# Expected Lifetime and Capacity

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**Abstract** We investigate sharp isoperimetric problems for random walks on weighted graphs. Symmetric weights on edges determine the one step transition probabilities for the random walk, measure of sets and capacity between sets. In that setup one can be interested in the exit time of the random walk from a set, i.e. to find for a fixed starting point the "optimal" set of given volume which maximizes the expected time when the walk leaves the set. A strongly related problem is to find a set of fixed volume which has minimal conductance with respect to a given set. In both problems the answer is less appealing than in the case of Euclidean space. As demonstrated by a simple counterexample, there is no unique optimal set. The Berman-Konsowa principle is used in the search for optimal sets. It allows to construct a new graph on which the calculation of conductance and mean exit time is tractable.

**Keywords** Isoperimetric inequality • Random walks • Berman-Konsowa principle

## 1 Introduction

Isoperimetric problems have a long and shining history in mathematics as well as in human culture. Pappus credited to Zenodorus the first statement of the two dimensional isoperimetric problem. Several other isoperimetric problems were formulated in the course of time. One can find a classical introduction in Polya's and Szegő's book [9] and further references in the nice survey of Caroll [5]. We are

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not going to present a review here. To find the optimal set for the maximal expected lifetime of a planar Brownian motion in a finite closed, connected domain of fixed area is a naturally arising similar problem (cf. [1, 5] and their references). In the light of recent developments in the study of diffusion processes in measure metric Dirichlet spaces (cf. [3, 7]) it is natural to rise the same question on such spaces.

Let us imagine that we have a sheep, a piece of grassland, and an electric fence of a given length. The sheep starts at a given point at time zero and performs a diffusion according to a fixed measure and a local, regular, Markovian Dirichlet form. We want to enclose the sheep with the fence in such a way that the sheep is hit by electricity as late as possible, in expectation. This scenario inspired Erin Pearse to coin the name "Brownian sheep" at the Cornell Conference on Analysis and Probability on Fractals in 2005.

In the present paper we make a very first step towards the solution of the Brownian sheep problem. We consider a discrete space-time counterpart of the problem, given by random walks on weighted graphs (for general introduction and background c.f. [6, 10]).

We provide a characterization of the optimal solution for:

1. The minimal capacity problem: given two sets $\Gamma$ and $F \subset \Gamma$ and a constant $M$, find a set $D \subset F$ with volume not larger than $M$ such that the capacity between $D$ and $\Gamma \backslash F$ is minimal.
2. The maximal lifetime problem: given a starting point of the walk and a constant $M$, find a set $F$ of volume not larger than $M$ such that the expected exit time of the walk is maximal.

The key tool for us is the Berman-Konsowa (B-K in the sequel) principle [4] (see also [8] for a nice interpretation), by which the problem can be reduced to star graphs.

The paper is organized as follows. In Sect. 2 we introduce basic notation and facts. In Sect. 3 we present the Berman-Konsowa principle, in Sect. 4 we discuss the problem of capacity and in Sect. 5 the problem of Brownian Sheep. Some technical details are collected in an Appendix.

## 2 Foundations

Let $(\Gamma, Q_\Gamma, \mu)$ be a connected weighted undirected graph with vertex set $\Gamma$, edge set $Q_\Gamma$ and a symmetric weight on edges $\mu_{x,y} = \mu_{y,x}$. The corresponding resistance is $R_{x,y} = 1/\mu_{x,y}$. For $x \in \Gamma$ let $\mu(x) = \sum_y \mu_{x,y}$.

