a famous theorem of McEliece and Posner [14], cf. also Berge, Simonovits [4].

Similar questions arise when investigating the independence ratio and the Hall-ratio of a graph.

The independence ratio of a graph is the ratio of its independence number and its number of vertices. Its asymptotic value with respect to the so-called Cartesian power is the ultimate independence ratio which is introduced by Hell, Yu and Zhou in [10]. Motivated by this concept Brown, Nowakowski and Rall considered in [5] the analogous, but significantly different parameter, the ultimate categorical independence ratio which is defined with respect to the so-called categorical power. This parameter was also investigated by Alon and Lubetzky in [3] and the characterization of independent sets in categorical power of graphs were considered by Alon, Dinur, Friedgut and Sudakov in [2]. The authors of [5] investigated graphs for which this limit equal to the independence ratio and they called such graphs self-universal. In that article it is proven that some graph families, for example Cayley graphs of Abelian groups have this property.

The paper [5] mentions complete multipartite graphs as one of those families of graphs for which the determination of the ultimate categorical independence ratio remained an open problem. It follows from a result of that article that if the size of the biggest part is larger than half the number of vertices then its ultimate categorical independence ratio equals to one. In this diploma thesis I proved that in every other cases, i.e., when none of the parts of the complete multipartite graph has size greather than half the number of vertices then the graph is self-universal.

The Hall-ratio is also closely related to the independence ratio, this parameter is the ratio of the number of vertices and the independence number maximized over all subgraphs of the graph. It was introduced in [7, 8] motivated by problems of list coloring. The (appropriately normalized) asymptotic values of this graph parameter for different graph powers were investigated by Simonyi in [18]. Considering for normal and co-normal power he proved that the corresponding limit equals to the similar limit one gets for the chromatic number.

It is a conjecture in [18] that the ultimate categorical Hall-ratio, which is the asymptotic value of the Hall-ratio with respect to the categorical power is also equal to the fractional chromatic number. In that article it is proven that this conjecture holds for perfect graphs, vertex-transitive graphs and also for wheel graphs. I proved the following generalization of the case of wheel graphs. If the conjecture holds for a graph then it holds also for the graph obtained by adding a vertex to the vertex set of the original graph and connecting it to all its vertices.

Investigating the Hall-ratio with respect to the so-called lexicographic power we get the ultimate lexicographic Hall-ratio, which is also introduced in [18]. It is shown there that this graph parameter is bounded from above by the fractional chromatic number, and conjectured that equality holds for all graphs. In this diploma thesis I proved this conjecture.

In the next section we review some concepts and definitions which will be used in this thesis. The ultimate categorical independence ratio is investigated in the third section. Then we deal with the Hall-ratio. After summarizing the related results the ultimate Hall-ratio is investigated with respect to the categorical and the lexicographic power in the last two sections, respectively.

2 Definitions, notations

In this section some well-known graph-theoretic concepts and notations will be reviewed.

Some standard notation

We use the following notations:

$V(G)$ is the *vertex set* of graph G,

$E(G)$ is the *edge set* of G,

$\alpha(G)$ is the *independence number* of G (i.e., the size of a largest independent set of G),

$\omega(G)$ is the *clique number* of G (i.e., the order of a largest complete subgraph of G),

$\chi(G)$ is the *chromatic number* of G (i.e., the least number of colors needed to color G, such that adjacent vertices get different colors),

$d_G(v)$ (or just $d(G)$) is the *degree* of vertex v in G (i.e., the number of vertices adjacent to v) and

$N_G(U)$ (or just $N(U)$) is the *neighborhood* of the vertex set U in G (i.e., the set of vertices adjacent to either of the elements of U).

Special graph families

A graph is *perfect* if the chromatic number of its induced subgraphs equal to the clique number of that subgraph. See [15] for several interesting properties of perfect graphs.

A graph is *vertex-transitive*, if its automorphism group is transitive.

The *wheel graph* consisting of a cycle of length m and an additional point joint to every vertex of the cycle is denoted by W_m.

Fractional chromatic number and fractional clique number

$S(G)$ denotes the set of the independent sets of graph G.

Definition. *Fractional chromatic number* of graph G defined as

$$\chi_f(G) = \inf \left\{ \sum_{U \in S(G)} f(U) \,:\, \text{f is a fractional coloring of G} \right\},$$

where *fractional coloring* of G is a function $f : S(G) \to \mathbb{R}_{+,0}$ for which

$$\forall v \in V(G) \,:\, \sum_{v \in U \in S(G)} f(U) \geq 1.$$

Definition. *Fractional clique number* of graph G defined as

$$\omega_f(G) = \sup \left\{ \sum_{v \in V(G)} g(v) \,:\, \text{g is a fractional clique of G} \right\},$$

where *fractional clique* of G is a function $g : V(G) \to \mathbb{R}_{+,0}$ for which

$$\forall U \in S(G) \,:\, \sum_{v \in U \in S(G)} g(v) \leq 1.$$

The following relations hold for these concepts (see in [16]):

The infimum and the supremum in the definitions are reached. The Duality theorem of linear programming implies that the fractional chromatic number and the fractional clique number equal for every graph G. Furthermore, it follows from the definitions that the chromatic number bounds from above the fractional chromatic number and the clique number bounds from below the fractional clique number. So,

$$\omega(G) \leq \omega_f(G) = \chi_f(G) \leq \chi(G)$$

See [16] for more details.

Graph products

We investigate the categorical and the lexicographic product in more detail, the definitions of these concepts are given in the corresponding sections. We notice some properties of other graph products. These are the normal, co-normal and Cartesian products which are defined as follows.

Definition. The *normal product* $F \odot G$ of two graphs F and G is defined on the vertex set $V(F \odot G) = V(F) \times V(G)$ with edge set $E(F \odot G) = \{\{(u_1, v_1), (u_2, v_2)\} : \{u_1, u_2\} \in E(F) \text{ and } \{v_1, v_2\} \in E(G), \text{ or } \{u_1, u_2\} \in E(F) \text{ and } v_1 = v_2, \text{ or } \{v_1, v_2\} \in E(G) \text{ and } u_1 = u_2\}$.
The nth normal power $G^{\odot n}$ is the n-fold normal product of G.

That is, $G^{\odot n}$ is defined on the n-length sequences of the original vertices and two such sequences are adjacent in $G^{\odot n}$ if and only if their elements at every coordinate are either equal or form an edge in G.

Definition. The *co-normal product* $F \cdot G$ of two graphs F and G is defined on the vertex set $V(F \cdot G) = V(F) \times V(G)$ with edge set $E(F \cdot G) = \{\{(u_1, v_1), (u_2, v_2)\} : \{u_1, u_2\} \in E(F) \text{ or } \{v_1, v_2\} \in E(G)\}$.
The nth co-normal power G^n is the n-fold co-normal product of G.

That is, G^n is defined on the n-length sequences of the original vertices and two such sequences are adjacent in G^n if and only if there is some coordinate where the corresponding elements of the two sequences form an edge in G.

It is easy to see that $G^n = \overline{\bar{G}^{\odot n}}$ holds for every graph G.

