Gallai colorings and domination in multipartite digraphs

András Gyárfás *
Computer and Automation Research Institute,
Hungarian Academy of Sciences,
1518 Budapest, P.O. Box 63, Hungary,
gyarfas@sztaki.hu

Gábor Simonyi †
Alfréd Rényi Institute of Mathematics,
Hungarian Academy of Sciences,
1364 Budapest, P.O. Box 127, Hungary,
simonyi@renyi.hu

Ágnes Tóth ‡
Department of Computer Science and Information Theory,
Budapest University of Technology and Economics,
1521 Budapest, P.O. Box 91, Hungary,
tothagi@cs.bme.hu

September 3, 2011

Abstract

Assume that D is a digraph without cyclic triangles and its vertices are partitioned into classes $A_1, \dots, A_t$ of independent vertices. A set $U = \cup_{i \in S} A_i$ is called a dominating set of size $|S|$ if for any vertex $v \in \cup_{i \notin S} A_i$ there is a $w \in U$ such that $(w, v) \in E(D)$. Let $\beta(D)$ be the cardinality of the largest independent set of D whose vertices are from different partite classes of D. Our main result says that there exists a $h = h(\beta(D))$ such that D has a dominating set of size at most h. This result is applied to settle a problem related to generalized Gallai colorings, edge colorings of graphs without 3-colored triangles.

*Research partially supported by the Hungarian Foundation for Scientific Research Grant (OTKA) No. K68322.

†Research partially supported by the Hungarian Foundation for Scientific Research Grant (OTKA) Nos. K76088 and NK78439.

‡Research partially supported by the Hungarian Foundation for Scientific Research Grant and by the National Office for Research and Technology (Grant number OTKA 67651).

1 Introduction

Investigating comparability graphs Gallai [9] proved an interesting theorem about edge-colorings of complete graphs that contain no triangle for which all three of its edges receive distinct colors. (Note that here and in the sequel edge-coloring just means a partition of the edge set rather than a proper coloring of it.) Such colorings turned out to be relevant and Gallai's theorem proved to be useful also in other contexts, see e.g., [3, 4, 5, 8, 10, 11, 13, 14, 15].

Honoring the above mentioned work of Gallai an edge-coloring of the complete graph is called a Gallai coloring if there is no completely multicolored triangle. Recently this notion was extended to other (not necessarily complete) graphs in [12].

A basic property of Gallai colored complete graphs is that at least one of the color classes spans a connected subgraph on the entire vertex set. In [12] it was proved that if we color the edges of a not necessarily complete graph G so that no 3-colored triangles appear then there is still a large monochromatic connected component whose size is proportional to $|V(G)|$ where the proportion depends on the independence number $\alpha(G)$.

In view of this result it is natural to ask whether one can also span the whole vertex set with a constant number of connected monochromatic subgraphs where the constant depends only on $\alpha(G)$. This question led to a problem about the existence of dominating sets in directed graphs that we believe to be interesting in itself. In this paper we solve this latter problem thereby giving an affirmative answer to the previous question.

The paper is organized as follows. In Subsection 1.1 we describe our digraph problem and state our results on it. The connection with Gallai colorings will be explained in Subsection 1.2. Section 2 contains the proofs of the results in Subsection 1.1. In Section 3 we further elaborate on a question the proofs give rise to.

1.1 Dominating multipartite digraphs

We consider multipartite digraphs, i.e., digraphs D whose vertices are partitioned into classes $A_1, \dots, A_t$ of independent vertices. (Note that in this paper we consider directed graphs without pairs of edges connecting the same two vertices in opposite direction.) Suppose that $S \subseteq [t]$. A set $U = \cup_{i \in S} A_i$ is called a dominating set of size $|S|$ if for any vertex $v \in \cup_{i \notin S} A_i$ there is a $w \in U$ such that $(w, v) \in E(D)$. The smallest $|S|$ for which a multipartite digraph D has a dominating set $U = \cup_{i \in S} A_i$ is denoted by $k(D)$. Let $\beta(D)$ be the cardinality of the largest independent set of D whose vertices are from different partite classes of D. (Such independent sets we sometimes refer to as transversal independent sets.) An important special case is when $|A_i| = 1$ for each $i \in [t]$. In this case $\beta(D) = \alpha(D)$ and $k(D) = \gamma(D)$, the usual domination number of D, the smallest number of vertices in D whose closed outneighborhoods cover $V(D)$. Our main result is the following theorem.

Theorem 1. *For every integer β there exists an integer $h = h(\beta)$ such that the following holds. If D is a multipartite digraph without cyclic triangles and $\beta(D) = \beta$, then $k(D) \leq h$.*

Notice that the condition forbidding cyclic triangles in D is important even when $|A_i| = 1$ for all i and $\beta(D) = 1$, i.e. for tournaments. It is well known that $\gamma(D)$ can be arbitrarily large for tournaments (see, e.g., in [2]), so $h(1)$ would not exist without excluding cyclic triangles.

From the proof of Theorem 1 we will get a factorial upper bound for $k(D)$ from the recurrence formula $h(\beta) = 3\beta + (2\beta+1)h(\beta-1)$. We have relatively small upper bounds on k only for $\beta = 1, 2$.

Theorem 2. *Suppose that D is a multipartite digraph without cyclic triangles. If $\beta(D) = 1$ then $k(D) = 1$ and if $\beta(D) = 2$ then $k(D) \leq 4$.*

Though the upper bound on $h(\beta)$ obtained from our proof of Theorem 1 is much weaker we could not even rule out the existence of a bound that is linear in β. We cannot prove a linear upper bound even in the special case when every partite class consists of only one vertex. Nevertheless, we treat this case also separately and provide a slightly better bound than the one following from Theorem 1. The class of digraphs we have here, i.e., those with no directed triangles, is called the class of *clique-acyclic digraphs*, see [1]. These digraphs has been well-studied also because of the Caccetta-Häggkvist Conjecture, see, e.g., in [6].