For sake of simplicity we will solve the problem on the *cable system* of the graph, i.e., all edges are considered as copies of the unit interval [2]. For an edge $(x, y)$ and $\alpha \in [0, 1]$ let $(\alpha, x, y)$ denote the point which splits the edge into $\alpha, 1-\alpha$ parts. We write $w_0 = (0, x, y)$ for $x$, $w_1 = (1, x, y)$ for $y$, and $w_\alpha = (\alpha, x, y)$ for points on the edge. Resistance and weight are proportional to the length of a subinterval:

$$R_{w_0,w_\alpha} = \alpha R_{x,y} \text{ and } \quad \mu_{w_0,w_\alpha} = \alpha \mu_{x,y}.$$

The basic space for our study is the set of all points of the unit intervals representing edges. It is denoted by $W$. We consider subsets $A \subset W$ which are unions of subintervals where adjacent endpoints are identified. We assume that such a set $A$ is convex in the following sense: if $w_{\bar{\alpha}} = (\bar{\alpha}, x, y) \in A$ then at least one of the vertices $x$ and $y$ is in $A$, and if lets say $x \in A$, then $w_\alpha = (\alpha, x, y) \in A$ for all $\alpha \in [0, \bar{\alpha}]$ as well.

In the sequel the investigated sets $A \subset W$ are assumed to be open and precompact. Let $\overline{A}$ denote the closure of the set and $\partial A = \overline{A} \backslash A$ the boundary of $A$. The boundary of a set is a discrete set of points on intervals. The set of edges crossing $\partial A$ will be denoted by $cA$.

The weights on edges define a measure $d\mu(\alpha, x, y) = \mu_{x,y} d\alpha$, with

$$\mu(A) = \sum_{x,y \in A \cap \Gamma} \mu_{x,y} + \sum_{\substack{(\alpha,x,y) \in \partial A \\ x \in A, y \notin A}} \alpha \mu_{x,y}.$$

We consider the usual random walk $X_n \in \Gamma$ on $(\Gamma, \mu)$ defined by the transition probability $P(x, y) = \mu_{x,y}/\mu(x)$. We assume that there is a $p_0 > 0$ such that for all $(x, y) \in Q_\Gamma$

$$P(x, y) \geq p_0. \tag{2.1}$$

As a consequence $\deg(x) \leq 1/p_0$ for all $x \in \Gamma$, i.e. the graph has bounded degree. $\Gamma$ can be infinite, however.

Now we define the killed random walk for a set $A$ which contains a finite number of vertices. We assign to $A$ a corresponding graph with vertex set $\Gamma_A = \Gamma \cap A \cup \partial A$ and the induced edges. On this graph we have a random walk which we will start at an interior vertex and kill at the first boundary vertex. The transition probabilities $P^A(x, y)$ are equal to $P(x, y)$ for $x, y \in \Gamma \cap A$. If $x \in \Gamma \cap A$ is adjacent with one boundary point $w_\alpha = (\alpha, x, y) \in \partial A$ then the interval $(x, y)$ is splitted into two parts and the transition probability modified accordingly:

$$P^A(x, w_\alpha) = \frac{\frac{1}{\alpha}\mu_{x,y}}{\sum_{z \neq y} \mu_{x,z} + \frac{1}{\alpha}\mu_{x,y}},$$

In other words new points are introduced as edge splitting points on the boundary of $A$ and the walk is defined inside $A$ as usual, choosing a neighbor proportional to the conductance. On vertices next to the boundary the walk tends to choose short edges with small $\alpha$, which get bigger weights by $1/\alpha$. The walk is killed as soon as it reaches a boundary point. The exit time of the random walk is

$$T_A = \min\{n : X_n \in \Gamma \backslash A\},$$

and the mean exit time for the walk starts in $x \in \Gamma$ is defined as

$$E_x(A) = \mathbb{E}^A(T_A | X_0 = x),$$

where $\mathbb{E}^A$ is the expected value with respect to the probability measure $P^A$ induced by the random walk $X_n$ starting at $X_0 = x \in \Gamma$ and killed when it leaves $A$.

*Remark 1* The notions of weight, capacity and resistance need a bit of explanation. Capacity is the reciprocal of resistance, shorter subintervals have smaller resistance and bigger capacity, while the weight assigned to the subinterval is proportional to its length. In that sense weight and capacity are not the same on subintervals while they numerically coincide on full intervals.