Definition. The *Cartesian product* $F \Box G$ of two graphs F and G is defined on the vertex set $V(F \Box G) = V(F) \times V(G)$ with edge set $E(F \Box G) = \{\{(u_1, v_1), (u_2, v_2)\} : \{u_1, u_2\} \in E(F) \text{ and } v_1 = v_2, \text{ or } \{v_1, v_2\} \in E(G) \text{ and } u_1 = u_2\}$.
The nth Cartesian power $G^{\Box n}$ is the n-fold Cartesian product of G.

Thus the Cartesian power is also given on the n-length sequences of the original vertices and two such sequences from an edge if and only if they differ at exactly one place and at that place the corresponding coordinates form an edge of the original graph.

About graph powers see [11].

Famous graph parameters

Definition. ([17]) The *Shannon capacity* of graph G is defined as

$$\Theta(G) = \lim_{n\to\infty} \sqrt[n]{\alpha(G^{\odot n})},$$

that is, as the normalized limit of the independence number under normal power.

Definition. ([19]) The *Witsenhausen rate* of graph G is defined as

$$R(G) = \lim_{n\to\infty} \sqrt[n]{\chi(G^{\odot n})},$$

that is, as the normalized limit of the chromatic number under normal power.

3 On the ultimate categorical independence ratio of complete multipartite graphs

In this section we investigate the independence ratio with respect to the categorical product.

3.1 The ultimate categorical independence ratio

Definition. For two graphs F and G, their *categorical* or *direct product* $F \times G$ is defined on the vertex set $V(F \times G) = V(F) \times V(G)$ with edge set $E(F \times G) = \{\{(u_1, v_1), (u_2, v_2)\} : \{u_1, u_2\} \in E(F) \text{ and } \{v_1, v_2\} \in E(G)\}$. The nth categorical power $G^{\times n}$ is the n-fold categorical product of G.

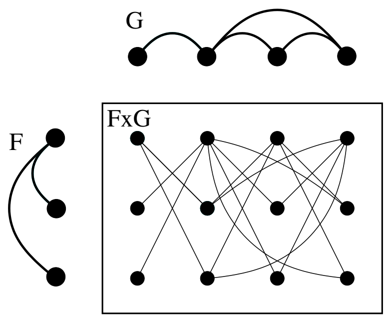

Figure 1: Categorical product of two graphs

That is, $G^{\times n}$ is defined on the n-length sequences of the original vertices and two such sequences are adjacent in $G^{\times n}$ if and only if their elements at every coordinate form an edge in G.

The *independence ratio* of a graph G is defined as

$$i(G) = \frac{\alpha(G)}{|V(G)|},$$

that is, as the ratio of the independence number and the number of vertices.

If I is an independent set in F then $I \times V(G)$ is an independent set in $F \times G$, thus

$$\alpha(F \times G) \geq \max\{\alpha(F)|V(G)|, \alpha(G)|V(F)|\}.$$

This fact implies that the sequence $\left\{i(G^{\times k})\right\}_{k=1}^{\infty}$ is nondecreasing, and as it is bounded from above by 1, it is convergent. The limit of the sequence is an interesting graph parameter which was introduced by Brown, Nowakowski and Rall in [5].

Definition. ([5]) The *ultimate categorical independence ratio* of a graph G is defined as

$$A(G) = \lim_{k\to\infty} i(G^k).$$

The ultimate independence ratio is an analogous but significantly different concept which is the asymptotic value of the independence ratio with respect to the Cartesian power. About this graph parameter see [9, 10, 20].

The following results were proven in [5] for the ultimate categorical independence ratio.
Upper bounds for $A(G)$:

Theorem 3.1. ([5]) *If G is an r-regular graph and $r \geq 1$ then $A(G) \leq \frac{1}{2}$.*

Theorem 3.2. ([5]) *Let G be a graph for which $H \cup K_{m_1} \cup \cdots \cup K_{m_p}$ is a spanning subgraph of G. If $\chi(H) \leq m_1 \leq m_2 \leq \cdots \leq m_p$ then $A(G) \leq A(H)$.*

Lower bounds for $A(G)$:

Theorem 3.3. ([5]) *If $i(G) > \frac{1}{2}$ then $A(G) = 1$.*

Theorem 3.4. ([5]) $A(G) \geq \frac{|I|}{|I \cup N_G(I)|}$ *holds for every independent subset I of G.*

Corollary 3.5. ([5]) *Let G be a graph for which $i(G) = \frac{1}{2}$. Then $A(G) = \frac{1}{2}$ if G has a perfect matching, and $A(G) = 1$ otherwise.*

Self-universal graphs

Graphs for which the ultimate categorical independence ratio equals to the independence ratio were investigated in [5] and [2].

Definition. ([5]) Graph G is *self-universal* if $A(G) = i(G)$.

In [5] the following results are proven.

Proposition 3.6. ([5]) *If G is a graph for which $G \cup G \cup \cdots \cup G$ (disjoint union of $|V(G)|$ copies of G) is a spanning subgraph of $G^{\times 2}$ then G is self-universal.*

Theorem 3.7. ([5]) *If G is a Cayley graph of an Abelian group then G is self-universal.*

(The *Cayley graph* associated with group G and generating set S is defined on the elements of G as vertex set, and two elements, g_1 and g_2 are adjacent if and only if $g_1^{-1} g_2 \in S$.)

Theorem 3.8. ([5]) *If graph G has an automorphism with a single orbit of size $|V(G)|$ then G is self-universal.*

The characterization of maximum-size independent sets in the power of a wide family of regular graphs was considered by Alon, Dinur, Friedgut and Sudakov in [2] with the help of Fourier analysis.

Theorem 3.9. ([2]) *Let G be a connected, r-regular graph on n vertices, let q be the smallest eigenvalue of the adjacency matrix of G (the largest eigenvalue is r). If $\frac{\alpha(G)}{n} = \frac{-q}{r-q}$ then G is self-universal.*

3.2 The case of complete multipartite graphs

In [5] the authors mentioned complete multipartite graphs as one of those families of graphs for which the ultimate categorical independence ratio is not determined in general, although several special cases are handled by their results.

In particular the following special cases covered by the results in [5]. It follows from Theorem 3.6 that the complete graphs are self-universal, $A(K_n) = i(K_n) = \frac{1}{n}$. Let $K_{n,m}$ be a complete bipartite graph, if $n = m$ then $A(K_{n,n}) = \frac{1}{2}$ follows from Corollary 3.5, otherwise (if $n \neq m$) we get $A(K_{n,m}) = 1$ by Theorem 3.3.

If G is a complete multipartite graph, for which the size of the largest part (which is also the independence number) is greater than half the number of vertices then by Theorem 3.3 we get $A(G) = 1$. We will prove that in every other case, i.e., when none of the parts of the complete multipartite graph has size greater than half the number of vertices we have $A(G) = i(G)$. In other words, for $i(G) \leq \frac{1}{2}$ a complete multipartite graph is self-universal.

The main result of this section is the following theorem. We get the result for the complete multipartite graphs as a simple corollary. The first condition, the one about the lower bound on the degrees is essential. The second condition is trivial, it must hold by Theorem 3.3.

