# Around a biclique cover conjecture 

Guantao Chen<br>Department of Mathematics and Statistics<br>Georgia State University<br>Atlanta, Georgia 30303<br>matgtc@langate.gsu.edu<br>\section*{Shinya Fujita}<br>Gunma National College of Technology<br>580 Toriba, Maebashi, Gunma, 371-8530 Japan<br>shinyaa@mti.biglobe.ne.jp<br>András Gyárfás<br>Computer and Automation Research Institute<br>Hungarian Academy of Sciences<br>1518 Budapest, P.O. Box 63, Hungary<br>gyarfas@sztaki.hu<br>Jenő Lehel<br>Department of Mathematical Sciences<br>The University of Memphis, Tennessee, USA<br>jlehel@memphis.edu<br>\section*{Ágnes Tóth}<br>Department of Computer Science and Information Theory<br>Budapest University of Technology and Economics<br>1521 Budapest, P.O. Box 91, Hungary<br>tothagi@cs.bme.hu

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#### Abstract

We address an old (1977) conjecture of a subset of the authors (a variant of Ryser's conjecture): in every $r$-coloring of the edges of a biclique $[A, B]$ (complete bipartite graph), $A \cup B$ can be covered by the vertices of at most $2 r-2$ monochromatic connected components. We reduce this conjecture to design-like conjectures, where the monochromatic components of the color classes are bicliques $[X, Y]$ with nonempty blocks $X$ and $Y$. It can be also assumed that each color class covers $A \cup B$ (spanning), moreover, no $X$-blocks or $Y$-blocks properly contain each other (antichain property). We prove this reduced conjecture for $r \leq 5$.

We show that the width (the number of bicliques) in every color class of any spanning $r$-coloring is at most $2^{r-1}$ (and this is best possible). On the other hand there exist spanning $r$-colorings such that the width of every color class is $\Omega\left(r^{3 / 2}\right)$.

We discuss the dual form of the conjecture which relates to transversals of intersecting and cross-intersecting $r$-partite hypergraphs.


## 1 Introduction, summary of results

A special case of a conjecture generally attributed to Ryser (appeared in his student, Henderson's thesis, [7]) states that intersecting $r$-partite hypergraphs have a transversal of at most $r-1$ vertices (see Conjecture 6 in Section 6). This conjecture is open for $r \geq 6$. It is trivially true for $r=2$, the cases $r=3,4$ are solved in [3] and in [2], and for the case $r=5$, see [2] and [11]. The following equivalent formulation is from [3],[5]:

Conjecture 1. In every r-coloring of the edges of a complete graph, the vertex set can be covered by the vertices of at most $r-1$ monochromatic connected components.

Gyárfás and Lehel discovered a bipartite version of this conjecture [3], [8]. A complete bipartite graph $G$ with non empty vertex classes $X$ and $Y$ is referred as a biclique $[X, Y]$ in this paper, and $X$ and $Y$ will be called the blocks of this biclique. Given an edge coloring, a monochromatic component means a connected component of the subgraph of any given color.

Conjecture 2. In every $r$-coloring of the edges of a biclique, the vertex set can be covered by the vertices of at most $2 r-2$ monochromatic components.

First we show here that Conjecture 2, if true, is best possible. Let $G^{*}=[A, B]$ be a biclique with $|A|=r-1,|B|=r$ !, and label the vertices of $A$ with $\{1,2, \ldots, r-1\}$ and those of $B$ with the $(r-1)$-permutations of the elements of $\{1,2, \ldots, r\}$. For $k \in A$ and $\pi=j_{1} j_{2} \ldots j_{r-1} \in B$, let the color of the edge $k \pi$ be $j_{k}$.

Since each vertex in $B$ is incident with $r-1$ edges of distinct color, every monochromatic component of $G^{*}$ is a star with $(r-1)$ ! leaves centered at $A$. Furthermore, $G^{*}$ has a vertex cover with $2 r-2$ monochromatic components, just take the $r$ monochromatic stars centered at vertex $r-1$, and add one edge from each vertex $k=1,2, \ldots, r-2$ of $A$.

Proposition 1. ([3]) The vertex set of $G^{*}$ cannot be covered with less than $2 r-2$ monochromatic components.

Proof. Let $\mathcal{C}$ be a cover of $V\left(G^{*}\right)=A \cup B$ by monochromatic stars centered in $A$. Let $a_{k}$ denote the number of monochromatic stars of $\mathcal{C}$ on vertex $k \in A$. We may assume that $a_{1} \leq a_{2} \leq \ldots, \leq a_{r-1}$.

We show first that $a_{i} \geq i+1$ holds for some $1 \leq i \leq r-1$. Suppose on the contrary that $a_{r-1}<r, a_{r-2}<r-1, \ldots, a_{1}<2$. Thus we can select a color $j_{r-1} \in\{1, \ldots, r\}$ different from the $a_{r-1}$ colors of all stars of $\mathcal{C}$ centered at $r-1$. Then we can select a new color $j_{r-2} \in\{1, \ldots, r\} \backslash\left\{j_{r-1}\right\}$ different from the $a_{r-2}$ colors of all stars of $\mathcal{C}$ centered at $r-2$, etc. Thus we end up by selecting $r-1$ distinct colors $j_{1}, \ldots, j_{r-1}$. This is a contradiction since the $(r-1)$-permutation $j_{1} j_{2} \ldots, j_{r-1} \in B$ is uncovered by $\mathcal{C}$.

Now let $a_{i} \geq i+1$, for some $1 \leq i \leq r-1$, then for the number of stars in $\mathcal{C}$ we have

$$
\sum_{k=1}^{r-1} a_{k}=\sum_{k=1}^{i-1} a_{k}+\sum_{k=i}^{r-1} a_{k} \geq(i-1)+(i+1)(r-i) .
$$

Because

$$
(i-1)+(i+1)(r-i)=-i^{2}+r i+r-1 \geq 2 r-2
$$

holds for every $1 \leq i \leq r-1$, the proposition follows.
It is worth noting that Conjecture 2 (similarly to Conjecture 1 ) becomes obviously true if the number of monochromatic components is just one larger than stated in the conjecture.

Proposition 2. ([3]) In every r-coloring of the edges of a biclique, the vertex set can be covered by the vertices of at most $2 r-1$ monochromatic components.

Proof. For an edge $u v$ of the biclique $G$, consider the monochromatic component (double star) formed by the edges in the color of $u v$ incident to $u$ or $v$. In all other colors consider the (at most $r-1$ ) monochromatic stars centered at $u$ and at $v$. This gives $2 r-1$ monochromatic components covering the vertices of $G$.

In Section 2 we show that Conjecture 2 can be reduced to design-like conjectures: one can assume that all components of all color classes are complete bipartite graphs. It is worth noting that similar reduction is not known for Conjecture 1.

We shall prove Conjecture 2 for $r=2,3,4,5$ in Sections 3 and 4, in fact in stronger forms defined in Section 2 (Propositions 4, 5 and Theorems 1, 2).