One can assume that the resistance is not uniform along the edges but there is a resistivity $\rho(s)$ along it and

$$R_{w_0,w_\alpha} = \int_0^\alpha \rho(s)\, ds.$$

This extension is not discussed here, but seems tractable and the whole machinery can be generalized to it without essential change.

**Problem 1 (Maximal exit time)** Let $x \in \Gamma$ and $M > 0$ be given. Find a set $F \ni x$, $F \subset W$ with volume $\mu(F) \leq M$ and maximal expected exit time $E_x(F)$.

**Problem 2 (Minimal capacity)** Let $F \subset W$ be a fixed set and $M > 0$ be given. Let $Cap(D, F)$ denote the capacity of the 'annulus' $F\backslash D$ for $D \subset F$, more precisely

$$Cap(D, F) = \inf_{f \in H} \sum_{w,w' \in F \cup \partial F \cup \partial D} \left(f(w) - f(w')\right)^2 \mu_{w,w'},$$

where $H = H(D, F)$ is the set of functions $f : W \to \mathbb{R}$, $f|_{\overline{D}} \geq 1$ and $f|_{\overline{\Gamma\backslash F}} = 0$. Here again the boundary crossing edges are splitted and only the parts in $\overline{F\backslash D}$ is considered. We seek for a set $D$ such that $D \subset F$, $\mu(D) \leq M$ and the capacity $Cap(D, F)$ is minimal.

## 3 The Berman-Konsowa Principle

The other model that we will use is the *path system* of the graph. Consider a pair of sets $(D, F)$, where $D \subset F$. Denote $L = L_{D,F}$ the set of all finite paths connecting $\partial_i D$ and $\partial_o F$ cropped at the boundary of the sets. Denote the ends of a path $l \in L_{D,F}$ by $d_l$ and $z_l$, respectively. The path-graph on $F\backslash\partial_i D$ will be defined between $y$'s and $\Gamma\backslash F$ and completed with common, unsplitted edges $(d, y)$ reaching $\partial D$, see Fig. 1. (If $y \in \partial_0 D$ but $y \notin F$ we consider the single edge $(d, y)$ as a path.)

We introduce $\mathcal{P}_{D,F}$ as the set of all probability measures on $L_{D,F}$, and let $Q_{D,F}$ be the edge set induced on $\overline{F}\backslash D$ by the original graph.

**Definition 3.1** A flow between $D$ and $\Gamma\backslash F$ is a function on $Q_{D,F}$. A flow function $\Phi$ is nonnegative and satisfies the following rules.

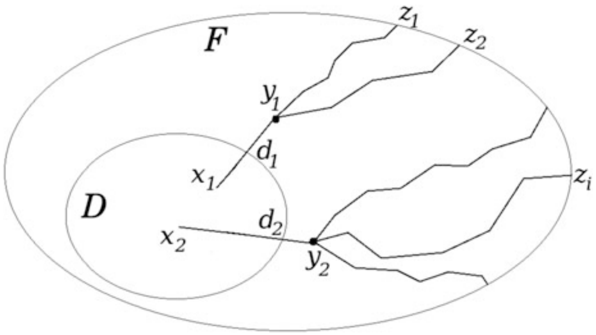


**Fig. 1** Paths starting inside $F \setminus D$

1. $\Phi(x,y)\,\Phi(y,x) = 0 \quad \forall (x,y) \in Q_{D,F}$, i.e. the flow is one-directional,
2. for $x \in F \backslash D$

$$\sum_{y:(x,y)\in Q_{D,F}} \Phi(x,y) = 0,$$

3.

$$\sum_{d\in D, y:(d,y)\in Q_{D,F}} \Phi(d,y) = \sum_{z\in\partial F, y:(y,z)\in Q_{D,F}} \Phi(y,z)\,, \tag{3.1}$$

4. $\Phi(x,d) = \Phi(z,y) = 0$ for all $x, y \in F\backslash D, d \in D, z \in \partial F$. In addition we say that $\Phi$ is a unit flow if $\sum_{d\in D, y:(d,y)\in Q_{D,F}} \Phi(d,y) = 1$.