Theorem 3.10. *Let G be a graph for which $\forall v \in V(G) : d(v) \geq |V(G)| - \alpha(G)$ and $i(G) \leq \frac{1}{2}$. Then $i(G^{\times k}) = i(G)$ holds for every integer $k \geq 1$.*

Corollary 3.11. *Let $G = K_{\ell_1,\ell_2,\ldots\ell_m}$ be a complete multipartite graph. Let $n = \sum_{i=1}^{m} \ell_i$ be the number of vertices and let $\ell = \max_{1\leq i\leq m} \ell_i$ be the size of the largest partite class. If $\ell \leq \frac{n}{2}$ then $A(G) = i(G) = \frac{\ell}{n}$, so G is self-universal, otherwise $A(G) = 1$.*

Proof of Corollary 3.11 from Theorem 3.10:

Proof. Since G is a complete multipartite graph $\alpha(G) = \ell$.
If $\ell > \frac{n}{2}$ then $i(G) = \frac{\ell}{n} > \frac{1}{2}$, thus $A(G) = 1$ follows from Theorem 3.3.
If $\ell \leq \frac{n}{2}$ then $i(G) = \frac{\ell}{n} \leq \frac{1}{2}$ and as G is a complete multipartite graph, degree of its vertices is at least $|V(G)| - \alpha(G)$ thus from Theorem 3.10 we get $A(G) = i(G) = \frac{\ell}{n}$. □

The conditions of Theorem 3.10 hold not just for complete multipartite graphs. For instance, the graph consisting of an 5-length cycle and three additional points joint to every vertex of the cycle also satisfies the conditions.

For the proof of Theorem 3.10 we need the following lemma.

Lemma 3.12. *Let G be a graph for which $\forall v \in V(G):\ d(v) \geq |V(G)| - \alpha(G)$ and $i(G) \leq \frac{1}{2}$ holds. If J is an independent set in $G^{\times k}$ for which $\frac{|J|}{|J \cup N(J)|} > i(G)$ holds then there is an independent set K in $G^{\times(k-1)}$ for which $\frac{|K|}{|K \cup N(K)|} > i(G)$ holds.*

Proof. Let G be a graph for which

$$\forall v \in V(G):\ d(v) \geq |V(G)| - \alpha(G) \tag{1}$$

and

$$i(G) \leq \frac{1}{2} \tag{2}$$

holds. Let J be an independent set in $G^{\times k}$ for which $\frac{|J|}{|J \cup N(J)|} > i(G)$.

Consider $G^{\times k}$ in the form $G \times G^{\times(k-1)}$. We denote the vertex set of $G^{\times(k-1)}$ by U and the vertex set of $G^{\times k}$ by V. Let U_1, U_2 and U_3 be the following subsets of U:

$$U_1 = \{v \in U\ :\ |J \cap (V(G) \times \{v\})| > \alpha(G)\}$$

$$U_2 = N(U_1)$$

$$U_3 = U \setminus (U_1 \cup U_2)$$

For $i = 1, 2, 3$ let $V_i = V(G) \times U_i$ and denote by J_i the corresponding subsets of J: $J_i = J \cap V_i$.

It follows from (1) that

$$\forall P \subseteq V(G), |P| > \alpha(G):\ N(P) = V(G) \tag{3}$$

because for every vertex v in $V(G)$ there are at most $\alpha(G)$ vertices which are non-adjacent to v, so there must be a vertex in P which is adjacent to v. The fact (3) and the definition of the categorical product implies that every vertex of $V(G) \times N(U_1)$ is a neighborhood of a vertex in J_1, i.e., $N(J_1) = V(G) \times N(U_1) = V_2$, thus U_1 is an independent set of $G^{\times(k-1)}$ and J_2 is empty.

It also follows that $U_1 \cap U_2 = \emptyset$, $V_1 \cap V_2 = \emptyset$, so $U = U_1 \cup U_2 \cup U_3$ is a partition of U, $V = V_1 \cup V_2 \cup V_3$ is a partition of V and $J = J_1 \cup J_3$ is a partition of J.

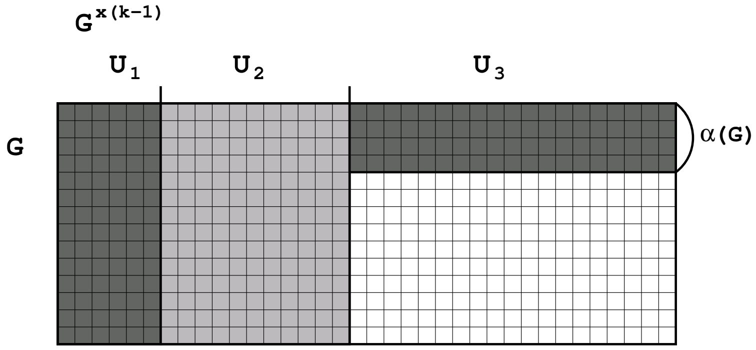

Figure 2: Partition of $G^{\times k}$ (the size of J is at most the size of the dark gray area and the size of $N(J)$ is at least the size of the bright gray area)

It is easy to see that the ratio $\frac{|J|}{|J \cup N(J)|}$ is a convex linear combination of $\frac{|J_1|}{|J_1 \cup N(J_1)|}$ (i.e., the fraction of J in $V_1 \cup V_2$) and $\frac{|J_3|}{|J_3 \cup (N(J_3) \cap V_3)|}$ (i.e., the fraction of J in V_3). Thus $\frac{|J|}{|J \cup N(J)|} > i(G)$ implies that

$$\frac{|J_1|}{|J_1 \cup N(J_1)|} > i(G) \tag{4}$$

or

$$\frac{|J_3|}{|J_3 \cup (N(J_3) \cap V_3)|} > i(G) \tag{5}$$

In the first case, when (4) holds, it follows from $N(J_1) = V_2$ that $\frac{|U_1|}{|U_1 \cup N(U_1)|} = \frac{|V_1|}{|V_1 \cup V_2|} \geq \frac{|J_1|}{|J_1 \cup N(J_1)|} > i(G)$. Since U_1 is an independent set in $G^{\times(k-1)}$, $K = U_1$ is a good choice and we the statement is proven.