Using a deep result of Alon [1], we show that the width (the number of bicliques) of every color class in any spanning $r$-coloring is at most $2^{r-1}$ and this is best possible (Theorem 3). We also show that there exist spanning $r$-colorings such that the width of every color class is $\Omega\left(r^{3 / 2}\right)$ (Theorem 5).

In Section 6 we formulate the dual forms of Conjectures 1, 2 and show their relation to transversals of intersecting and cross-intersecting $r$-uniform hypergraphs.

## 2 Equivalent conjectures, notations

We shall see that Conjecture 2 is equivalent to further design-like conjectures, thus an $r$ coloring will be also called a partition of the edge set into $r$ subgraphs.

A bi-equivalence graph is a bipartite graph whose connected components are bicliques; the width of a bi-equivalence graph is the number of its components.

Conjecture 3. If a biclique is partitioned into $r$ bi-equivalence graphs, then its vertex set can be covered by at most $2 r-2$ biclique components.

Since the bi-equivalence graphs in Conjecture 3 can be color classes of an $r$-coloring, validity of Conjecture 2 implies that Conjecture 3 is also true.

On the other hand, suppose we have an $r$-coloring of a biclique $G=[X, Y]$ such that some monochromatic component $C$, say in color 1 , is not a biclique. Let $x \in X, y \in Y$ be non-adjacent vertices in $C$, w.l.o.g. $x y$ has color 2. Observe that the $2(r-2)$ monochromatic stars in colors $3, \ldots, r$ centered at $x$ and at $y$, plus the component $C$, and the component in color 2 containing $x y$ cover $V(G)$, leading to a cover with at most $2 r-2$ monochromatic components. Thus Conjecture 2 follows from Conjecture 3.

Let us call a bi-equivalence graph partition $G_{1}, \ldots, G_{r}$ of biclique $G$ a spanning partition if each vertex $v \in V(G)$ is included in every $V\left(G_{i}\right), i=1, \ldots, r$. Notice that it is enough to prove Conjecture 3 for spanning partitions. Indeed, assuming that $v \notin V\left(G_{1}\right)$ and $v w \in$ $E\left(G_{2}\right)$, just take the at most $r-2$ bicliques from $G_{3}, \ldots, G_{r}$ that contain $v$ and add the at most $r$ bicliques from $G_{1}, G_{2}, \ldots, G_{r}$ that contain $w$, together they form a cover of all vertices of $G$ with at most $2 r-2$ bicliques. Thus we have the following equivalent form of Conjecture 3.

Conjecture 4. If a biclique has a spanning partition into $r$ bi-equivalence graphs, then its vertex set can be covered by at most $2 r-2$ biclique components.

Let a biclique $[X, Y]$ be partitioned into the bi-equivalence graphs $G_{1}, G_{2}, \ldots, G_{r}$. Then we will say that $i$ is the color of the edges in $G_{i}(i=1, \ldots, r)$. Any connected component of $G_{i}$ is a biclique, its vertex classes will called blocks in color $i$.

Denote by $B_{i}\left[u_{1}, \ldots, u_{k}\right]$ the connected component of $G_{i}$ which contains the vertices $u_{1}, \ldots, u_{k}$, if they are in the same component of $G_{i}$, and in this case let $X_{i}\left[u_{1}, \ldots, u_{k}\right]=$ $X \cap V\left(B_{i}\left[u_{1}, \ldots, u_{k}\right]\right)$ and $Y_{i}\left[u_{1}, \ldots, u_{k}\right]=Y \cap V\left(B_{i}\left[u_{1}, \ldots, u_{k}\right]\right)$ be the corresponding blocks. Otherwise set $B_{i}\left[u_{1}, \ldots, u_{k}\right]=\emptyset, X_{i}\left[u_{1}, \ldots, u_{k}\right]=Y_{i}\left[u_{1}, \ldots, u_{k}\right]=\emptyset$.

Note that $B_{i}[u] \neq \emptyset$ for any $u \in V(G)$ in a spanning partition. In the sequel we will also use the fact that the blocks $X_{i}[u]$ and $X_{i}[v]$ are either disjoint or equal for any color $i \in\{1,2, \ldots, r\}$ and any vertices $u, v \in V(G)$.

Let us call a spanning bi-equivalence graph partition $G_{1}, \ldots, G_{r}$ of biclique $G$ an antichain partition if no blocks properly contain each other, that is if no colors $i, j \in\{1, \ldots, r\}$ and no vertices $u, v \in V(G)$ exist such that $X_{i}[u] \subsetneq X_{j}[v]$ or $Y_{i}[u] \subsetneq Y_{j}[v]$.

If $v \in X$ and $\left|X_{i}[v]\right|=1$ (or $v \in Y$ and $\left|Y_{i}[v]\right|=1$ ) then we call vertex $v$ a singleton block in color $i$. Note that if a coloring has the antichain property, then a singleton block in some color is a singleton in every color, in this case we just say that $v$ is a singleton.

It turns out that it is enough to prove Conjecture 4 for antichain partitions. Indeed, assume that in a spanning partition there are two blocks properly containing each other, that is $X_{1}[z] \subsetneq X_{2}[x]$, for some biclique components $B_{1}[z]$ and $B_{2}[x]$. Assume that $x \notin X_{1}[z]$, and let $y \in Y_{1}[z]$. The color of the edge $x y$ is neither 1 nor 2, w.l.o.g. it is 3 . Because $B_{3}[y]=B_{3}[x]$ and $X_{1}[y]=X_{1}[z] \subseteq X_{2}[x]$, the collection

$$
\left\{B_{i}[x]: i \in\{1,2, \ldots, r\}\right\} \cup\left\{B_{i}[y]: i \in\{1,2, \ldots, r\} \backslash\{1,3\}\right\}
$$

is a cover with at most $2 r-2$ monochromatic components. Thus we obtain the following equivalent form of Conjecture 2 .

Conjecture 5. If a biclique has an antichain partition into r bi-equivalence graphs, then its vertex set can be covered by at most $2 r-2$ biclique components.

Our example in Proposition 1 showing that Conjecture 2 is sharp is not an antichain partition (not even a spanning partition). It is not impossible that for antichain partitions (or even for spanning partitions) stronger result holds.

Question 1. Suppose that a biclique has an antichain partition into r bi-equivalence graphs. Can one cover its vertex set by at most $r$ biclique components?

For $r=2,3,4$ the answer to Question 1 is affirmative (see Sections 3, 4). Note that one-factorizations of $K_{r, r}$ show that one cannot expect a cover with less than $r$ biclique components.

Finally we note an important reduction used extensively in the proofs later. We call a pair $u, v \in A$ or $u, v \in B$ equivalent if in every bi-equivalence graph of the bi-equivalence graph partition of the biclique $G, u$ and $v$ belong to the same block. We may assume w.l.o.g. that there is no pair of equivalent vertices, and in this case we say that the coloring is reduced. Indeed, if there were two vertices $u, v$ such that $u v \notin E(G)$ and for every $w \in V(G)$ with $u w, v w \in E(G)$, the edges $u w$ and $v w$ have the same color, then $v$ could be added to any monochromatic component of $G-\{v\}$ containing $u$. Hence if Conjecture 5 holds for $G-\{v\}$ then it also holds for $G$.