Theorem 3. *Let $f(1) = 1$ and for $\alpha \geq 2$, $f(\alpha) = \alpha + \alpha f(\alpha - 1)$. If D is a clique-acyclic digraph then $\gamma(D) \leq f(\alpha(D))$.*

Apart from the obvious case $\alpha(D) = 1$ (when D is a transitive tournament) we know the best possible bound only for $\alpha(D) = 2$.

Theorem 4. *If D is a clique-acyclic digraph with $\alpha(D) = 2$, then $\gamma(D) \leq 3$.*

Note that Theorem 4 is sharp as shown by the cyclically oriented pentagon. Moreover, the union of t vertex disjoint cyclic pentagons shows that we can have $\alpha(D) = 2t$ and $\gamma(D) = 3t$. Thus in case a linear upper bound would be valid at least in the special case of clique-acyclic digraphs, it could not be smaller than $\frac{3}{2}\alpha(D)$. There are some easy subcases though when the bound is simply $\alpha(D)$.

Proposition 5. *If D is acyclically oriented or D is a clique-acyclic perfect graph then $\gamma(D) \leq \alpha(D)$.*

Note that Proposition 5 is sharp in the sense that every graph G has a clique-acyclic orientation resulting in digraph D with $\gamma(D) = \alpha(G) = \alpha(D)$. Indeed, an acyclic orientation of G where every vertex of a fixed maximum independent set has indegree zero shows this. It is worth noting the interesting result of Aharoni and Holzman [1] stating that a clique-acyclic digraph always has a fractional kernel, i.e., a fractional independent set, which is also fractionally dominating.

We will see in Section 2 from the proof of Theorems 1 and 2 that the dominating sets we find there contain two kinds of partite classes. The first kind could be substituted by just one vertex in it, while the second kind is chosen not so much to dominate others but because it is itself not dominated by others. That is, apart from a bounded number of exceptional partite classes we will dominate the rest of our digraph with a bounded number of vertices. In Section 3 we will prove another theorem showing that the exceptional classes are indeed needed.

1.2 Application to Gallai colorings

Recall that Gallai colorings are originally defined as edge-colorings of complete graphs where no triangle gets three different colors. As already mentioned earlier, one of the basic properties of Gallai colorings is that at least one color spans a connected subgraph, i.e. forms a component covering all vertices of the underlying complete graph. In [12] the notion was extended to arbitrary graphs and it was proved that in this setting there is still a large monochromatic connected component. More precisely the following was proved.

Theorem 6. ([12]) *Suppose that the edges of a graph G are colored so that no triangle is colored with three distinct colors. Then there is a monochromatic component in G with at least $\frac{|V(G)|}{\alpha^2(G)+\alpha(G)-1}$ vertices.*

Another, in a sense stronger possible generalization of the above basic property of Gallai colorings is also suggested by Theorem 6. The first author proposed the following problem at a workshop at Fredericia in November, 2009.

Problem 1. Suppose that the edges of a graph G are colored so that no triangle is colored with three distinct colors. Is it true that the vertices of G can be covered by the vertices of at most k monochromatic components where k depends only on $\alpha(G)$?

We remark that an example in [12] shows that even if the k of Problem 1 exists, it must be at least $\frac{c\alpha^2(G)}{\log \alpha(G)}$ where c is a small constant.

Theorem 1 implies an affirmative answer to Problem 1. Let $g(1) = 1$ and for $\alpha \geq 2$, let $g(\alpha) = g(\alpha - 1) + h(\alpha)$ where h is the function given by Theorem 1.

In the sequel we will use the notation $G[A]$ that denotes the subgraph of graph G induced by $A \subseteq V(G)$.

Theorem 7. *Suppose that the edges of a graph G are colored so that no triangle is colored with three distinct colors. Then the vertices of G can be covered by the vertices of at most $g(\alpha(G))$ monochromatic components. In case $\alpha(G) = 2$ at most five components are enough.*

Note that the last statement of Theorem 7 generalizes Theorem 6 in the case $\alpha(G) = 2$.

Proof. For $\alpha(G) = 1$ the result is obvious by Gallai's theorem. For $\alpha(G) \geq 2$, suppose that $v \in V(G)$ and let X be the set of vertices in G that are not adjacent to v. By induction, the subgraph $G[X]$ can be covered by the vertices of $g(\alpha(G) - 1)$ monochromatic components. Let t be the number of colors used on edges of G incident to v and let A_i be the set of vertices incident to v in color i. Observe that the condition on the coloring implies that edges of G between A_i, A_j are colored with either color i or color j whenever

$1 \leq i < j \leq t$. Thus orienting all edges of color i outward from A_i for every i, all edges of G between different classes A_j are oriented. Moreover, in this orientation there are no cyclic triangles. Thus Theorem 1 is applicable to the oriented subgraph H spanned by the union of the classes A_j after the edges inside the A_j's are removed. We obtain at most $h(\alpha(G))$ dominating sets A_i and each set $v \cup A_i$ together with the vertices that A_i dominates form a connected subgraph of G in color i. Thus all vertices of G can be covered by at most $g(\alpha(G) - 1) + h(\alpha(G)) = g(\alpha(G))$ connected components. In case of $\alpha(G) = 2$ we can use Theorem 2 to get a covering with at most five monochromatic components. □

Remark 1. In [11] it was proved that in a Gallai coloring of a complete graph there is a monochromatic spanning tree with height at most two. This result can also be generalized for non-complete graphs. From the prevoius proof we easily obtain that each of the $g(\alpha(G))$ monochromatic components which cover the vertex set of G have a spanning tree with height at most two. ♢

2 Proofs

We will use the following notation throughout. If D is a digraph and $U \subseteq V(D)$ is a subset of its vertex set then $N_+(U) = \{v \in V(D) : \exists u \in U \ (u, v) \in E(D)\}$ is the *outneighborhood* of U. The *closed outneighborhood* $\hat{N}_+(U)$ of U is meant to be the set $U \cup N_+(U)$. When $U = \{u\}$ is a single vertex we also write $N_+(u)$ and $\hat{N}_+(u)$ for $N_+(U)$ and $\hat{N}_+(U)$, respectively. When $(u, v) \in E(D)$, we will often say that *u sends an edge to v*.