We define a new network $(\Gamma_L, Q_L, \mu_L)$ based on the path system $L = L_{D,F}$. That will be the set of paths connecting $D$ and $\Gamma\backslash F$ with vertex and edge replicas of the original graph, to ensure that the path have no common vertices except at their endpoints. The objects of the new graph will be labeled by $l \in L$. Each $l \in L$ is a sequence of edges. We redefine the vertex set. For each $x \in F\backslash D$ let $x_l$ be a vertex in $\Gamma_L$ if $x \in l \cap (F\backslash D)$, formally: $\Gamma_L = \{x_l : x \in F\backslash D \cap l \text{ and } l \in L\}$. Edges are kept along the paths. We associate a new resistance $R^l_{x,y}$ to each edge on $l$ with respect to a probability measure $\mathbb{P} \in \mathcal{P}_{D,F}$. If $(x,y) \in Q_{D,F}$ the flow can be decomposed into separate flows along disjoint paths

$$\Phi_{\mathbb{P}}(x,y) = \sum_{l':l'\ni(x,y)} \mathbb{P}\left(l'\right)$$

$$\mu^l_{x,y} = \mu_{x,y}\frac{\mathbb{P}(l)}{\Phi_{\mathbb{P}}(x,y)}$$

$$R^l_{x,y} = \left(\mu^l_{x,y}\right)^{-1} = R_{x,y}\frac{\Phi_{\mathbb{P}}(x,y)}{\mathbb{P}(l)}. \tag{3.2}$$

The path $l$ has resistance $r_l = \sum_{(x,y)\in l} R^l_{x,y}$ and its conductance is $\mathrm{Cap}^{\mathbb{P}}(l) = 1/r_l$. Finally the capacity or conductance determined by $\mathbb{P}$ between $D$ and $\partial F$ is

$$\mathrm{Cap}^{\mathbb{P}}(D,F) = \sum_{l\in L_{D,F}} \mathrm{Cap}^{\mathbb{P}}(l).$$

*Remark 2* Let us observe that the edge weights are shared between the paths, it is contained proportional to the probability measure. For each edge

$$\mu^l_{x,y} = \mu_{x,y}\frac{\mathbb{P}(l)}{\sum_{l':l'\ni(x,y)}\mathbb{P}(l')},$$

and consequently

$$\sum_{l:l\ni(x,y)} \mu^l_{x,y} = \mu_{x,y}.$$

**Theorem 3.2 (Berman-Konsowa principle)**

$$\mathrm{Cap}(D,F) = \max_{\mathbb{P}\in\mathcal{P}_{D,F}} \mathrm{Cap}^{\mathbb{P}}(D,F).$$

In what follows this nice path system will play a particular role. Let us mention that the capacity potential defines an important unit flow which minimizes the energy dissipation of the network. Let $\tau_C = \min\{k : X_k \in C\}$ be the hitting time of the set $C$ and $v(y) = P\left(\tau_D < \tau_{\Gamma\setminus F} | X_0 = y\right)$. The natural flow generated by the properly adjusted external source is

$$\Phi(x,y) = (v(x) - v(y))_+ \mu_{x,y}, \tag{3.3}$$

where $a_+ = \max\{a, 0\}$.

## 4 Sets with Minimal Capacity

Let $D$ be an optimal solution of Problem 2. Then we may assume that for all $w \in \partial D$, $w = (\alpha, x, y)$ with $\alpha \in (0,1)$, i.e., the boundary points of $D$ are internal points of edges. We can assume even more, that there is a small $\varepsilon > 0$ such that

$$\alpha \in (\varepsilon, 1-\varepsilon).$$

If it is not the case, given that $\partial D$ is finite, with an arbitrary small change of the volume $M$ that can be ensured.