In a reduced $r$-coloring of a biclique, the number of vertices is bounded by a function of $r$. In fact, one can easily see the following.

Proposition 3. Suppose a biclique $[A, B]$ has a partition into $r$ bi-equivalence graphs and no two vertices of $A$ are equivalent. Then $|A| \leq r$ !, and equality is possible.

Proof. It is easy to check that the partition of $G^{*}$ into bi-equivalence graphs in Proposition 1 is a reduced one, hence the second statement follows.

To see the first statement, the case $r=1$ is obvious. Assuming it is true for some $r \geq 1$, suppose indirectly that $|A| \geq(r+1)!+1$ in some partition into $r+1$ bi-equivalence graphs. Then for any fixed $v \in B$ there are $r!+1$ edges of the same color from $v$, say in color $r+1$, to $Y \subset A$. Let $X$ be the set of vertices in $B$ that send edges in at least two different colors to $Y$. By the assumption $X \neq \emptyset$ and since color class $r+1$ is a bi-equivalence graph, $[X, Y]$ has no edge of color $r+1$. This means no two vertices of $Y$ are equivalent in the induced $r$-partition on $[X, Y]$, and thus $|Y|>r!$ contradicts the inductive hypothesis.

## 3 Bi-equivalence partitions for $r=2,3$ and 4

In the present section we prove Conjecture 2 for the small cases in strongest possible form.
Proposition 4. If a biclique $[X, Y]$ is partitioned into at most two bi-equivalence graphs, then each has at most two (non trivial) connected components.

Proof. Assume on the contrary that $x_{j} y_{j}, j=1,2,3$, are three edges from three distinct connected components of $G_{1}$, where $x_{j} \in X$ and $y_{j} \in Y$. Then the path $\left(x_{1}, y_{2}, x_{3}, y_{1}\right)$ is in $G_{2}$, but the color of $x_{1} y_{1}$ is not 2 . Hence $G_{2}$ is not bi-equivalence graph, a contradiction.

Proposition 5. Let a biclique $[X, Y]$ be partitioned into three bi-equivalence graphs. If one of those has more than three non trivial components, then some of the other two is spanning and has two connected components.

Proof. Assume on the contrary that $x_{j} y_{j}, j=1,2,3,4$, are four edges from four distinct connected components of $G_{1}$, where $x_{j} \in X$ and $y_{j} \in Y$.

The subgraph of the biclique on the vertex set $\left\{x_{1}, x_{2}, x_{3}, y_{1}, y_{2}, y_{3}\right\}$ contains a 6 -cycle $C$ whose edges are colored with 2 and 3 . Since the color classes are bi-equivalence graphs, $C$ has no monochromatic path of length more than two.

First assume that $C$ has three edges of color 2 (the other three are colored with 3 ). W.l.o.g. we assume that $x_{4} y_{1}, x_{4} y_{2} \in E\left(G_{2}\right)$. Since the bi-equivalence property, we have $x_{1} y_{2}, x_{2} y_{1} \in E\left(G_{3}\right)$. Since $C$ have three edges in $G_{2}$, we may assume $y_{1} x_{3}, y_{2} x_{3} \in E\left(G_{2}\right)$. Observe that the edges $x_{1} y_{3}, x_{2} y_{3}$ of $C$ are colored differently from the set $\{2,3\}$ hence the color of $x_{4} y_{3}$ is neither 2 nor 3 , a contradiction.

Therefore $C$ has four edges in one color and two edges in the other color. W.l.o.g. we assume that the colors follow each other along the cycle $C=\left(x_{1}, y_{3}, x_{2}, y_{1}, x_{3}, y_{2}, x_{1}\right)$ as $2,2,3,2,2,3$. Then for every vertex $x \in X \backslash\left(X_{1}\left[x_{1}\right] \cup X_{1}\left[x_{2}\right] \cup X_{1}\left[x_{3}\right]\right)$ we obtain that $x y_{1}, x y_{2} \in E\left(G_{2}\right)$. Observe that this is also true for every $x \in X_{1}\left[x_{3}\right]$, since the (2,3)-coloring pattern along the 6 -cycle $C^{\prime}=\left(C-x_{3}\right)+x$ uniquely determines the color of the two edges at $x$.

In the same way one obtains that $X \backslash\left(X_{1}\left[x_{1}\right] \cup X_{1}\left[x_{2}\right]\right)$ and $Y_{1}\left[y_{1}\right] \cup Y_{1}\left[y_{2}\right]$ induce a biclique in $G_{2}$, since, for $i=1,2$, any vertex $y \in Y_{1}\left[y_{i}\right]$ can replace $y_{i}$ in the cycle $C$ without altering the (2,3)-coloring pattern along the modified cycle. By symmetry of $X$ and $Y$, we obtain that $Y \backslash\left(Y_{1}\left[y_{1}\right] \cup Y_{1}\left[y_{2}\right]\right)$ and $X_{1}\left[x_{1}\right] \cup X_{1}\left[x_{2}\right]$ induce a biclique of $G_{2}$ as well.

Therefore $G_{2}$ is spanning and has two connected components.

Theorem 1. If a biclique has an antichain partition into 4 bi-equivalence graphs, then its vertex set can be covered by at most 4 connected components of the same color, or equivalently, one of the bi-equivalence graphs has width at most 4.
Proof. Let $G_{i}, i=1,2,3,4$, be the bi-equivalence graphs in a reduced antichain partition of a biclique $[X, Y]$.

Claim 1: if $\left|X_{i}[u]\right| \leq 2$ for every color $i$ and vertex $u$, then $G_{1}$ has 4 components. To see this let $y \in Y$ and set $U=\bigcup_{i=2}^{4} X_{i}[y]$. Let $s$ be the number of components of $G_{1}$ that intersect $U$ at a single vertex. If $x \in X_{i}[y]$, for some $i \in\{2,3,4\}$, and $B_{1}[x] \cap U=\{x\}$, then $X_{1}[x]=\{x\}$ and hence by the antichain property, $X_{i}[y]=\{x\}$ follows. Thus for the number of components of $G_{1}$ different from $B_{1}[y]$ we obtain $s+2(3-s) / 2=3$, and the claim follows.

Due to Claim 1 we may assume that there are three distinct vertices, $x_{1}, x_{2}, x_{3} \in X$ in some block of $G_{1}$. Let

$$
Y\left(c_{1}, c_{2}, c_{3}\right)=\left\{y \in Y \mid y x_{i} \text { is colored with } c_{i}, i=1,2,3\right\}
$$

The three-tuple $\left(c_{1}, c_{2}, c_{3}\right)$ will be called the type of the subset $Y\left(c_{1}, c_{2}, c_{3}\right)$. In terms of this notation $Y(1,1,1) \neq \emptyset$. When the wildcard character $*$ is used for a color, then the color of the corresponding edge between $\left\{x_{1}, x_{2}, x_{3}\right\}$ and the set of that type is undetermined (e.g. $Y(3,3,4) \subseteq Y(3, *, 4)$ is true $)$.