We first deal with the case $\beta(D) = 1$ and prove the first statement of Theorem 2. As it will be used several times later, we state it separately as a lemma.

Lemma 8. *Let D be a multipartite digraph with no cyclic triangle. If $\beta(D) = 1$ then $k(D) = 1$.*

Proof. Let K be a partite class for which $|\hat{N}_+(K)|$ is largest. We claim that K is a dominating set. Suppose on the contrary, that there is a vertex l in a partite class $L \neq K$, which is not dominated by K. Since all edges between distinct partite classes are present in D with some orientation, l must send an edge to all vertices of K. Furthermore, if a vertex m in a partite class $M \neq K, L$ is an outneighbor of some $k \in K$ then it is also an outneighbor of l, otherwise m, l and k would form a cyclic triangle. Thus $\hat{N}_+(K) \subseteq \hat{N}_+(L)$. Moreover, $l \in \hat{N}_+(L) \setminus \hat{N}_+(K)$, so $|\hat{N}_+(L)| > |\hat{N}_+(K)|$ contradicting the choice of K. This completes the proof of the lemma.

□

In the following two subsections we prove Theorems 2 and 1, respectively.

2.1 At most 2 independent vertices

To prove the second statement of Theorem 2 we will need the following stronger variant of Lemma 8.

Lemma 9. *Let D be a multipartite digraph with no cyclic triangle and $\beta(D) = 1$. Then there is a partite class K which is a dominating set, and there is a vertex $k \in K$ such that $V(D) \setminus (K \cup L) \subseteq N_+(k)$ for some partite class $L \neq K$.*

Thus Lemma 9 states that the dominating partite class K has an element that alone dominates almost the whole of D, there may be only one exceptional partite class L whose vertices are not dominated by this single element of K.

For proving Lemma 9, the following observations will be used, where X, Y, Z will denote partite classes.

Observation 10. *Let D be a multipartite digraph with no cyclic triangle and $\beta(D) = 1$. Suppose that for vertices $x_1, x_2 \in X$ and $y \in Y$ the edges (x_2, y) and (y, x_1) are present in D. Then for every $z \in Z \neq X, Y$ with $(x_1, z) \in E(D)$ we also have $(x_2, z) \in E(D)$.*

Proof. Assume on the contrary that for some $z \in Z$ the orientation is such that we have (x_1, z), $(z, x_2) \in E(D)$. Then the edge connecting z and y cannot be oriented either way: $(z, y) \in E(D)$ would give a cyclic triangle on vertices z, y, x_1, while $(y, z) \in E(D)$ would create one on y, z, x_2. (Figure 1 illustrates the statement of this observation.)

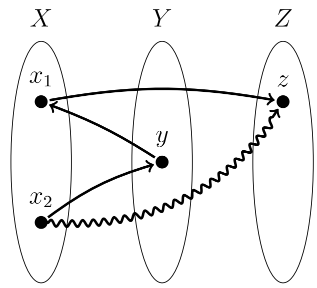

Figure 1: A simple configuration: if x_1 dominates z then x_2 also dominates z.

□

Observation 11. *Let D be a multipartite digraph with no cyclic triangle and $\beta(D) = 1$. Suppose that for vertices $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ the edges (x_1, y_2), (y_2, x_2), (x_2, y_1), (y_1, x_1) are present in D forming a cyclic quadrangle. Then in every partite class $Z \neq X, Y$ the outneighborhood of these four vertices is the same.*

Proof. Let z be an element of $Z \cap N_+(x_1)$. By $(y_1, x_1) \in E(D)$ we must have $z \in Z \cap N_+(y_1)$, otherwise y_1, x_1, z would form a cyclic triangle. Thus we have $Z \cap N_+(x_1) \subseteq Z \cap N_+(y_1)$. Now shifting the role of vertices along the oriented quadrangle backwards we similarly get $Z \cap N_+(x_1) \subseteq Z \cap N_+(y_1) \subseteq Z \cap N_+(x_2) \subseteq Z \cap N_+(y_2) \subseteq Z \cap N_+(x_1)$ proving that we have equality everywhere. (Figure 2 illustrates the statement of this observation.)

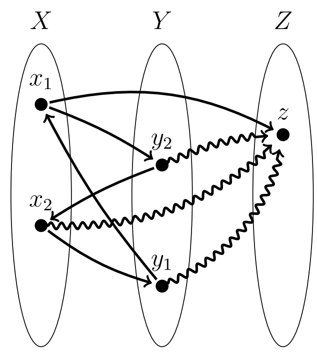

Figure 2: If x_1 dominates z then x_2, y_1, y_2 also dominate z.

□

Note that in Observation 11, as $\beta(D) = 1$, the inneighborhood of the vertices x_1, x_2, y_1, y_2 is also the same, so these vertices split to out- and inneighborhood in the same way every partite class $Z \neq X, Y$.

Proof of Lemma 9. We know from Lemma 8 that there is a partite class K which is a dominating set. (Figure 3 shows the main steps of the proof.)

Let k be an element of K for which $|N_+(k)|$ is maximal. If k itself dominates all the vertices not in K then we are done. (In that case we do not even need an exceptional class L.) Otherwise, there is a vertex l_1 in a partite class $L \neq K$ for which the edge between l_1 and k is oriented towards k. As $L \subseteq N_+(K)$, there must be a vertex $k_1 \in K$ which sends an edge to l_1.

Using Observation 10 for the vertices k, k_1 and l_1, we obtain that k_1 sends an edge not just to l_1 but to every vertex in $N_+(k) \setminus L$. By the choice of k this implies the existence of a vertex $l_2 \in L$ for which $(k, l_2), (l_2, k_1) \in E(D)$. Thus the vertices k, l_2, k_1, l_1 form a cyclic quadrangle. Applying Observation 11 this implies that these four vertices have the same outneighborhood in $V(D) \setminus (K \cup L)$.