In the second case, when (5) holds we investigate the structure of J further. Let A and B be the following subsets in U_3:

$$A = \{v \in U_3 \,:\, J_3 \cap (V(G) \times \{v\}) \neq \emptyset\}$$

$$B = N(A) \cap U_3$$

We prove that $|A| > |B|$. From (1) we get that $\forall w \in B$: $N(J_3) \cap (V(G) \times \{w\}) \geq |V(G)| - \alpha(G)$, so $|N(J_3) \cap V_3| \geq |B|(|V(G)| - \alpha(G))$. On the other hand, it holds by the definition of U_1, U_2 and U_3 that $\forall v \in U_3$: $|J \cap (V(G) \times \{v\})| \leq \alpha(G)$, so $|J_3| \leq |A|\alpha(G)$. Thus $\frac{|J_3|}{|N(J_3)\cap V_3|} \leq \frac{|A|\alpha(G)}{|B|(|V(G)|-\alpha(G))}$. Furthermore, $|A| \leq |B|$ would imply that $\frac{|J_3|}{|N(J_3)\cap V_3|} \leq \frac{\alpha(G)}{(|V(G)|-\alpha(G))}$, so $\frac{|J_3|}{|J_3\cup(N(J_3)\cap V_3)|} \leq \frac{\alpha(G)}{|V(G)|} = i(G)$, which contradicts (5). Hence

$$|A| > |B| \tag{6}$$

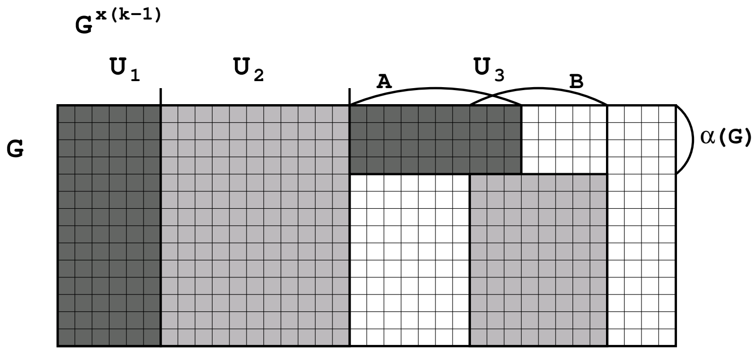

Figure 3: Partition of $G^{\times k}$, the structure of J and $N(J)$

We know that $\frac{|J|}{|J\cup N(J)|} > i(G) = \frac{\alpha(G)}{|V(G)|}$ which means that $\frac{|J|}{|N(J)|} > \frac{\alpha(G)}{|V(G)|-\alpha(G)}$. Let L be the following subset of $V(G^{\times k})$.

$$L = V(G) \times M, \text{ where } M = U_1 \cup (A \setminus B)$$

We prove that $\frac{|L|}{|L\cup N(L)|} > i(G)$.

$$\frac{|J|}{|N(J)|} = \frac{|J_1| + |J_3|}{|N(J_1)| + |N(J_3) \cap V_3|} \leq \frac{|V(G)||U_1| + \alpha(G)|A|}{|V(G)||U_2| + (|V(G)| - \alpha(G))|B|} \tag{7}$$

By the definition of B we get $N(A\setminus B) \subseteq (B\setminus A)\cup U_2$, in fact, $N(A\setminus B)\setminus U_2 = B \setminus A$ and $N(U_1) = U_2$, so

$$\frac{|L|}{|N(L)|} = \frac{|V(G) \times (U_1 \cup A \setminus B)|}{|V(G) \times (U_2 \cup B \setminus A)|} = \frac{|V(G)|(|U_1| + |A \setminus B|)}{|V(G)|(|U_2| + |B \setminus A|)}. \tag{8}$$

The difference between the right hand side of (8) and (7)

in the numerator:	$-\alpha(G)\|A \cap B\|$	$+$	$(\|V(G)\| - \alpha(G))\|A \setminus B\|$,
in the denominator:	$-(\|V(G)\| - \alpha(G))\|A \cap B\|$	$+$	$\alpha(G)\|B \setminus A\|$.

Recall that the numerators belong to the independent sets and the denominators belong to the neighborhood of the independent sets.
Furthermore,

$$\frac{-\alpha(G)|A \cap B|}{-(|V(G)| - \alpha(G))|A \cap B|} = \frac{\alpha(G)}{|V(G)| - \alpha(G)} \tag{9}$$

and

$$\frac{(|V(G)| - \alpha(G))|A \setminus B|}{\alpha(G)|B \setminus A|} > 1 > \frac{\alpha(G)}{|V(G)| - \alpha(G)} \tag{10}$$

because (6) implies that $|A \setminus B| > |B \setminus A|$ and from (2) it follows that $\alpha(G) < |V(G)| - \alpha(G)$.
Since the right hand side of (8) is a convex linear combination of the right hand side of (7), the left hand side of (9) and left hand side (10), each of which is bounded from below by $\frac{\alpha(G)}{|V(G)|-\alpha(G)}$ we have $\frac{|L|}{|N(L)|} > \frac{\alpha(G)}{|V(G)|-\alpha(G)}$ which means that

$$\frac{|L|}{|L \cup N(L)|} > i(G). \tag{11}$$

From the structure of L and from (11) we get that $\frac{|M|}{|M \cup N(M)|} = \frac{|L|}{|L \cup N(L)|} > i(G)$. Since U_1 and $A \setminus B = A \setminus N(A)$ are independent and $N(A \setminus B) \cap U_1 = \emptyset$, we have that M is an independent set. Thus setting $K = M$ we found the independent set in $G^{\times(k-1)}$ the existence of which is claimed by the lemma.

□

Proof of Theorem 3.10. Suppose indirectly that there is a positive integer k for which $i(G^{\times k}) \neq i(G)$. As the sequence $\{i(G^{\times k})\}_{k=1}^{\infty}$ is nondecreasing we get that there is an independent set I in $G^{\times k}$ for which $i(G) < \frac{|I|}{|V(G^{\times k})|} = \frac{|I|}{|I \cup N(I)|}$. By the iterative use of Lemma 3.12 we obtain that there is an independent set J in G for which $\frac{|J|}{|J \cup N(J)|} > i(G) = \frac{\alpha(G)}{|V(G)|}$. As $J \leq \alpha(G)$ and $\forall v \in V(G) : d(v) \geq |V(G)| - \alpha(G)$ implies that $N(J) \geq |V(G)| - \alpha(G)$ we get that $\frac{|J|}{|J \cup N(J)|} \leq i(G)$. Hence we got contradiction, the statement of the theorem is true. □

4 Asymptotic values of the Hall-ratio

Motivated by problems of list coloring the *Hall-ratio* of a graph G is investigated in [7, 8] where it is defined as

$$\rho(G) = \max\left\{\frac{|V(H)|}{\alpha(H)} \,:\, H \subseteq G\right\},$$

that is, as the ratio of the number of vertices and the independence number maximized over all subgraphs of G.

It is clear that $\omega(G) \leq \rho(G)$ by considering any maximal clique as a subgraph. Furthermore, $\rho(G) \leq \chi_f(G) \leq \chi(G)$ which follows from the simple fact that $\chi_f(G) \geq \chi_f(H) \geq \frac{|V(H)|}{\alpha(H)}$, where H is a subgraph of G.

The asymptotic values of the Hall-ratio for different graph powers were investigated by Simonyi in [18]. In particular, he considered the (appropriately normalized) asymptotic values of the Hall-ratio for the exponentiations called normal, co-normal, Cartesian, categorical and lexicographic, respectively.

In that article the asymptotic value of the Hall-ratio with respect to the normal and co-normal power is determined with the help of the method of types (cf. [6]).

Theorem 4.1. ([18]) *It holds for the ultimate normal Hall-ratio that*

$$\lim_{n\to\infty} \sqrt[n]{\rho(G^{\odot n})} = R(G)$$

(Recall that $R(G)$ is the Witsenhausen rate of graph G, cf. Definitions, notations.)