Consider the Berman-Konsowa path system $L_{\partial_o D,F}$ and let us extend each path $l$ which connect $y_l \in \partial_o D$ to $F$ with the edge segment $(d_l, y_l)$, where $d_l \in \partial D$ and $d_l = (\alpha_l, x_l, y_l)$ for some $0 \le \alpha_l \le 1$. Then the resistance from $d_l$ can be calculated as follows. We have

$$\mathrm{Cap}_L\,(y_l,\partial F)=\sum_{\tilde{l}:y_l\in\tilde{l}}\mathrm{Cap}\left(y_l,f_{\tilde{l}}\right),$$

where $f_l\in\partial F\cap l$ and $R_L\,(y_l,\partial F)=1/\mathrm{Cap}_L\,(y_l,\partial F)$

$$R_L\,(d_l,\partial F)=R_{d_l,y_l}+R_L\,(y_l,\partial F)\,.$$

Finally, $\mathrm{Cap}_L\,(d_l,\partial F)=1/R_L\,(d_l,\partial F)$ and we have

$$\mathrm{Cap}\,(D,F)=\sum_{d\in\partial D}\mathrm{Cap}_L\,(d,\partial F)\,.$$

Let us recognize, that the path system we have used here is smaller than the path system in the original B-K construction, since the border crossing edges are not split. For that reason we will refer to this construction as reduced B-K path system.

In order to investigate the optimal set of Problem 2 we use the Lagrange method and consider small perturbations of the optimal set. Let us consider a function $\xi:cD\to(0,1)$ which defines the boundary of the set $D_\xi$ with $w=(\xi\,(x,y)\,,x,y)\in\partial D_\xi$.

We consider the reduced B-K path system over $(D,F)$ and fix the resistances $R^l_{x,y}$ defined in (3.2) by the capacity potential and optimal flow (3.3) . We reserve $\mathbb{P}$ for the optimal distribution and $\tilde{\mathbb{P}}$ will denote an arbitrary other one on the fixed set of paths $L$. We shall consider in many cases a fixed set of paths $L$ with different weights, in that case the resistances, conductances on the path system with respect to the probability $\mathbb{P},\tilde{\mathbb{P}}$ will be denoted by $R,\mathrm{Cap}$, and $\tilde{R}=\tilde{R}_L=R^{\tilde{\mathbb{P}}}_L$, $\widetilde{\mathrm{Cap}}=\widetilde{\mathrm{Cap}}_L=\mathrm{Cap}^{\tilde{\mathbb{P}}}_L$, respectively. We shall drop the sub and superscripts if it does not cause ambiguity.

*Remark 3* The Berman-Konsowa principle says that for any set $D\subset F$ and any weight system $\tilde{\mathbb{P}}$ with the corresponding $\widetilde{\mathrm{Cap}}_L$

$$\mathrm{Cap}\,(D,F)=\max_{\mathbb{P}}\mathrm{Cap}^{\mathbb{P}}_L\,(D,F)\geq\widetilde{\mathrm{Cap}}_L\,(D,F)\,.$$

In particular if $D$ is optimal, and $\tilde{D}$ is another set in $F$ then

$$\mathrm{Cap}\left(\tilde{D},F\right)\geq\mathrm{Cap}\,(D,F)\geq\widetilde{\mathrm{Cap}}_L\,(D,F)\,.$$

**Lemma 4.1** *If $D\subset F,\mu\,(D)\leq M$ minimizes the capacity on the path system $L_{D,F}$ (with weights defined by the optimal $\mathbb{P}$) then $D$ is optimal for Problem 2.*