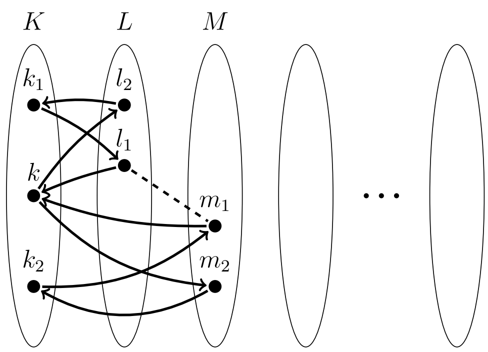

Figure 3: Two cyclic quadrangles give a contradiction.

We claim that $N_+(k)$ contains all vertices of $D \setminus (K \cup L)$. Assume on the contrary, that there is a vertex m_1 in a partite class $M \neq K, L$ which is not dominated by k. We can argue similarly as we did for l_1. Namely, since $M \subseteq N_+(K)$ there is some $k_2 \in K$ (perhaps identical to k_1) dominating m_1. Applying Observation 10 to the vertices k, m_1 and k_2, we obtain $(N_+(k) \setminus M) \subseteq N_+(k_2)$. Then by the choice of k we must have a vertex $m_2 \in M$ for which $(k, m_2), (m_2, k_2) \in E(D)$. So vertices k, m_2, k_2, m_1 also form a cyclic quadrangle, and Observation 11 gives us that $Z \cap N_+(k) = Z \cap N_+(m_2) = Z \cap N_+(k_2) = Z \cap N_+(m_1)$ for all partite classes $Z \neq K, M$.

The contradiction will be that the edge between l_1 and m_1 should be oriented both ways. Indeed, since $(l_1, k) \in E(D)$ and in L the inneighbors of k and m_1 are the same, we must have $(l_1, m_1) \in E(D)$. However, $(m_1, k) \in E(D)$ and the fact that k and l_1 split M in the same way implies $(m_1, l_1) \in E(D)$. This contradiction completes the proof of the lemma.

□

Now we are ready to prove the second statement of Theorem 2.

Proof of Theorem 2. We have already proven the first statement of the theorem. To prove the second part let D be a multipartite digraph without cyclic triangles and $\beta(D) = 2$. We use induction on the number of vertices. The base case is obvious. Let p be a vertex of D and consider the subdigraph $\hat{D} := D \setminus \{p\}$. (One can follow the proof on Figure 4.)

By induction $k(\hat{D}) \leq 4$. Let K, L, M and N be four partite classes of $\hat{D}$ that form a dominating set in $\hat{D}$. If $p \in \hat{N}_+(K \cup L \cup M \cup N)$ then we are done, the same four sets also dominate D. If $p \notin \hat{N}_+(K \cup L \cup M \cup N)$ then we will choose four other partite classes that will dominate D. First we choose P, the class of p. We partition every other partite class into three parts according to how it is connected to p. For any class Z, let Z_1 denote the set of vertices in Z dominated by p, let Z_2 be the set of vertices in Z nonadjacent to p, and let Z_3 denote the set of remaining vertices of Z, i.e., those which send an edge to p. We will refer to Z_i as the i-th part of the partite class Z, where $i = 1, 2, 3$. Note that K_3, L_3, M_3, N_3 are all empty, otherwise we would have $p \in \hat{N}_+(K \cup L \cup M \cup N)$.

Let D_2 be the subdigraph of D induced by the vertices in the second part of the partite classes of $D \setminus P$ in their partition above. This graph is also a multipartite digraph with no cyclic triangle and $\beta(D_2) = 1$. The latter follows from the fact that the vertices of D_2 are all nonadjacent to p and $\beta(D) = 2$. Thus by Lemma 8 the vertices of D_2 can be dominated by one partite class Q_2, the second part of some partite class Q of D. We choose Q to be the second partite class in our dominating set. Observe that all vertices of D not dominated so far, i.e., those not in $\hat{N}_+(P \cup Q)$ should belong to the third part of their partite classes. Let u be such a vertex. (If there is none, then we are done.) We know $u \notin K \cup L \cup M \cup N$ as none of these four classes has a third part. Since $K \cup L \cup M \cup N$ is a dominating set in $\hat{D}$ there is a vertex k in one of these four classes for which (k, u) is an edge of D. No vertex in the first part of a class can send an edge to a vertex lying in the third part of some other class, otherwise the latter two vertices would form a cyclic triangle with p. Thus, since K, L, M, N has no third parts, k must be in the second part of one of them.

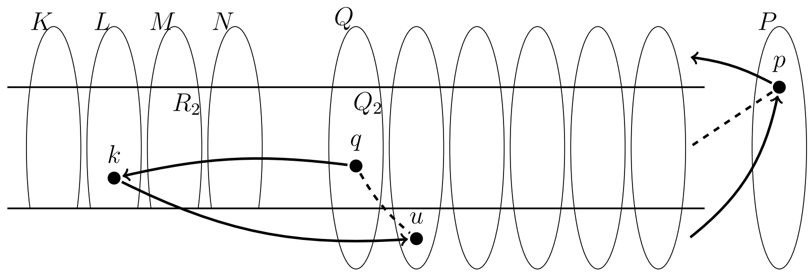

Figure 4: Domination of a multipartite digraph D with $\beta(D) = 2$.