Theorem 4.2. ([18]) *It holds for the ultimate co-normal Hall-ratio that*

$$\lim_{n\to\infty} \sqrt[n]{\rho(G^{n})} = \chi_f(G)$$

For the Cartesian power the value of $\lim\limits_{n\to\infty} \rho(G^{\Box n})$ is the ultimate Cartesian Hall-ratio. The following relations are known about this graph parameter by [18]. A result of [9] implies that this graph parameter equals to the reciprocal value of the ultimate (Cartesian) independence ratio, which is mentioned in the previous section. Furthermore, it follows from the results in [9, 10] that it lies in the interval $[\chi_f(G), \chi(G)]$ and it is proven in [20] that it equals to the value of $\lim_{n\to\infty} \chi_f(G^{\Box n})$.

For the categorical power the value of $\lim \rho(G^{\times n})$ is known only for special graph families, it is conjectured that the limit of the sequence is also the fractional chromatic number.

Considering the Hall-ratio with respect to the lexicographic power we get the ultimate lexicographic Hall-ratio. For the value of $\lim_{n\to\infty} \sqrt[n]{\rho(G^{\circ n})}$ only bounds are proven, its exact value is not known. It is a conjecture in [18] that it equals to the fractional chromatic number.

In the next two sections we investigate the asymptotic value of the Hall-ratio for the categorical and the lexicographic power further.

5 On the ultimate categorical Hall-ratio

In this section we investigate the Hall-ratio with respect to the categorical power.

Definition. ([18]) The *ultimate categorical Hall-ratio* of graph G is

$$h_\times(G) = \lim_{n\to\infty} \rho(G^{\times n}).$$

In [18] the categorical product was called direct product, but we use this name to correspond with the third section.

The limit of the sequence $\{\rho(G^{\times i})\}_{i=1}^{\infty}$ exists, because it is monoton increasing and bounded from above. We get the following upper bound for this graph parameter from the simple fact that $\chi_f(G^{\times n}) = \chi_f(G)$ (see the proof for example in [18]).

Proposition 5.1. ([18]) *It holds for every graph G that*

$$h_\times(G) \le \chi_f(G).$$

It is a conjecture in [18] that equality holds in the latter formula.

To prove that $h_\times(G) \ge \chi_f(G)$ for a graph G it is enough to prove $\rho(G^{\times k}) \ge \chi_f(G)$ for some value of k by the monotonity of the sequence $\{\rho(G^{\times i})\}_{i=1}^{\infty}$. This observation implies the following propositions.

Proposition 5.2. ([18]) *If G is a perfect graph or vertex transitive graph then $h_\times(G) = \chi_f(G)$.*

Proposition 5.3. ([18]) $h_\times(W_m) = \chi_f(W_m)$, *where W_m denotes the wheel graph on $m+1$ vertices.*

By using a similar argument to that which was used in the proof of the latter result we prove the following generalization.

Proposition 5.4. *Let G be a graph for which $h_\times(G) = \chi_f(G)$ holds. Let $\hat{G}$ be the graph we obtain from G by connecting each of its vertices to an additional vertex. Then $h_\times(\hat{G}) = \chi_f(\hat{G})$ holds, too.*

Proof. $h_\times(G) = \lim_{n\to\infty} \rho(G^{\times n}) = \chi_f(G)$ means that

$$\forall \varepsilon > 0 : \exists n_0(\varepsilon) : \forall n \geq n_0 : \rho(G^{\times n}) \geq \chi_f(G) - \varepsilon \tag{12}$$

by definition of the limit and by the monoton increasing property of the sequence $\{\rho(G^{\times i})\}_{i=1}^{\infty}$.

Adding a new vertex w to G increases the fractional chromatic number by 1, as it does not lie in a common independent set with any other vertex of the graph. Therefore $\chi_f(\hat{G}) = \chi_f(G) + 1$.

Thus we have to show that $h_\times(\hat{G}) = \lim_{n\to\infty} \rho(\hat{G}^{\times n}) = \chi_f(\hat{G}) = \chi_f(G) + 1$, i.e.,

$$\forall \varepsilon > 0 : \exists \hat{n}_0(\varepsilon) : \forall n \geq \hat{n}_0 : \rho(\hat{G}^{\times n}) \geq \chi_f(G) + 1 - \varepsilon. \tag{13}$$

It is enough to find for all ε a suitable $\hat{n}_0$ for which $\rho(\hat{G}^{\times n_0}) \geq \chi_f(G) + 1 - \varepsilon$. It follows from (12) that for all $\varepsilon > 0$ there is an n_0 and $H \subseteq G^{\times n_0}$, for which $\frac{|V(H)|}{\alpha(H)} \geq \chi_f(G) - \varepsilon$ holds. Denote by k the number of vertices and by α the independence number of H. Let $v_1, v_2, \dots v_k$ be the vertices of H and let $v_1, v_2, \dots v_\alpha$ be the vertices of a maximum size independent set in H. Let $\hat{H}$ be the subgraph of $\hat{G}^{\times 2n_0}$ induced on the vertex set $P_1 \cup P_2 \cup Q$, where

$$P_1 = \{(v_1, w^{n_0}), (v_2, w^{n_0}), \dots (v_\alpha, w^{n_0})\},$$

$$P_2 = \{(w^{n_0}, v_1), (w^{n_0}, v_2), \dots (w^{n_0}, v_\alpha)\} \text{ and}$$

$$Q = \{(v_{\alpha+1}, v_{\alpha+1}), (v_{\alpha+2}, v_{\alpha+2}), \dots (v_k, v_k)\}.$$

The number of vertices of $\hat{H}$ is $k+\alpha$. Its independence number is less than or equal to α, because on the vertex set $P_1 \cup P_2$ we get a complete bipartite graph, thus every independent set of $\hat{H}$ can contain vertices only from P_1 or only from P_2, but on the set $P_1 \cup Q$ and $P_2 \cup Q$ the induced graph isomorph to H.

It follows that $\frac{|V(\hat{H})|}{\alpha(\hat{H})} \geq \frac{k+\alpha}{\alpha} = \frac{k}{\alpha} + 1 = \frac{|V(H)|}{\alpha(H)} + 1 \geq \chi_f(G) + 1 - \varepsilon$, thus $\hat{n}_0 = 2n_0$ is a good choice to satisfy (13).

Thus we have proved that $h_\times(\hat{G}) = \lim_{n\to\infty} \rho(\hat{G}^{\times n}) = \chi_f(\hat{G})$. $\square$

6 On the ultimate lexicographic Hall-ratio

In this section we investigate the Hall-ratio with respect to the lexicographic power.

6.1 The ultimate lexicographic Hall-ratio

Definition. For two graphs F and G, their *lexicographic product* $F \circ G$ is defined on the vertex set $V(F \circ G) = V(F) \times V(G)$ with edge set $E(F \circ G) = \{\{u_1v_1, u_2v_2\} : \{u_1, u_2\} \in E(F)$, or $u_1 = u_2$ and $\{v_1, v_2\} \in E(G)\}$. The nth lexicographic power $G^{\circ n}$ is the n-fold lexicographic product of G.

That is, $G^{\circ n}$ is defined on the n-length sequences of the original vertices and two such sequences are adjacent in $G^{\circ n}$ if and only if they are adjacent in the first coordinate where they differ.

The lexicographic product is associative but it is not commutative unlike the other products we mentioned.