In a bi-equivalence graph partition certain types cannot coexist as is expressed in the next rule: If $a, b$ are distinct colors, then at least one of the sets $Y(a, a, *)$ and $Y(a, b, *)$ must be empty. Indeed, if $y_{1} \in Y(a, a, *)$ and $y_{2} \in Y(a, b, *)$, then $\left(y_{2}, x_{1}, y_{1}, x_{2}\right)$ is a path belonging to some biclique of $G_{a}$, hence the edge $x_{2} y_{2}$ must have color $a$, and not $b$.

Claim 2: there is no (nonempty) three of a kind type in $Y \backslash Y(1,1,1)$. Assume on the contrary that $Y(2,2,2) \neq \emptyset$. Because $x_{1}$ and $x_{2}$ are not equivalent, we have $Y(3,4, *) \neq \emptyset$, $Y(4,3, *) \neq \emptyset$, and therefore, $Y(3,3, *)=\emptyset, Y(4,4, *)=\emptyset$. Moreover, this must hold for any pair $x_{i}, x_{j}, 1 \leq i<j \leq 3$, which is clearly impossible (by the pigeon hole principle).

Claim 3: at least one of $Y(2,2,3)$ and $Y(2,2,4)$ is empty. To see this, assume $Y(2,2,3) \neq \emptyset$ and $Y(2,2,4) \neq \emptyset$. Then by the argument above we have

$$
Y=Y(1,1,1) \cup Y(2,2, *) \cup Y(3,4, *) \cup Y(4,3, *) .
$$

In particular $Y(*, *, 3) \cup Y(*, *, 4) \subseteq Y(2, *, *)$, violating the antichain property.
Now w.l.o.g. assume that either $Y(2,2,3) \neq \emptyset$ or no (nonempty) pair type exists in $Y \backslash Y(1,1,1)$. In both cases every (nonempty) type in $Y \backslash Y(1,1,1)$ has a color 3. Then the components $B_{3}\left[x_{i}\right], i=1,2,3$, form a cover provided $Y_{3}[z] \cap(Y \backslash Y(1,1,1)) \neq \emptyset$, for all $z \in X$. If some $z$ does not satisfy this, then by the antichain property, $Y(1,1,1)=Y_{3}[z]$, and $B_{3}\left[x_{i}\right], i=1,2,3$, and $B_{3}[z]$ together form a cover.

## 4 Bi-equivalence partitions for $r=5$

In this section we shall verify Conjecture 5 , for $r=5$, in a stronger form. Actually we will show that under the appropriate conditions there is a cover with at most $2 r-2=8$ monochromatic components in the same color, or equivalently, one of the bi-equivalence graphs of the partition has width at most 8 .

Theorem 2. If a biclique has an antichain partition into 5 bi-equivalence graphs, then its vertex set can be covered by at most 8 components of the same color.

Let $G_{i}, i=1,2,3,4,5$, be the bi-equivalence graphs in a reduced antichain partition of the biclique $[X, Y]$. For the proof we need two technical lemmas.

Lemma 1. If each $G_{i}, i=1, \ldots, 5$, has width at least 6 , then $[X, Y]$ contains at most two singletons in each vertex class.

Proof. Suppose on the contrary that one class has three singletons, say $x_{1}, x_{2}, x_{3} \in X$ with $\left|X_{i}\left[x_{j}\right]\right|=1$, for every $1 \leq i \leq 5$, and $1 \leq j \leq 3$. Then taking any $y \in Y_{1}\left[x_{1}\right]$, we may assume that $y x_{2} \in E\left(G_{2}\right)$ and $y x_{3} \in E\left(G_{3}\right)$. In particular, we obtain that $X=$ $\left\{x_{1}, x_{2}, x_{3}\right\} \cup X_{4}[y] \cup X_{5}[y]$.

For any $z \in X_{4}[y]$, we have $X_{5}[z] \cap X_{5}[y]=\emptyset$, hence by the antichain property, $X_{5}[z]=$ $X_{4}[y]$. Therefore $G_{5}$ has five components: $B_{5}\left[x_{1}\right], B_{5}\left[x_{2}\right], B_{5}\left[x_{3}\right], B_{5}[z], B_{5}[y]$, a contradiction.

Lemma 2. Let each $G_{i}, i=1, \ldots, 5$, have width at least 9. If $[X, Y]$ contains at most two singletons in both of its vertex classes, then there is a color $i$ and a vertex $u$ for which $\left|X_{i}[u]\right| \geq 9$ or $\left|Y_{i}[u]\right| \geq 9$.

Proof. Assume that for every color $i$ and vertex $u$ we have $\left|X_{i}[u]\right| \leq t$ and $\left|Y_{i}[u]\right| \leq t$. Let $G_{1}$ be the graph with the maximum number of edges among $G_{i}, i=1, \ldots, 5$. The trivial inequality $|E(G)| \leq 5\left|E\left(G_{1}\right)\right|$ will give us a first lower bound on $t$.

For a vertex $u \in X$ we have $Y=Y_{1}[u] \cup Y_{2}[u] \cup Y_{3}[u] \cup Y_{4}[u] \cup Y_{5}[u]$. As $\left|Y_{i}[u]\right| \leq t$ we get $|Y| \leq 5 t$. Similarly it follows that $|X| \leq 5 t$. Since $G$ contains at most two singletons, and the width of $G_{1}$ is at least 9 we have $5 t \geq|Y| \geq 2 \cdot 1+7 \cdot 2=16$, therefore $t \geq 4$.

Let $\underline{x}$ and $y$ be vectors which contain the sizes of the components of $G_{1}$ in $X$ and in $Y$, respectively. Our assumptions on $G_{1}$ mean that the length of $\underline{x}$ and $y$ is at least 9 , they have at most two elements equal to 1 , and all their elements are at most $t$. Using this notation $\left|E\left(G_{1}\right)\right|=\underline{x} \cdot \underline{y}$, and $|E(G)|=|X||Y|=(\underline{x} \cdot \underline{1})(\underline{y} \cdot \underline{1})$, where $\underline{1}$ is the constant 1 vector with appropriate length. We are going to investigate $\operatorname{diff}(\underline{x}, \underline{y})=|E(G)|-5\left|E\left(G_{1}\right)\right|=$ $(\underline{x} \cdot \underline{1})(\underline{y} \cdot \underline{1})-5(\underline{x} \cdot \underline{y})$, and determine its minimum over all possible values of $\underline{x}$ and $\underline{y}$. If this function is positive for some $t$, then there is no partition of $G$ into graphs with the above conditions for the given value of $t$.

In the first steps we minimize $\operatorname{diff}(\underline{x}, \underline{y})$, for any fixed $|X|$ and $|Y|$, that is we maximize $\left|E\left(G_{1}\right)\right|=\underline{x} \cdot \underline{y}$.