Lemma 9 implies that there is a vertex $q \in Q_2$ with $V(D_2) \cap \hat{N}_+(q)$ containing $V(D_2)$ except one exceptional class R_2. We choose R, the partite class of R_2, to be the third partite class in our dominating set. If $u \notin \hat{N}_+(R)$ then k must be an outneighbor of q. Observe that (u, q) cannot be an edge of D, otherwise q, k and u would form a cyclic triangle. But (q, u) cannot be an edge either, as $u \notin N_+(Q)$. Thus u and every so far undominated vertex is nonadjacent to q. Thus the set U of undominated vertices induces a subgraph $D[U]$ with $\beta(D[U]) = 1$, otherwise adding q we would get $\beta(D) \geq 3$. But then by Lemma 8 all vertices in U can be dominated by one additional, fourth class. □

Remark 2. It is not difficult to show that we only need the partite class R for the domination if it coincides with K, L, M or N. (Otherwise k cannot be an element of R hence q surely sends an edge to k and nonadjacent to every $u \notin \hat{N}_+(P \cup Q)$.) Also, obviously if in D_2 we do not need the exceptional partite class, that is the vertex q dominates every other partite class except for Q_2, then we can dominate D with three partite classes. Moreover, it is easy to see that in the proof of Lemma 9 if there is a vertex $l_1 \in L \neq K$ which is not dominated by $k \in K$ then we can change the roles of the dominating vertex and the exceptional partite class, namely it is also true that $V(D) \setminus (L \cup K) \subseteq N_+(l_1)$. From this it follows that in the proof of Theorem 2 if $R \in \{K, L, M, N\}$ but $Q \notin \{K, L, M, N\}$ then P, R and one additional partite class for the undominated vertices are enough for domination. Thus we only need four partite classes in the dominating set if both Q and R are equal to one of the dominating parite classes of $D \setminus \{p\}$. This observation may be useful in deciding whether there is a multipartite digraph D with no cyclic triangle for which $\beta(D) = 2$ and $k(D) = 4$. ◊

2.2 General case

Surprisingly, our proof of Theorem 1 is not a direct generalization of the argument proving Theorem 2 in the previous subsection. In fact, in a way it is conceptually simpler.

Proof of Theorem 1. We have seen that $h(1) = 1$ (and $h(2) = 4$) is an upper bound for $k(D)$ if $\beta(D) = 1$ (and if $\beta(D) = 2$). Now we prove that $h(\beta) = 3\beta + (2\beta + 1)h(\beta - 1)$ is an upper bound on $k(D)$ if $\beta(D) = \beta \geq 2$. Let D be a multipartite digraph without cyclic triangles and $\beta(D) = \beta$. (See Figure 5.) Let $k_1, k_2, \ldots, k_{2\beta}$ be vertices of D, each from a different partite class, such that $|\hat{N}_+(\cup_{i=1}^{2\beta}\{k_i\})|$ is maximal. Let the partite class of k_i be K_i for all i

and let $\mathcal{K}$ denote $\cup_{i=1}^{2\beta}\{k_i\}$. First we declare the 2β partite classes of these vertices k_i to be part of our dominating set. Next we partition every other partite class into $2\beta+2$ parts. For an arbitrary partite class $Z \neq K_i$ $(i = 1, \ldots, 2\beta)$ we denote by Z_0 the set $Z \cap N_+(\mathcal{K})$. For $i = 1, 2, \ldots, 2\beta$ let Z_i be the set of vertices in $Z \setminus Z_0$ that are not sending an edge to k_i, but are sending an edge to k_j for all $j < i$. Finally, we denote by $Z_{2\beta+1}$, the remaining part of Z, that is the set of those vertices of Z that send an edge to all vertices $k_1, k_2, \ldots, k_{2\beta}$. (As in the proof of Theorem 2 we will refer to the set Z_i as the i-th part of Z.) The subgraph D_i of D induced by the i-th parts of the partite classes of $D \setminus (\cup_{i=1}^{2\beta} K_i)$ is also a multipartite digraph with no cyclic triangle. For $1 \leq i \leq 2\beta$ it satisfies $\beta(D_i) \leq \beta - 1$, since adding k_i to any transversal independent set of D_i we get a larger transversal independent set. So by induction on β, each of these 2β digraphs D_i can be dominated by at most $h(\beta-1)$ partite classes. We add the appropriate $2\beta h(\beta-1)$ partite classes to our dominating set.

If $\beta(D_{2\beta+1}) \leq \beta - 1$ also holds then the whole graph can be dominated by choosing $h(\beta-1)$ additional partite classes. Otherwise let $\mathcal{L} = \{l_1, l_2, \ldots, l_\beta\}$ be an independent set of size β with all its vertices in $V(D_{2\beta+1})$ belonging to distinct partite classes (of D), that are denoted by $L_1, L_2, \ldots, L_\beta$, respectively. We claim that in the remaining part of $D_{2\beta+1}$, i.e., in $D_{2\beta+1} \setminus (\cup_{i=1}^{\beta} L_i)$ there is no other independent set of size β with all elements belonging to different partite classes. Assume on the contrary that $m_1 \in M_1, m_2 \in M_2, \ldots, m_\beta \in M_\beta$ form such an independent set $\mathcal{M}$. As $\mathcal{L}$ is a maximal transversal independent set, every element of a partite class different from $L_1, \ldots, L_\beta$ is connected to at least one of the l_i's. And since every element of $\mathcal{L}$ sends an edge to all the vertices $k_1, \ldots, k_{2\beta}$, we must have $N_+(\mathcal{K}) \setminus (\cup_{i=1}^{\beta} L_i) \subseteq N_+(\mathcal{L})$ otherwise a cyclic triangle would appear. (The latter is because if k_i $(i \in \{1, 2, \ldots, 2\beta\})$ sends an edge to v, and l_j $(j \in \{1, 2, \ldots, \beta\})$ sends an edge to k_i, moreover l_j is connected with v then the edge between l_j and v must be oriented towards v.)