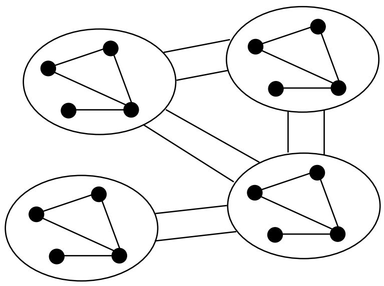

Figure 4: Square of a graph for the lexicographic power

Definition. ([18]) The *ultimate lexicographic Hall-ratio* of graph G is

$$h_\circ(G) = \lim_{n\to\infty} \sqrt[n]{\rho(G^{\circ n})}.$$

As $E(F \odot G) \subseteq E(F \circ G) \subseteq E(F \cdot G)$ holds for every graph F and G Theorem 4.1 and Theorem 4.2 implies that the value of $h_\circ(G)$ falls into the interval $[R(G), \chi_f(G)]$. While $R(G) \leq \chi_f(G)$ always holds, one can have strict inequality. An example is the five length cycle, C_5, the Witsenhausen rate of which happens to be $\sqrt{5} < \frac{5}{2} = \chi_f(C_5)$. The equality $R(C_5) = \sqrt{5}$ follows from the results of [13] combined with those of [19]. (As the latter suggests, determining $R(G)$ is in general not easier than determining the Shannon capacity of a graph.)

We remark that the lower bound $R(G)$ is sometimes better but some other times worse than the easy lower bound $\rho(G)$, cf. [18]. Thus we know the following bounds for the ultimate lexicographic Hall-ratio.

Theorem 6.1. ([18])

$$\max\{\rho(G), R(G)\} \leq h_\circ(G) \leq \chi_f(G)$$

For some types of graphs the upper and lower bounds are equal, so this formula gives the exact value of the ultimate lexicographic Hall-ratio. For instance, if G is a perfect graph, then $\chi_f(G) = \chi(G) = \omega(G) \leq \rho(G)$. If G is a vertex-transitive graph, then $\chi_f(G) = \frac{|V(G)|}{\alpha(G)} \leq \rho(G)$. (The proof of the fact that $\chi_f(G) = \frac{|V(G)|}{\alpha(G)}$ holds for vertex-transitive graphs, can be found for example in [11].)

The independence ratio (and thus also its reciprocal value) is multiplicative for the lexicographic product, because $|V(G^{\circ n})| = |V(G)|^n$ and $\alpha(G^{\circ n}) = (\alpha(G))^n$. (The proof of the latter easy statement can be found for example in [11].) The inequalities of Theorem 6.1 imply that the Hall-ratio is not multiplicative in general for this product.

The length of the interval $[\max\{\rho(G), R(G)\}, \chi_f(G)]$ is also positive in general (the 5-wheel is an example for it, see the next subsection). It was conjectured in [18], that in fact, $h_\circ(G)$ always coincides with the larger end of the above interval. The main goal of this section is to prove this conjecture.

In the next subsection we prove the conjecture for the 5-wheel, W_5 to show the basic idea of the proof in a simple case. Then we prove it generally with a lot of technical computation. (We do not use the result for W_5 in the proof of the general case.)

6.2 The case of wheel graphs

If m is even or less than 5 then the wheel graph W_m is perfect thus $h_\circ(W_m) = \chi_f(W_m)$. The 5-wheel, W_5 is the smallest wheel graph for which Theorem 6.1 does not imply the value of $h_\circ(W_5)$. It is clear that $\rho(W_5) = 3$. It is easy to find a coloring of $C_5^{\odot 2}$ with 5 colors and it can be completed to a coloring of $W_5^{\odot 2}$ with 12 colors so $\chi(W_5^{\odot 2}) \leq 12$. Since $\chi(G^{\odot n}) \leq (\chi(G))^n$ (see [11] for the proof) and by the definition of $R(G)$ we get $R(W_5) \leq \sqrt{12}$. Furthermore, $\chi_f(W_5) = \chi_f(C_5) + 1 = \frac{7}{2} > \max\{3, \sqrt{12}\}$ so the length of the interval $[\max\{\rho(W_5), R(W_5)\}, \chi_f(W_5)]$ is positive. In this section we prove that $h_\circ(W_5)$ equals to the larger end of the interval.
($h_\circ(W_m) = \chi_f(W_m)$ can be proven for every wheel graph without further complications.)

Proposition 6.2. *$h_\circ(W_5) = \chi_f(W_5)$.*

Proof. It is enough to prove that $h_\circ(W_5) \geq \chi_f(W_5) = \frac{7}{2}$.

Let $p(n, \alpha)$ be the number of vertices maximized over all subgraphs of $W_5^{\circ n}$ with independence number at most α, that is

$$p(n, \alpha) = \max\{|V(H)| \,:\, H \subseteq W_5^{\circ n}, \alpha(H) \leq \alpha\}$$

and let

$$q(n, \alpha) = \frac{p(n, \alpha)}{\alpha},$$

where n and α are positive integer.

It holds by the definition of $p(n, \alpha)$ and $q(n, \alpha)$ that

$$\rho(W_5^{\circ n}) = \max\{q(n, \alpha) \,:\, \alpha \in \mathbb{Z}_+\}$$

and so

$$h_\circ(W_5) = \lim_{n\to\infty} \sqrt[n]{\rho(W_5^{\circ n})} = \lim_{n\to\infty} \max\{\sqrt[n]{q(n, \alpha)} \,:\, \alpha \in \mathbb{Z}_+\}.$$

If we substitute the vertices of W_5 by subgraphs of $W_5^{\circ(n-1)}$ which independence number is at most α for the points of the cycle and 2α for the center point then we get a subgraph of $W_5^{\circ n}$ which independence number is at most 2α. (We notice that the optimal fractional clique of W_5 equals to $\frac{1}{2}$ for the points of the cycle and it equals to 1 for the center point.)

Thus

$$p(n, 2\alpha) \geq p(n-1, 2\alpha) + 5p(n-1, \alpha).$$

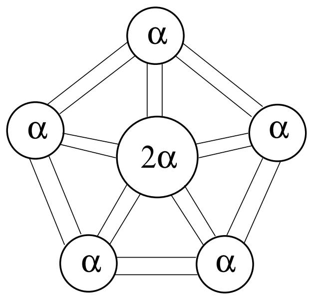

Figure 5: The construction for W_5

By the definition of $q(n, \alpha)$ we get
$q(n, 2\alpha) = \frac{p(n,2\alpha)}{2\alpha} \geq \frac{1}{2\alpha}\left(p(n-1, 2\alpha) + 5p(n-1, \alpha)\right) =$
$= q(n-1, 2\alpha) + \frac{5}{2}q(n-1, \alpha) = \frac{7}{2}\left(\frac{2}{7}q(n-1, 2\alpha) + \frac{5}{7}q(n-1, \alpha)\right).$
Thus $q(n, 2\alpha)$ is bounded from below by $\frac{7}{2}$ times value of the convex combination of $q(n-1, 2\alpha)$ and $q(n-1, \alpha)$.