Step 1: We may assume that the length of $\underline{x}$ is equal to 9 , and so the length of $y$ is also 9. Otherwise we could join two components of $G_{1}$ and increase the number of edges. So we have $\underline{x}=\left(x_{1}, \ldots, x_{9}\right)$ and $\underline{y}=\left(y_{1}, \ldots, y_{9}\right)$.

Step 2: We can reorder the components of $G_{1}$ such that $\underline{y}$ is ordered non-increasingly. After that we may assume that the elements of $\underline{x}$ are also ordered non-increasingly. Otherwise we could swap two elements with $x_{i}<x_{j}$ for $1 \leq i<j \leq 9$ and this operation would not decrease the value of $\underline{x} \cdot \underline{y}$. (The increment is $\left(x_{j}-x_{i}\right)\left(y_{i}-y_{j}\right) \geq 0$.) Hence $y_{1} \geq y_{2} \geq \cdots \geq y_{9}$ and $x_{1} \geq x_{2} \geq \cdots \geq x_{9}$.

Step 3: We can not decrease the number of edges of $G_{1}$ neither if we increase an element $x_{i}$ of $\underline{x}$ by some constant $c$ and decrease $x_{j}$ for $j>i$ by the same constant. (The increment is $c\left(y_{i}-y_{j}\right) \geq 0$.)
By repeated use of this operation (observing the condition that each element of $\underline{x}$ and $\underline{y}$ is at most $t$, and these vectors contain at most two elements equal to 1) we obtain that $x_{1}=\cdots=x_{p}=t, t>x_{p+1} \geq 2, x_{p+2}=\cdots=x_{7}=2, x_{8}=x_{9}=1$ and similarly $y_{1}=\cdots=y_{q}=t, t>y_{q+1} \geq 2, y_{q+2}=\cdots=y_{7}=2, y_{8}=y_{9}=1$. From $|X| \leq 5 t$ it follows that $p<5$, and similarly we get $q<5$.

Thus for a given $|X|$ and $|Y|$, the maximum value $\left|E\left(G_{1}\right)\right|=\underline{x} \cdot \underline{y}$ is determined by the vectors $\underline{x}, \underline{y}$ standardized as above. In the next steps we minimize diff $(\underline{x}, \underline{y})$ by changing $|X|$ and $|Y|$.

Step 4: If $x_{p+1} \neq 2$ then let $\underline{x}^{-}$and $\underline{x}^{+}$be vectors almost the same as $\underline{x}$, but at the ( $p+1$ )-th position they have $x_{p+1}-1 \geq 2$ and $x_{p+1}+1 \leq t$, respectively. We claim that $\operatorname{diff}\left(\underline{x}^{-}, \underline{y}\right)$ or $\operatorname{diff}\left(\underline{x}^{+}, \underline{y}\right)$ is not greater than $\operatorname{diff}(\underline{x}, \underline{y})$. Indeed, $\operatorname{diff}(\underline{x}, \underline{y})-\operatorname{diff}\left(\underline{x}^{-}, \underline{y}\right)=$ $\operatorname{diff}\left(\underline{x}^{+}, \underline{y}\right)-\operatorname{diff}(\underline{x}, \underline{y})=|Y|-5 y_{p+1}$ which means that $\operatorname{diff}(\underline{x}, \underline{y})$ is a middle element of an arithmetic progression between $\operatorname{diff}\left(\underline{x}^{-}, \underline{y}\right)$ and $\operatorname{diff}\left(\underline{x}^{+}, \underline{y}\right)$. Thus we may assume that $x_{p+1}=2$ and similarly $y_{q+1}=2$. Furthermore we assume that $q=p+r$, where $r \geq 0$.

Step 5: Now we can express $\operatorname{diff}(\underline{x}, \underline{y})$ as a function of $p$ and $r$ in the following way:

$$
\begin{aligned}
\operatorname{diff}(\underline{x}, \underline{y})= & (\underline{x} \cdot \underline{1})(\underline{y} \cdot \underline{1})-5(\underline{x} \cdot \underline{y}) \\
= & (t p+2(7-p)+2)(\bar{t}(p+r)+2(7-p-r)+2) \\
& -5\left(t^{2} p+2 t r+4(7-p-r)+2\right)
\end{aligned}
$$

where the coefficient of $r$ is $p(t-2)^{2}+6(t-2)>0$, as $t \geq 4$. Therefore diff $(\underline{x}, \underline{y})$ is minimal if $r=0$, that is $p=q$, and so $\underline{x}=\underline{y}$. In this case $\operatorname{diff}(\underline{x}, \underline{x})=p^{2}\left(t^{2}-4 t+4\right)+p\left(-5 t^{2}+\right.$ $32 t-44)+106$, which has extremum if $\frac{\mathrm{d}}{\mathrm{d} p} \operatorname{diff}(\underline{x}, \underline{x})=0$ which gives $p=\frac{5 t^{2}-32 t+44}{2\left(t^{2}-4 t+4\right)}$. (This extremum is a minimum since $\frac{\mathrm{d}^{2}}{\mathrm{~d} p^{2}} \operatorname{diff}(\underline{x}, \underline{x})=2\left(t^{2}-4 t+4\right)=2(t-2)^{2}>0$, because $t \geq 4$.)

From the above formula we get $p=1.5$, for $t=8$, which gives that the minimum value of $\operatorname{diff}(\underline{x}, \underline{y})$ for any $\underline{x}, \underline{y}$ is at least $25>0$. (Actually the minimum is 34 which is taken on the integer values $p=\overline{1}$ and $p=2$.) Thus $|E(G)| \leq 5\left|E\left(G_{1}\right)\right|$ cannot hold for $t=8$, which completes the proof.

Proof of Theorem 2. Applying Lemmas 1 and 2, it follows that there is a block containing at least nine distinct vertices, say $x_{i} \in X_{1}\left[x_{1}\right]$, for every $i=1,2, \ldots, 9$. Similarly to the proof of Theorem 1, for a sequence of given colors $c_{1}, \ldots, c_{9}$, let

$$
Y\left(c_{1}, \ldots, c_{9}\right)=\left\{y \in Y \mid y x_{i} \text { is colored with } c_{i}, i=1, \ldots, 9\right\} .
$$

The nine-tuple $\left(c_{1}, \ldots, c_{9}\right)$ will be called the type of the subset $Y\left(c_{1}, \ldots, c_{9}\right) \subseteq Y$. In terms of this notation Lemmas 1 and 2 imply that $Y(1, \ldots, 1) \neq \emptyset$. Again, when the wildcard character $*$ is used for the $i$-th color position in a type, then the color of the corresponding edges to $x_{i}$ are undetermined.

In a bi-equivalence graph partition certain types cannot coexist as is expressed in the next rule.

Type rule. If $a, b$ are distinct colors, then at least one of the sets $Y(a, a, *, \ldots, *)$ and $Y(a, b, *, \ldots, *)$ must be empty.