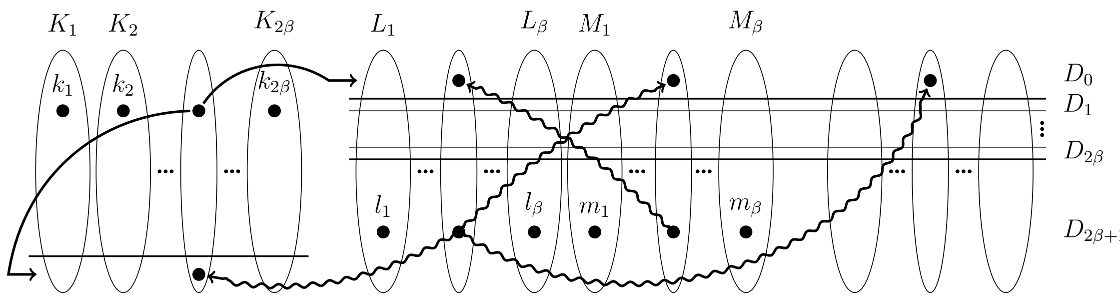

Figure 5: Domination of a multipartite digraph in the general case.

Similarly, we have $N_+(\mathcal{K}) \setminus (\cup_{i=1}^{\beta} M_i) \subseteq N_+(\mathcal{M})$. Thus if such an $\mathcal{M}$ exists then $\hat{N}_+(\mathcal{K}) \subseteq N_+(\mathcal{L} \cup \mathcal{M})$ while $\hat{N}_+(\mathcal{L} \cup \mathcal{M})$ also contains the additional vertices belonging to $\mathcal{L} \cup \mathcal{M}$. This contradicts the choice of $\mathcal{K}$. (Note that $\mathcal{L} \cup \mathcal{M}$ dominates also the vertices in $(K_1 \cup \cdots \cup K_{2\beta}) \cap (N_+(k_1) \cup \cdots \cup N_+(k_{2\beta}))$.) Thus if we add the classes $L_1, \ldots, L_\beta$ to our dominating set, the still not dominated part of D can be dominated by $h(\beta-1)$ further classes. So we constructed a dominating set of D containing at most $2\beta+2\beta h(\beta-1)+\beta+h(\beta-1) = 3\beta+(2\beta+1)h(\beta-1)$ partite classes. This proves the statement. $\square$

Note that we have proved a little bit more than stated in Theorem 1. Namely, we showed that there is a set of at most $h_1(\beta)$ vertices of D which dominates the whole graph except perhaps their own partite classes and at most $h_2(\beta)$ other exceptional classes. From the proof we obtain the recursion formula $h_1(\beta) \leq 2\beta + (2\beta + 1)h_1(\beta - 1)$ and $h_2(\beta) \leq \beta + (2\beta + 1)h_2(\beta - 1)$.

2.3 Clique-acyclic digraphs

For the proof of Theorem 3 we will use the following theorem due to Chvátal and Lovász [7].

Theorem CL ([7]). *Every directed graph D contains a semi-kernel, that is an independent set U satisfying that for every vertex $v \in D$ there is an $u \in U$ such that one can reach v from u via a directed path of at most two edges.*

Proof of Theorem 3. The statement is trivial for $\alpha(D) = 1$, since a transitive tournament is dominated by its unique vertex of indegree 0. We use induction on $\alpha = \alpha(D)$. Assume the theorem is already proven for $\alpha - 1$. Consider D with $\alpha(D) = \alpha$ and a semi-kernel U in D that exists by Theorem CL. (Figure 6 illustrates the proof.)

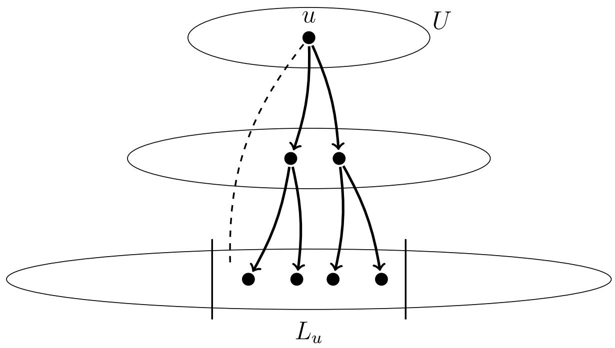

Figure 6: Domination of a clique-acyclic digraph.

We define a set S with $|S| \leq f(\alpha)$ elements dominating each vertex. Let $U \subseteq S$. Then S already dominates the outneighborhood of U. Denote by T the second outneighborhood of U (i.e., the set of all vertices not in U and not yet dominated). Observe that for every vertex $w \in T$ there is a vertex $u \in U$ such that neither (u, w) nor (w, u) is an edge. Indeed, let u be the vertex of U from which w can be reached by traversing two directed edges. Then $(w, u) \notin E(D)$ otherwise we would have a cyclic triangle. But $(u, w) \notin E(D)$ is immediate from knowing that w is not in the first outneighborhood of U. Partition T into $|U| \leq \alpha$ classes L_u indexed by the elements of U where $w \in L_u$ means that u and w are nonadjacent. Thus all vertices in each class L_u are independent from the same vertex in U implying that the induced subgraph $D[L_u]$ has independence number at most $\alpha - 1$. Thus $D[L_u]$ can be dominated by at most $f(\alpha - 1)$ vertices. Add these to S for every $u \in U$. So all vertices can be dominated by at most $\alpha + \alpha f(\alpha - 1) = f(\alpha)$ vertices completing the proof.

□

For $\alpha(D) = 2$ the above theorem gives $\gamma(D) \leq f(2) = 4$. Compared to this the improvement of Theorem 4 is only 1, but as already mentioned, the cyclically oriented five-cycle shows that $\gamma(D) \leq 3$ is the best possible upper bound.

The proof of Theorem 4 goes along similar lines as the proof we had for the second statement of Theorem 2.

Proof of Theorem 4. We use induction on the number of vertices in D. Let p be a vertex of D, and partition the remaining vertices of D into three parts. (See Figure 7.) Let V_1 be the set of vertices that are dominated by p, V_2 the set of vertices nonadjacent to p, and let V_3 be the set of vertices which send an edge to p. We assume by induction that $D \setminus \{p\}$ can be dominated by three vertices. (The base case is obvious.) If at least one of these is located in V_3 then p is also dominated by them and we are done. Otherwise we create a new dominating set.