*Proof* Let $\tilde{D}\subset F,\mu\left(\tilde{D}\right)\leq M$ be another set and $\tilde{L}$ be the path system defined by $(\tilde{D},F)$. Then from

$$\mathrm{Cap}^{\mathbb{P}}_L\left(\tilde{D},F\right)\geq\mathrm{Cap}^{\mathbb{P}}_L\,(D,F)$$

and from the capacity definition and Remark 3, we have the statement:

$$\mathrm{Cap}\left(\tilde{D},F\right)=\widetilde{\mathrm{Cap}}_{\tilde{L}}\left(\tilde{D},F\right)\geq \mathrm{Cap}_{L}^{\mathbb{P}}\left(\tilde{D},F\right)\geq \mathrm{Cap}_{L}^{\mathbb{P}}\left(D,F\right)=\mathrm{Cap}\left(D,F\right).$$

□

For each path $l\in L$, we introduce the resistance $r_l$ and the weight $\mu_l$ of the whole path:

$$r_l=\sum_{(z,v)\in l}R_{z,v}^{l}\quad\text{and}\quad \mu_l=\sum_{(z,v)\in l}\mu_{z,v}^{l},$$

where $(z,v)\in l$ are the edges of path $l$.

The proof of the following statement is given in the Appendix.

**Proposition 4.2** *If $D$ is optimal, then in the path system for each $l\in L_{D,F}$*

$$\mu^{l}\left(x_l,y_l\right)r_l=\text{const},\tag{4.1}$$

*where $(x_l,y_l)\in cD$ denotes the crossing edge of $l$.*

*Remark 4*

1. The simplest case is to fix $x\in\Gamma$ and look for a set $D$ with $x\in D\subset F$, $\mu(D)\leq M$ which minimizes $\mathrm{Cap}(x,D)$.
2. The following example shows that an optimal set $D$ need not be unique. Consider two copies of $L_i=\{0_i,1_i,2_i,3_i,4_i,5_i\}$, $i=1,2$ with edges between direct neighbours and join them by setting $0_1=0_2$. We switch to continuous setup. Let

$$m_i(s)=\begin{cases}2\text{ if }s\in[0,1]\\4\text{ if }s\in(1,2]\\3\text{ if }s\in(2,5]\end{cases}$$

be the mass density along $L_i$. Denote the mass and resistance of the ray from 0 to a point $x\in[0,5]$ by $m$ and $r$. Then

$$m(x)=\begin{cases}2x & \text{if }0\leq x<1\\ 2+4(x-1) & \text{if }1<x\leq 2\\ 6+3(x-2) & \text{if }2<x\leq 5\end{cases}$$

$$r(x)=\begin{cases}\frac{x}{2} & \text{if }0\leq x<1\\ 1/2+\frac{1}{4}(x-1) & \text{if }1<x\leq 2\\ \frac{3}{4}+\frac{1}{3}(x-2) & \text{if }2<x\leq 5\end{cases}.$$

For the calculation of capacity we pick a point $x$ on the first ray for which $m(x)$ is the total mass from 0 to it. We allocate the rest of the mass to the point $y$ on the second ray, that is $m(y) = 12 - m(x)$ if $M = 12$. We define the inverse function $p$ of the function $m$:

$$p(t) = \begin{cases} t/2 & \text{if } 0 \le t < 2 \\ \frac{1}{4}(t+2) & \text{if } 2 < t \le 6 \\ t/3 & \text{if } 6 < t \le 15 \end{cases}.$$

The capacity is $\mathrm{Cap}(x, D) = g(t) = 1/r(p(t)) + 1/r(p(12-t))$ expressed in the mass $t$ used on the first ray, namely $t = m(x)$. It is easy to see, that this function has two global minimal solutions.

3. This example shows that no unique optimal solution can be guaranteed. The last resort is provided by the observation made in Lemma 4.1. If a set is optimal on its own B-K path system, it can be found by the Lagrange method, and it is optimal with respect to the original problem. Also let us recall Remark 3 which helps to sort out non-optimal sets. Since if we find a weight system $\mathbb{P}'$ over $L$ on which the candidate set $D^*$ is not optimal then it can not be optimal.