We investigate the values of $q(n, \alpha)$ for $n = 1, 2, 3, \ldots \infty$, $\alpha = 1, 2, 4, \ldots 2^n$. For $\alpha = 1$ we get that $q(n, 1) = p(n, 1) = 3^n$, because $p(n, 1) = \omega(W_5^{\circ n}) = \omega(W_5)^n = 3^n$. For $\alpha = 2^n$ we get that $q(n, 2^n) = 3^n$, because $\alpha(W_5^{\circ n}) = \alpha(W_5)^n = 2^n$ so $p(n, 2^n) = |V(W_5^{\circ n})| = 6^n$ thus $q(n, 2^n) = \frac{6^n}{2^n} = 3^n$. A lower bound for $q(n, \alpha)$ for other values of α can be computed from $q(n, 2\alpha) \geq \frac{7}{2}\left(\frac{2}{7}q(n-1, 2\alpha) + \frac{5}{7}q(n-1, \alpha)\right)$. By this inequality the sum of the values of $q(n, \alpha)$ for $\alpha = 1, 2, 4, \ldots 2^n$ is asymptotically equals to $(\frac{7}{2})^n$. Since we summarized just n values of $q(n, \alpha)$, the maximal value of them asymptotically equals to $\frac{7}{2}^n$.

Thus we have proved $h_\circ(W_5) = \lim_{n\to\infty} \max\{\sqrt[n]{q(n, \alpha)} : \alpha \in \mathbb{Z}_+\} \geq \frac{7}{2}$. $\square$

6.3 The general case

In this subsection we prove that the ultimate lexicographic Hall-ratio equals to the fractional chromatic number for every graph.

Theorem 6.3.

$$h_\circ(G) = \chi_f(G)$$

We know $h_\circ(G) \le \chi_f(G)$ thus it is enough to prove the reverse inequality.

Preparing for the proof we introduce some notations.

Let $p_G(n, \alpha)$ be the number of vertices maximized over all subgraphs of $G^{\circ n}$ with independence number at most α, that is

$$p_G(n, \alpha) = \max\{|V(H)| \,:\, H \subseteq G^{\circ n}, \alpha(H) \le \alpha\}$$

and let

$$q_G(n, \alpha) = \frac{p_G(n, \alpha)}{\alpha},$$

where n is a positive integer, α is a positive real number.

Clearly, $p_G(n, \alpha) = p_G(n, [\alpha])$ and $q_G(n, \alpha) \le q_G(n, [\alpha])$. In spite of this fact it will be useful that $p_G(n, \alpha)$ is defined for any positive real α values.

Now we are going to prove some technical lemmas.

The ultimate lexicographic Hall-ratio can be expressed by the values of $q_G(n, \alpha)$ as follows.

Lemma 6.4.

$$h_\circ(G) = \lim_{n\to\infty} \max\{\sqrt[n]{q_G(n, \alpha)} \,:\, \alpha \in \mathbb{R}_+\} \tag{14}$$

Proof. The Hall-ratio of the nth lexicographic power of G can be calculated by the above terms the following simple way:

$$\rho(G^{\circ n}) = \sup\{q(n, \alpha) \,:\, \alpha \in \mathbb{R}_+\}.$$

Since $p_G(n, \alpha)$ is a bounded, monotone increasing function and $q_G(n, \alpha)$ is the fraction of this and the strictly monotone increasing identity function, the above supremum is always reached. Since $q_G(n, \alpha) \le q_G(n, [\alpha])$, it is reached

at some integer value of α, so the maximum value belongs to one of the subgraph of $G^{\circ n}$.
Thus we get $h_\circ(G) = \lim_{n\to\infty} \sqrt[n]{\rho(G^{\circ n})} = \lim_{n\to\infty} \max\{ \sqrt[n]{q_G(n,\alpha)} \,:\, \alpha \in \mathbb{R}_+\}$. $\square$

Thus our aim is to show that $\lim_{n\to\infty} \max\{ \sqrt[n]{q_G(n,\alpha)} \,:\, \alpha \in \mathbb{R}_+\} \geq \chi_f(G)$.

Let $g : V(G) \to \mathbb{R}_{+,0}$ be an optimal fractional clique of G. That is, it is a fractional clique: $\forall U \in S(G)$: $\sum_{v\in U} g(v) \leq 1$ (recall that $S(G)$ denotes the set of the independent sets of G), and it is optimal: $\sum_{v\in V(G)} g(v) = \chi_f(G)$.

Lemma 6.5.

$$q_G(n,\alpha) \geq \sum_{v\in V(G)} g(v) q_G\left(n-1, g(v)\alpha\right)$$

Proof. Every subgraph of $G^{\circ n}$ can be imagined as if the vertices of G would be substituted by subgraphs of $G^{\circ(n-1)}$. Furthermore, every independent set of $G^{\circ n}$ can be thought of as having the vertices of an independent set of G substituted by independent sets of (the above subgraphs of) $G^{\circ(n-1)}$.
If we substitute every vertex v of G by a subgraph of $G^{\circ(n-1)}$ with independence number at most $g(v)\alpha$, then we get a subgraph of $G^{\circ n}$ with independence number at most $\max_{U\in S(G)} \sum_{v\in U} g(v)\alpha \leq \alpha \max_{U\in S(G)} \sum_{v\in U} g(v) \leq \alpha$, because g is a fractional clique of G.
Thus we get

$$p_G(n,\alpha) \geq \sum_{v\in G} p_G\left(n-1, g(v)\alpha\right).$$

It follows from this inequality and the definition of $q_G(n,\alpha)$ that
$q_G(n,\alpha) = \frac{p_G(n,\alpha)}{\alpha} \geq \frac{1}{\alpha} \sum_{v\in G} p_G\left(n-1, g(v)\alpha\right) = \sum_{v\in G} \frac{g(v)\alpha}{\alpha} \frac{p_G(n-1,g(v)\alpha)}{g(v)\alpha} =$
$= \sum_{v\in V(G)} g(v) q_G\left(n-1, g(v)\alpha\right).$ $\square$

Next we bound $q_G(n,\alpha)$ function from below, it will be important for later calculations. Let us define function $r_G(n,\alpha)$ as follows.

$$r_G(1,\alpha) = \begin{cases} c_G, & \text{if } 1 \leq \alpha \leq m = |V(G)| \\ 0, & \text{otherwise} \end{cases}$$

where c_G is a positive constant, which bounds $q_G(1,\alpha)$ from below for all $1 \le \alpha \le m = |V(G)|$. Such c_G exists, for example $c_G = \frac{1}{m}$ is a good choice. For $n \ge 2$ let

$$r_G(n,\alpha) = \sum_{v \in V(G)} g(v) r_G\left(n-1, g(v)\alpha\right).$$

By Lemma 6.5 and by the construction of $r_G(n,\alpha)$ it holds for all positive integer n and all positive real number α that

$$r_G(n,\alpha) \le q_G(n,\alpha). \tag{15}$$

Thus it is enough to show that $\lim\limits_{n\to\infty} \max\{\sqrt[n]{r_G(n,\alpha)} \,:\, \alpha \in \mathbb{R}_+\} \ge \chi_f(G)$.