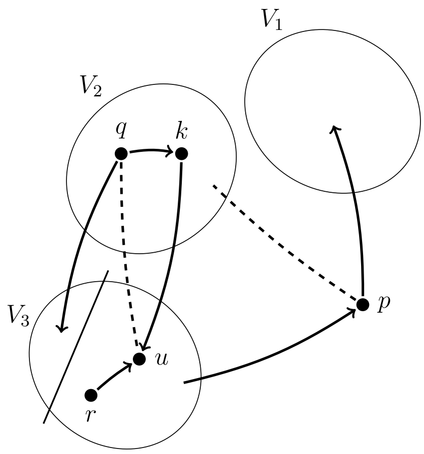

Figure 7: Domination of a clique-acyclic digraph D with $\alpha(D) = 2$.

First we choose p, and by p we dominate all the vertices in V_1. Observe that any two vertices in V_2 must be connected, because two nonadjacent vertices of V_2 and p would form an independent set of size 3. Thus $D[V_2]$ is a transitive tournament and so it can be dominated by just one vertex, let it be $q \in V_2$. Let U be the set of remaining undominated vertices. That is, $U = V_3 \setminus N_+(q)$. Consider an arbitrary element $u \in U$. We know that u is dominated by a vertex of the dominating set of $D \setminus \{p\}$. Let this vertex be k, it does not belong to V_3 as we assumed above. We also have $k \notin V_1$, otherwise there is a cyclic triangle on the vertices p, k, and u. So $k \in V_2$, and thus q sends an edge to k. Since u is undominated, (q, u) is not an edge of D. With the edge (u, q), we would get a cyclic triangle on u, q and k. So u and all the vertices in U are nonadjacent to q, therefore $\alpha(D[U]) = 1$ and thus U can be dominated by one vertex r. Thus all vertices of D are dominated by the 3-element set $\{p, q, r\}$. This completes the proof.

□

To prove Proposition 5 we formulate the following simple observation. Let $\chi(F)$ denote the chromatic number of graph F.

Observation 12. *Let D be a directed graph and $\bar{D}$ the complementary graph of the undirected graph underlying D. If D is clique-acyclic, then $\gamma(D) \leq \chi(\bar{D})$.*

Proof. It follows from the definition of $\chi(\bar{D})$ that the vertex set of D can be covered by $\chi(\bar{D})$ complete subgraphs of D. Since D is clique-acyclic, all these complete subgraphs can be dominated by one of their vertices. Thus all vertices are dominated by these $\chi(\bar{D})$ chosen vertices. □

Proof of Proposition 5. If the orientation of D is acyclic, then consider those vertices that have indegree zero. Let these form the set U_0. Delete these vertices and all vertices they dominate. Let set U_1 contain the indegree zero vertices of the remaining graph, and delete the vertices in $U_1 \cup N_+(U_1)$. Proceed this way to form the sets $U_2, \dots, U_s$, where finally there are no remaining vertices after U_s and its neighbors are deleted. It follows from the construction that $U_0 \cup U_1 \cup \dots \cup U_s$ is an independent set and dominates all vertices not contained in it.

The second statement immediately follows from Observation 12 and the fact that $\chi(\bar{D}) = \alpha(D)$ if D is perfect, an immediate consequence of the Perfect Graph Theorem [16]. □

3 On the exceptional classes

As already mentioned in the Introduction and also after the proof of Theorem 1, the statement of Theorem 1 could be formulated in a somewhat stronger form. Namely, we do not only dominate our multipartite digraph D by $h(\beta)$ partite classes, we actually dominate almost all of D by $h_1(\beta)$ vertices, where "almost" means that there is only a bounded number $h_2(\beta)$ of partite classes not dominated this way. The first appearance of this phenomenon is in Lemma 9 where we showed that if $\beta(D) = 1$ then a single vertex dominates the whole graph except at most one class. To complement this statement we show below that this exceptional class is indeed needed, we cannot expect to dominate the whole graph by a constant number of vertices. In other words, if we want to dominate with a constant number of singletons (and not by simply taking a vertex from each partite class), then we do need exceptional classes already in the $\beta(D) = 1$ case.

For a bipartite digraph D with partite classes A and B let $\gamma_A(D)$ denote the minimum number of vertices in A that dominate B and similarly let $\gamma_B(D)$ denote the minimum number of vertices in B dominating A. Let $\gamma_0(D) = \min\{\gamma_A(D), \gamma_B(D)\}$.

Theorem 13. *There exists a sequence of oriented complete bipartite graphs $\{D_k\}_{k=1}^{\infty}$ satisfying $\gamma_0(D_k) > k$.*

We note that the existence of D_k with n vertices in each partite class and satisfying $\gamma_0(D_k) > k$ follows by a standard probabilistic argument provided that $2\binom{n}{k}(1-2^{-k})^n < 1$. Our proof below is constructive, however.

Proof. We give a simple recursive construction for D_k in which we blow up the vertices of a cyclically oriented cycle C_{2k+2} and connect the blown up versions of originally nonadjacent vertices that are an odd distance away from each other by copies of the already constructed digraph D_{k-1}.

Let D_1 be a cyclic 4-cycle, i.e., a cyclically oriented $K_{2,2}$. It is clear that neither partite class in this digraph can be dominated by a single element of the other partite class. Thus $\gamma_0(D_1) > 1$ holds.