Indeed, if $y_{1} \in Y(a, a, *, \ldots, *)$ and $y_{2} \in Y(a, b, *, \ldots, *)$, then $\left(y_{2}, x_{1}, y_{1}, x_{2}\right)$ is a path belonging to $G_{a}$, hence the edge $x_{2} y_{2}$ must have color $a$, and not $b$.

Notice that the Type rule remains valid when permuting colors and/or when relabelling the vertices $x_{1}, x_{2}, \ldots, x_{9}$, that is when the colors in the types are moved to different positions. Thus, for instance, types $(*, 5, *, \ldots, *, 3)$ and $(*, 3, *, \ldots, *, 3)$ cannot coexist.

We will need a simple corollary of the antichain property as follows:
Starring rule. If $Y_{c}[w] \subseteq Y\left(c_{1}, \ldots, c_{9}\right)$, for some $w \in X$, then equality must hold because $Y\left(c_{1}, \ldots, c_{9}\right) \subseteq Y\left(c_{1}, *, \ldots, *\right)=Y_{c_{1}}\left[x_{1}\right]$, in other words vertex $w$ "stars" the set $Y\left(c_{1}, \ldots, c_{9}\right)$ in color $c$.

In the sequel when we write "w.l.o.g. we assume", we mean: "by appropriately permuting the colors and relabelling $x_{1}, x_{2}, \ldots, x_{9}$ we may assume".

We shall proceed with investigating the partition of $Y^{\prime}=Y \backslash Y(1, \ldots, 1)$ into different types. Note that if $Y\left(c_{1}, \ldots, c_{9}\right) \subseteq Y^{\prime}$, then we have $c_{i} \neq 1$, for every $i=1, \ldots, 9$.

Distinguishing rule 1. If $Y(2,2, *, \ldots, *) \neq \emptyset$ and $Y(3,3, *, \ldots, *) \neq \emptyset$, then

$$
Y(4,4, *, \ldots, *) \cup Y(5,5, *, \ldots, *)=\emptyset
$$

furthermore,

$$
Y(4,5, *, \ldots, *) \neq \emptyset, \quad Y(5,4, *, \ldots, *) \neq \emptyset .
$$

To see this recall that no equivalent vertices exist in the coloring, in particular $x_{1}, x_{2}$ must be distinguished by the components in colors 4 and 5 . If $Y(4,4, *, \ldots, *) \neq \emptyset$, then by the Type rule, $B_{i}\left[x_{1}\right]=B_{i}\left[x_{2}\right]$ for every $i=1,2,3,4$, implying $B_{5}\left[x_{1}\right]=B_{5}\left[x_{2}\right]$, hence $x_{1}, x_{2}$ would be equivalent. An immediate corollary of Distinguishing rule 1 is stated for convenience as follows.

Distinguishing rule 2. At least one of $Y(2,2,2, *, \ldots, *)$ and $Y(3,3,3, *, \ldots, *)$ must be empty.

Returning to the proof let $Y\left(c_{1}, \ldots, c_{9}\right) \subseteq Y^{\prime}$. Since $c_{i} \in\{2,3,4,5\}$, some color must repeat at least three times. We shall consider the following three cases:

1) there is a (nonempty) type in $Y^{\prime}$ such that a color repeats more than four times;
2) no (nonempty) type in $Y^{\prime}$ repeats a color more than four times, and there is a (nonempty) type repeating a color four times;
3) no (nonempty) type in $Y^{\prime}$ repeats a color more than three times.

Case 1: there is a (nonempty) type in $Y^{\prime}$ such that a color repeats more than four times, say $Y(2,2,2,2,2, *, \ldots, *) \neq \emptyset$.

Observe that color 2 cannot repeat seven times. Indeed, in every (nonempty) type in $Y^{\prime}$ different from $(2,2,2,2,2,2,2, *, *)$ color 2 is not used on the first seven positions, by the Type rule. Hence one color among 3,4 , and 5 must repeat at least three times contradicting Distinguishing rule 2. Thus we may assume that $Y\left(2,2,2,2,2, *, c_{7}, *, *\right) \neq \emptyset$, where $c_{7} \neq 2$.

A similar pigeon hole argument shows that in every (nonempty) type in $Y^{\prime}$ different from $(2,2,2,2,2, *, *, *)$ each of the three colors $3,4,5$ must be used on the first five positions, otherwise Distinguishing rule 2 is violated. Thus w.l.o.g. we assume that $c_{7}=3$.

Observe that by the Type rule, $Y_{3}\left[x_{7}\right] \subseteq Y(2,2,2,2,2, *, \ldots, *)$, thus by the Starring rule, $Y_{3}\left[x_{7}\right]=Y(2,2,2,2,2, *, \ldots, *)$ follows. Then we obtain that

$$
Y^{\prime}=\left(\bigcup\left\{Y_{3}\left[x_{i}\right] \mid 1 \leq i \leq 5\right\}\right) \cup Y_{3}\left[x_{7}\right]
$$

If the six connected components $B_{3}\left[x_{i}\right], 1 \leq i \leq 5$ and $B_{3}\left[x_{7}\right]$ do not cover $X$, then there is an uncovered vertex $w \in X$ which stars $Y(1, \ldots, 1)$ in color 3, by the Starring rule. In this case $B_{3}\left[x_{i}\right], 1 \leq i \leq 5, B_{3}\left[x_{7}\right]$, and $B_{3}[w]$ cover $Y$ (thus the whole vertex set of $G$ ).

Consequently, in either case $G_{3}$ has width at most 7.
Case 2: no (nonempty) type in $Y^{\prime}$ repeats a color more than four times, and there is a (nonempty) type repeating a color four times, say $Y\left(2,2,2,2, c_{5}, \ldots, c_{9}\right) \neq \emptyset$, where $c_{5}, \ldots, c_{9} \neq 2$. We also know that among the five colors, $c_{5}, \ldots, c_{9}$, there are two distinct colors, w.l.o.g. we assume that $c_{5}=3$ and $c_{6}=4$.

Assume now that in every (nonempty) type in $Y^{\prime}$ different from $(2,2,2,2, *, \ldots, *)$ color 3 is used somewhere on the first four positions. Then a similar argument that we used in Case 1 shows that the width of $G_{3}$ is at most 6 . By the same reason repeated for color 4 , it
remains to consider the situation when, for each color 3 and 4, there is a (nonempty) type in $Y^{\prime}$ different from $(2,2,2,2, *, \ldots, *)$ missing 3 and 4 on the first four positions, respectively.

Since a color cannot repeat three times on the first four positions, we have that $Y(4,4,5,5, *, \ldots, *) \neq \emptyset$, moreover $Y(a, b, c, d, *, \ldots, *) \neq \emptyset$, where among $a, b, c, d$ both colors 3 and 5 repeat twice. By the Type rule, either $a=b=5, c=d=3$ or $c=d=5, a=$ $b=3$. In each case Distinguishing rule 1 is violated.

Case 3: no (nonempty) type in $Y^{\prime}$ repeats a color more than three times.
Then by the pigeon hole principle, each (nonempty) type in $Y^{\prime}$ has a color repeated three times. Furthermore, if a type uses just three colors, then each of its three colors is repeated exactly three times.