To make the calculations simpler, we express α as m^β, that is $\beta = \log_m \alpha$ and introduce

$$s_G(n,\beta) = r_G(n, m^\beta),$$

where n is a positive integer, β is a real number. Since this transformation does not change the maximum value of the function (just the place of it), it holds that

$$\max\{\sqrt[n]{r_G(n,\alpha)} \,:\, \alpha \in \mathbb{R}_+\} = \max\{\sqrt[n]{s_G(n,\beta)} \,:\, \beta \in \mathbb{R}\} \tag{16}$$

Thus it is enough to prove that $\lim\limits_{n\to\infty} \max\{\sqrt[n]{s_G(n,\beta)} \,:\, \beta \in \mathbb{R}\} \ge \chi_f(G)$.

Observe that the following equalities hold:

$$s_G(1,\beta) = \begin{cases} c_G, & \text{if } 0 \le \beta \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$s_G(n,\beta) = \sum_{v \in V(G)} g(v) s_G\left(n-1, \log_m g(v) + \beta\right), n \ge 2.$$

We get the formula for $s_G(1,\beta)$ from the definition of the function $s_G(n,\beta)$. The second equality follows by writing

$s_G(n,\beta) = r_G(n,m^\beta) = \sum\limits_{v \in V(G)} g(v) r_G\left(n-1, g(v)m^\beta\right) =$

$= \sum\limits_{v \in V(G)} g(v) s_G\left(n-1, \log_m(g(v)m^\beta)\right) = \sum\limits_{v \in V(G)} g(v) s_G\left(n-1, \log_m g(v) + \beta\right).$

Lemma 6.6.

$$\lim_{n\to\infty} \max\{\sqrt[n]{s_G(n,\beta)}\ :\ \beta\in\mathbb{R}\} \geq \chi_f(G) \tag{17}$$

Proof. Let us determine the integral of the function $s_G(n,\beta)$.

$\int\limits_{\beta=-\infty}^{\infty} s_G(1,\beta)\,\mathrm{d}\beta = c_G$

$\int\limits_{\beta=-\infty}^{\infty} s_G(n,\beta)\,\mathrm{d}\beta = \int\limits_{\beta=-\infty}^{\infty} \sum\limits_{v\in V(G)} g(v)s_G\left(n-1,\log_m g(v)+\beta\right)\,\mathrm{d}\beta =$

$= \sum\limits_{v\in V(G)} \left(g(v) \int\limits_{\beta=-\infty}^{\infty} s_G(n-1,\log_m g(v)+\beta)\,\mathrm{d}\beta \right) =$

$= \sum\limits_{v\in V(G)} \left(g(v) \int\limits_{\beta=-\infty}^{\infty} s_G(n-1,\beta)\,\mathrm{d}\beta \right) =$

$= \left(\sum\limits_{v\in V(G)} g(v) \right) \int\limits_{\beta=-\infty}^{\infty} s_G(n-1,\beta)\,\mathrm{d}\beta =$

$= \chi_f(G) \int\limits_{\beta=-\infty}^{\infty} s_G(n-1,\beta)\,\mathrm{d}\beta,\ n\geq 2$

Hence,

$$\int\limits_{\beta=-\infty}^{\infty} s_G(n,\beta)\,\mathrm{d}\beta = c_G(\chi_f(G))^{n-1}.$$

For a function $f(x)$ we call the support of $f(x)$, denote by $T(f(x))$, the set of reals x for which $f(x)\neq 0$. Let us determine $T(s_G(n,\beta))$.
$T(s_G(1,\beta)) = [0,1]$. Let g_G be any real value satisfying $g_G \leq \log_m g(v) \leq 0$ for all $v\in V(G)$. Such g_G exists, for example $g_G = \min\{\log_m g(v)\ :\ v\in V(G)\}$ is a good choice. Thus $T(s_G(n,\beta)) \subseteq [0, 1-(n-1)g_G]$.

It is clear from the above discussion that $\int_{\beta=-\infty}^{\infty} s_G(n,\beta)\,\mathrm{d}\beta$ asymptotically equals to $(\chi_f(G))^n$ (i.e., the limit of their fraction equals 1 as n goes to infinity). The length of the support of $s_G(n,\beta)$ can be bounded from above by a linear function of n, let this function be $d_G n$ where d_G is a constant. These facts imply that $\lim\limits_{n\to\infty} \max\{\sqrt[n]{s_G(n,\beta)}\ :\ \beta\in\mathbb{R}\} \geq \chi_f(G)$, for suppose indirectly that there is an $\varepsilon > 0$, for which $\forall n > N \in \mathbb{N}_+$, $\forall \beta \in \mathbb{R}$: $s_G(n,\beta) < (\chi_f(G)-\varepsilon)^n$, then $\int_{\beta=-\infty}^{\infty} s_G(n,\beta)\,\mathrm{d}\beta < d_G n(\chi_f(G)-\varepsilon)^n$ but $\lim\limits_{n\to\infty} \frac{d_G n(\chi_f(G)-\varepsilon)^n}{\chi_f(G)^n} = \lim\limits_{n\to\infty} (1-\frac{\varepsilon}{\chi_f(G)})^n = 0$.

□

By now we have essentially proved Theorem 6.3, it needs only to be summarized.

Proof of Theorem 6.3. The preceding lemmas imply that
$h_\circ(G) = \lim_{n\to\infty} \max\{ \sqrt[n]{q_G(n,\alpha)} \,:\, \alpha \in \mathbb{R}_+\} \geq$
$\geq \lim_{n\to\infty} \max\{ \sqrt[n]{r_G(n,\alpha)} \,:\, \alpha \in \mathbb{R}_+\} =$
$= \lim_{n\to\infty} \max\{ \sqrt[n]{s_G(n,\beta)} \,:\, \beta \in \mathbb{R}\} \geq \chi_f(G),$
we used (14), (15), (16) and (17), respectively.

Thus we have proved

$$h_\circ(G) = \chi_f(G).$$

□

Remarks

There are graphs for which the sequence $\{ \sqrt[n]{\rho(G^{\circ n})}\}_{n=1}^{\infty}$ does not reach its limit $\chi_f(G)$ for any finite n. The 5-wheel, W_5 is a graph for which no t attains $\sqrt[t]{\rho(W_5^{\circ t})} = \chi_f(W_5) = \frac{7}{2}$. This is because if there was such a t then there must be a subgraph H of $W_5^{\circ t}$ for which $\frac{|V(H)|}{\alpha(H)} = \left(\frac{7}{2}\right)^t = \frac{7^t}{2^t}$, but this fraction is irreducible and $|V(H)| \leq |V(W_5^{\circ t})| = 6^t$.

It is known from the theorem of McEliece and Posner [14] that the normalized asymptotic value of the chromatic number with respect to the co-normal product is the fractional chromatic number. This theorem with the result proven here implies Theorem 4.2 of [18], since $\rho(G) \leq \chi_f(G) \leq \chi(G)$ holds for every graph G and the lexicographic power of a graph is a subgraph of its the co-normal power. These results imply also that the normalized asymptotic value of each of the Hall-ratio, the fractional chromatic number and the chromatic number with respect to both the co-normal and the lexicographic power equals to the fractional chromatic number. (These relations were already known except for the asymptotic value of the Hall-ratio for the lexicographic power.)

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