Assume we have already constructed D_{k-1} satisfying $\gamma_0(D_{k-1}) > k-1$. Let the two partite classes of D_{k-1} be $A_{k-1} = \{a_1, \ldots, a_m\}$ and $B_{k-1} = \{b_1, \ldots, b_m\}$. Now we construct D_k as follows. (The construction of D_2 is shown on Figure 8.) Let the vertex set of D_k be $V(D_k) = A_k \cup B_k$, where

$$A_k := \{(j, a_i) : 1 \leq j \leq k+1, 1 \leq i \leq m\},$$

$$B_k := \{(j, b_i) : 1 \leq j \leq k+1, 1 \leq i \leq m\}.$$

There will be an oriented edge from vertex (j, a_i) to (r, b_s) if either $j = r$, or $j \not\equiv r+1 \pmod{k+1}$ and $(a_i, b_s) \in E(D_{k-1})$. All other edges between A_k and B_k are oriented towards A_k, i.e., this latter set of edges can be described as

$$\{((r, b_s), (j, a_i)) \,:\, j \equiv r+1 \pmod{k+1} \quad \text{or} \quad ((b_s, a_i) \in E(D_{k-1}) \text{ and } j \neq r)\}.$$

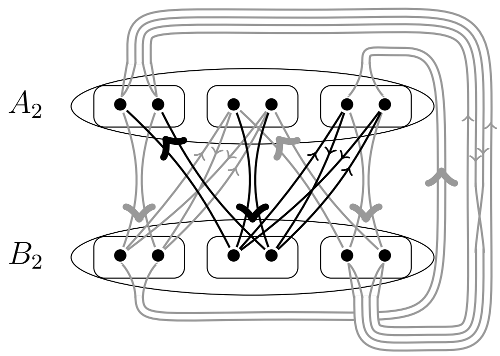

Figure 8: The construction of D_2.

It is only left to prove that $\gamma_0(D_k) > k$. Let us use the notation $A_k(j) = \{(j, a_i) : 1 \leq i \leq m\}$, $B_k(j) = \{(j, b_i) : 1 \leq i \leq m\}$. Consider a set K of k vertices of A_k, we show it cannot dominate B_k. There must be an $r \in \{1, \ldots, k+1\}$ by pigeon-hole for which $K \cap A_k(r) = \emptyset$ and $K \cap A_k(r+1) \neq \emptyset$. (Addition here is meant modulo $(k+1)$.) Fix this r. We claim that some vertex in $B_k(r)$ will not be dominated by K. Indeed, the vertex $(r+1, a_i) \in K \cap A_k(r+1)$ does not send any edge into $B_k(r)$, so we have only at most $k-1$ vertices in K that can dominate vertices in $B_k(r)$ and all these vertices are in $A_k \setminus A_k(r)$. Notice that the induced subgraph of D_k on $B_k(r) \cup A_k \setminus A_k(r)$ admits a digraph homomorphism (that is an edge-preserving map) into D_{k-1}. Indeed, the projection of each vertex to its second coordinate gives such a map by the definition of D_k. So if the above mentioned $k-1$ vertices would dominate the entire set $B_k(r)$, then their homomorphic images would dominate the homomorphic image of $B_k(r)$ in D_{k-1}. The latter image is the entire set B_{k-1} and by our induction hypothesis it cannot be dominated by $k-1$ vertices of A_{k-1}. Thus we indeed have $\gamma_{A_k}(D_k) > k$.

The proof of $\gamma_{B_k}(D_k) > k$ is similar by symmetry. Thus we have $\gamma_0(D_k) > k$ as stated. □

References

[1] R. Aharoni and R. Holzman, Fractional kernels in digraphs, *J. Combin. Theory Ser. B.*, **73** (1998), 1–6.

[2] N. Alon and J. H. Spencer, *The Probabilistic Method*, Third edition, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley and Sons, Hoboken, NJ, 2008.

[3] R.N. Ball, A. Pultr and P. Vojtěchovský, Colored graphs without colorful cycles, *Combinatorica*, **27** (2007), 407–427.

[4] K. Cameron and J. Edmonds, Lambda composition, *J. Graph Theory*, **26** (1997), 9–16.

[5] K. Cameron, J. Edmonds and L. Lovász, A note on perfect graphs, *Period. Math. Hungar.*, **17** (1986), 441–447.

[6] M. Chudnovsky, P. Seymour and B. Sullivan, Cycles in dense digraphs, *Combinatorica*, **28** (2008), 1–18.

[7] V. Chvátal and L. Lovász, Every directed graph has a semi-kernel, *Hypergraph Seminar* (Proc. First Working Sem., Ohio State Univ., Columbus, Ohio, 1972; dedicated to Arnold Ross), Lecture Notes in Math., Vol. 411, Springer, Berlin, 1974, pp. 175.

[8] S. Fujita, C. Magnant and K. Ozeki, Rainbow Generalizations of Ramsey Theory: A Survey, *Graphs Combin.*, **26** (2010), 1–30.

[9] T. Gallai, Transitiv orientierbare Graphen, *Acta Math. Sci. Hungar.*, **18** (1967) 25–66. English translation by F. Maffray and M. Preissmann, in: J. L. Ramírez Alfonsín and B. A. Reed (editors), Perfect Graphs, John Wiley and Sons, 2001, 25–66.

[10] V. Gurvich, Decomposing complete edge-chromatic graphs and hypergraphs. Revisited, *Discrete Applied Math.*, **157** (2009), 3069–3085.

[11] A. Gyárfás and G. Simonyi, Edge colorings of complete graphs without tricolored triangles, *J. Graph Theory*, **46** (2004), 211–216.

[12] A. Gyárfás and G. N. Sárközy, Gallai colorings of non-complete graphs, *Discrete Math.*, **310** (2010), 977–980.

[13] A. Gyárfás, G. N. Sárközy, A. Sebő and S. Selkow, Ramsey-type results for Gallai colorings, *J. Graph Theory*, **64** (2010), 233–243.

[14] J. Körner and G. Simonyi, Graph pairs and their entropies: Modularity problems, *Combinatorica*, **20** (2000), 227–240.

[15] J. Körner, G. Simonyi and Zs. Tuza, Perfect couples of graphs, *Combinatorica*, **12** (1992), 179–192.

[16] L. Lovász, Normal hypergraphs and the perfect graph conjecture, *Discrete Math.*, **2** (1972), 253–267.