Let $Y(c, c, c, *, \ldots, *) \neq \emptyset$, for some $c=2,3,4$, or 5 . If each (nonempty) type uses color $c$ at some position, then either the connected components $B_{c}\left[x_{i}\right], 3 \leq i \leq 9$ cover $X$, or some $w \in X$ stars $Y(1, \ldots, 1)$ in color $c$, hence $B_{c}\left[x_{i}\right], 3 \leq i \leq 9$ and $B_{c}[w]$ cover $Y$ (thus the whole vertex set of $G$ ). In each situation $G_{c}$ has width at most 8 . We claim that this must happen for some $c$.

Assume that color 2 repeats three times in some (nonempty) type, and some other (nonempty) type misses color 2. W.l.o.g. let $T_{2}=(3,3,3,4,4,4,5,5,5)$ be a (nonempty) type. By repeating the same idea, we see that, for every $c=3,4,5$, some (nonempty) type $T_{c}$ misses $c$.

Thus $T_{3}$ has three triplets in colors $2,4,5$ at some positions. The last three positions of $T_{3}$ is not a triplet in 5 due to Distinguishing rule 2 and the Type rule. W.l.o.g. assume that $T_{3}=(5,5, *, 5, *, \ldots, *)$. Then again, by Distinguishing rule 2 and the Type rule, it follows that $T_{3}=(5,5,4,5,2,2,4,4,2)$.

Finally, for the possible positions of the three 5's of $T_{4}$ with respect to $T_{2}$ and $T_{3}$, we conclude as before that $T_{4}=(*, *, 5, *, 5,5, *, *, *)$. This contradicts Distinguishing rule 1 on positions 5 and 6 and completes the proof of Theorem 2 .

## 5 Homogeneous covering

In 1998 Guantao Chen asked whether a stronger version of Conjecture 4 can be true, i.e. whether $2 r-2$ biclique components of the same bi-equivalence graph $G_{i}, 1 \leq i \leq r$, can cover $[X, Y]$. Call such cover a homogeneous cover. Although this is not true in general (see Theorem 5 below), the question introduces interesting variants of the cover problem.

Given $r$, let $g(r)$ be the smallest $m$ such that in every biclique $B$ with a spanning partition into $r$ bi-equivalence graphs $G_{1}, \ldots, G_{r}$, there is a partition class $G_{i}$ with width at most $m$. We shall prove that $g(r)$ exists, in a stronger form: for every $r$, there is a smallest $m=h(r)$ such that in every spanning partition of a biclique into $r$ bi-equivalence graphs, the width of every partition class is at most $m$.

Theorem 3. $h(r)=2^{r-1}$.

Proof. To see that $h(r) \geq 2^{r-1}$ consider the following easy recursive construction to partition a biclique into $r$ bi-equivalence graphs such that the maximum width is $2^{r-1}$. The case $r=1$ is obvious. Given such a spanning partition of $B=K_{n, n}$ into $r$ bi-equivalence classes, take two vertex disjoint copies of $B$ and place two bicliques crosswise as the $r+1$-th partition. This way a spanning partition of $K_{2 n, 2 n}$ is obtained into $r+1$ bi-equivalence graphs and the width of every partition class is doubled - apart from the $(r+1)$-th class which has width two.

To prove the other direction, $h(r) \leq 2^{r-1}$, we need some definitions. An equivalence graph is a graph whose components are complete graphs. Let eq(G) denote the minimum number of spanning equivalence graphs needed to cover the edge set of a graph $G$. Similarly, for any bipartite graph $G$, let eqbi( $G$ ) denote the minimum number of spanning bi-equivalence graphs needed to cover the edges of $G$. Let $G^{+}$denote the graph obtained from the bipartite graph $G$ by adding to $E(G)$ all pairs inside the partite classes of $G$. Let $K_{t, t}^{-}=K_{t, t}-t K_{2}$, i.e. $K_{t, t}^{-}$is a balanced biclique from which a perfect matching is removed. We need the next two straightforward propositions.

Proposition 6. For any bipartite graph $G$, eqbi $(G) \geq e q\left(G^{+}\right)-1$.
Proof. Consider an optimal cover of $E(G)$ with eqbi $(G)$ spanning bi-equivalence graphs and turn them into spanning equivalence graphs by adding all missing edges to all biclique components. These plus one more spanning equivalence graph formed by the two vertex classes of $G$ cover all edges of $G^{+}$thus eqbi( $\left.G\right)+1$ is an upper bound of $e q\left(G^{+}\right)$.

Proposition 7. If $B$ is a biclique and $G=B-E(H)$, where $H$ is a spanning bi-equivalence subgraph of $B$ with $t \geq 2$ components, then eqbi $(G)=\operatorname{eqbi}\left(K_{t, t}^{-}\right)$.

Proof. Suppose $X_{i}, Y_{i}$ are the bicliques of $H, i=1,2, \ldots, t$ and $x_{i} y_{i}$ are the removed edges of $K_{t, t}$.

If $\left\{H_{l}: 1 \leq l \leq s\right\}$ is a spanning partition of $K_{t, t}^{-}$into bi-equivalence graphs, define $G_{l}$ by adding all edges of all bipartite graphs $\left[X_{i}, Y_{j}\right]$ whenever $x_{i} y_{j}$ is an edge of biclique of $H_{l}$. This defines $\left\{G_{l}: 1 \leq l \leq s\right\}$ as a spanning partition of $G$ into bi-equivalence graphs showing that $e q b i(G) \leq s$.

To see the reverse inequality, consider an arbitrary cover of $G$ by spanning bi-equivalence graphs $G_{1}, \ldots, G_{k}$. Let $T$ be the subset of $2 t$ vertices of $V(G)$ containing one vertex from each partite class of each bipartite component of $H$. For any $1 \leq l \leq k$, define $H_{l}$ as the induced subgraph of $G_{l}$ on $T$. Then $\left\{H_{l}: 1 \leq l \leq k\right\}$ is a spanning partition of $K_{t, t}^{-}$into bi-equivalence graphs showing that $k \geq e q b i\left(K_{t, t}^{-}\right)$.

The main tool is the following result of Alon [1].
Theorem 4. ([1]) Suppose that the maximum degree of the complement of a graph $G$ is $d$ and $|V(G)|=n$. Then $e q(G) \geq \log _{2} n-\log _{2} d$.

Suppose indirectly that $B$ is a biclique with a spanning partition into bi-equivalence graphs $G_{1}, \ldots, G_{r}$ such that some of them, say $G_{1}$ has width $t>2^{r-1}$. Let $G=B-G_{1}$. Using Propositions 6, 7 and Theorem 4, we obtain that

$$
\operatorname{eqbi}(G)=e q b i\left(K_{t, t}^{-}\right) \geq e q\left(\left(K_{t, t}^{-}\right)^{+}\right)-1 \geq \log _{2}(2 t)-\log _{2} 1-1>\log _{2}\left(2^{r}\right)-1=r-1
$$

which is a contradiction since the $r-1$ bi-equivalence graphs $G_{2}, \ldots, G_{r}$ partition $G=B-G_{1}$. Consequently $t \leq 2^{r-1}$, and $h(r)=2^{r-1}$ follows. This concludes the proof of Theorem 3 .

The following construction gives a lower bound for $g(r)$.
Theorem 5. There are spanning r-partitions of bicliques such that the width of every partition class is $\Omega\left(r^{3 / 2}\right)$.
Proof. Let $s \geq 3$ be an integer, set $r=(s-2) s$ and $p=\binom{s-1}{2}$. We shall construct a spanning $r$-partition of the biclique $K_{s p, s p}$ into bi-equivalence graphs such that each class will be the disjoint union of one copy of the biclique $K_{p, p}$ and $s-1$ copies of the matching $p K_{2}$. Notice that each of those $r$ classes has width $p(s-1)+1 \geq c r^{3 / 2}$, with constant $c$.

The construction is as follows. Let us color the edges of a Hamiltonian cycle of $K_{s, s}$ red, and all the other edges of $K_{s, s}$ blue. Each of the $s^{2}-2 s=r$ blue edges can be uniquely extended with $s-1$ red edges into a 1 -factor of $K_{s, s}$. Therefore, each red edge belongs to the same number, $r(s-1) / 2 s=p$ such 1-factors. Now we replace each vertex by a set of $p$ elements, every blue edge with a copy of $K_{p, p}$, and every red edge with $p$ pairwise disjoint copies of $p K_{2}$.

What we know about the functions $g$ and $h$ is $c r^{3 / 2} \leq g(r) \leq h(r)=2^{r-1}$, and it is a challenging question how they separate.

Although there are no homogeneous covers with $2 r-2$ bicliques in general for spanning partitions, they might exist for antichain partitions, in fact we proved this in Sections 3 and 4 for $r \leq 5$.

Question 2. Suppose that a biclique has an antichain partition into $r$ bi-equivalence graphs. Is it true that some of them has width at most $2 r-2$ ?

## 6 The dual form, transversals of $r$-partite intersecting hypergraphs

Conjectures 1 and 4 can be translated into dual forms as conjectures about transversals of $r$ partite $r$-uniform intersecting hypergraphs. To do that, one should consider the $r$ partitions defined by the monochromatic connected components of an $r$-colored complete or complete bipartite graph as hyperedges over the vertex set and consider the dual of this hypergraph. This approach already turned out to be very useful, for example results of Füredi established in [4] can be applied. A survey on the subject is [6].

An $r$-uniform hypergraph $H$ is defined by a finite set $V(H)$ called the vertex set of $H$, and by a set $E(H)$ of r-sets of $V(H)$ called edges of $H$. An $r$-uniform hypergraph $H$ is called $r$-partite if there is a partition $V(H)=V_{1} \cup \ldots, \cup V_{r}$ such that $\left|e \cap V_{i}\right|=1$, for all $i=1, \ldots, r$ and $e \in E(H)$. A hypergraph $H$ is called intersecting if $e \cap f \neq \emptyset$ for any $e, f \in E(H)$. A set $T \subseteq V(H)$ is called a transversal of $H$ provided $e \cap T \neq \emptyset$, for all $e \in E(H)$; the minimum cardinality of a transversal of $H$ is the transversal number of $H$ denoted by $\tau(H)$.

The dual of Conjecture 1 is Ryser's conjecture for intersecting hypergraphs in its usual form as follows:

Conjecture 6. If $\mathcal{H}$ is an intersecting $r$-partite hypergraph then $\tau(\mathcal{H}) \leq r-1$.
There are infinitely many examples of intersecting $r$-partite hypergraphs with transversal number equal to $r-1$. Take a finite projective plane of order $q$, then truncate it by removing one point and the incident $q+1$ lines. The remaining lines taken as edges define an intersecting $(q+1)$-partite hypergraph with transversal number equal to $q$. (Note that the truncated projective plane is the dual of an affine plane.) A related question, finding $f(r)$, the minimum number of edges among intersecting $r$-partite hypergraphs with transversal number at least $r-1$, was addressed in $[10]$, where it was shown that $f(3)=3, f(4)=6$, and $f(5)=9$.

Concerning our biclique cover conjectures, the dual of a spanning partition of a complete bipartite graph into $r$ bi-equivalence graphs gives two $r$-partite hypergraphs, $\mathcal{H}_{1}, \mathcal{H}_{2}$ on the same vertex set such that for every $h_{1} \in E\left(\mathcal{H}_{1}\right), h_{2} \in E\left(\mathcal{H}_{2}\right),\left|h_{1} \cap h_{2}\right|=1$ holds, moreover at each vertex there is at least one edge from both hypergraphs. We call such hypergraph pairs 1 -cross intersecting. Then Conjecture 4 reads as follows:

Conjecture 7. Let $\mathcal{H}_{1}, \mathcal{H}_{2}$ be a pair of 1-cross intersecting r-partite hypergraphs. Then $\tau\left(\mathcal{H}_{1} \cup \mathcal{H}_{2}\right) \leq 2 r-2$.

To illustrate the advantage of the dual formulation, here is a quick proof showing that $h(r)$ is bounded (although with a bound weaker than the one in Theorem 3).
Proposition 8. Let $\mathcal{H}_{1}, \mathcal{H}_{2}$ be a pair of 1-cross intersecting r-partite hypergraphs. Then each partite class contains at most $\binom{2(r-1)}{r-1}$ vertices.

Proof. Let $v_{1}, \ldots, v_{p}$ be the vertices of a partite class of $\mathcal{H}_{1}, \mathcal{H}_{2}$. For each $v_{i}$ select $f_{i}^{1} \in$ $E\left(\mathcal{H}_{1}\right), f_{i}^{2} \in E\left(\mathcal{H}_{2}\right)$ such that $v_{i} \in f_{i}^{1} \cap f_{i}^{2}$, and set $g_{i}=f_{i}^{1} \backslash\left\{v_{i}\right\}, h_{i}=f_{i}^{2} \backslash\left\{v_{i}\right\}$. Then the pairs ( $g_{i}, h_{i}$ ) form a cross-intersecting $r$-1-uniform family (in fact a very special one). It is well known (see Exercise 13.32 in [9]) that such hypergraphs have at most $\binom{2(r-1)}{r-1}$ edges.

Notice that 1-cross intersecting $r$-partite hypergraph pairs with $\mathcal{H}_{1}=\mathcal{H}_{2}$ form a test case for Conjecture 6. It was conjectured by Lehel ([8], Problem 3) that in this case Ryser's conjecture is true in a stronger form (and the bound of Proposition 8 is essentially better):
Conjecture 8. Suppose that an intersecting r-partite hypergraph $\mathcal{H}$ has no isolated vertices and its edges pairwise intersect in precisely one vertex. Then some partite class of $\mathcal{H}$ contains at most $r-1$ elements, in particular $\tau(\mathcal{H}) \leq r-1$.

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