## 5 The Exit Time

We are going to find optimal sets which maximize $E_x(F)$ with $x \in F$, where $x \in \Gamma$ and $\mu(F) \le M$ fixed. In order to find an optimal set we try to maximize simultaneously $R(D_m, F)$ where $D_m$ is the level set of the Green function $G^F(x, y)$ of volume $m$, $0 \le m \le M$. We defer the statement of the result to the end of this section to avoid repetition of technical notation. As in Sect. 4 we assume that the boundary points of the optimal set are inside the edges, i.e., $\varepsilon$-separated from the endpoints.

From now on we work on the reduced path system and weights are defined by the optimal flow. The path system is $\{l_i\}_{i=1}^n$, denote $z_i = l_i \cap \partial F$. Let

$$e_{l_i} = E_i(z_i) = \mathbb{E}(T_{z_i} | Z_0 = x)$$

the exit time on the path $l_i$ of the random walk $Z_n$ on $l_i$ determined by the weights on $l_i$. Denote $m_i = m_{l_i}$ the volume of the path $l_i$ and $R_i = R_{l_i}$ the resistance of it.

**Definition 5.1** The local Green function (Green kernel) $G^F$ $\left(g^F(x, y)\right)$ is defined by the transition probabilities $P_n^F(x, y)$ of the random walk, killed on exiting the set $F \subset \Gamma$ is the following:

$$G^F(x, y) = \sum_{n=1}^{\infty} P_n^F(x, y),$$

$$g^F(x,y) = G^F(x,y)/\mu(y).$$

In the following we summarize some known facts about the Green function and the exit time of random walks (for more details see [11]). It is know that

$$E_x(F) = \sum_{y \in F} g^F(x,y)\,\mu(y).$$

Furthermore, on the graph and on the cable system for any $w \in F$

$$\begin{aligned} G^F(x,w) &= g^F(x,w)\,d\mu(w), \\ E_x(F) &= \int_F g^F(x,w)\,d\mu(w), \end{aligned} \tag{5.1}$$

where

$$g^F(x,w) = R(H_w, \partial F),$$

where $H_w = \{v : g^F(x,v) > g^F(x,w)\}$ is the super-level set with boundary of the equipotential surface $B_w$. (Let us remark here that $g^F$ on the cable system is linear extension of $g^F$ on the graph.) On the other hand we know that in the path decomposition we have for a given $l_i$ that the Green kernel $g_i^{z_i}(x,w) = g_{l_i}^{z_i}(x,w) = R_i(w,z_i)$ and similarly to (5) we have that

$$\begin{aligned} e_{l_i} &= \int_{l_i} g_i^{z_i}(x,w)\,d\mu(w) = \int_{l_i} R_i(w,z_i)\,d\mu(w), \\ \frac{R}{R_i} &= \frac{R(x,F)}{R_i(x,z_i)} = \frac{R(B_w,\partial F)}{R_i(w,z_i)} \end{aligned}$$

consequently for all path $l_i$

$$g^F(x,w) = R(B_w,F) = \frac{R}{R_i} r_i(w,z_i) = \frac{R}{R_i} g_i^{z_i}(x,w). \tag{5.2}$$

Since the path system splits each edge, we have

$$d\mu(w) = \sum_{i:\, w \in l_i} d\mu_i(w), \tag{5.3}$$

where in general $d\mu_i = P(l_i)\,d\mu$ and in particular $P(l_i) = \frac{R}{R_i}$. Here $z_i$'s are not necessarily different. In the next step we shall join the paths which have common endpoints, i.e., the boundary crossing edge is shared by them.

$$\begin{aligned} E_x(F) &= \int_F g^F(x,y)\,d\mu(y) = \sum_i \int_{l_i} g^F(x,y)\,d\mu_i(y) \\ &= \sum_z \sum_{l_i \ni z} \int_{l_i} g^F(x,y)\,d\mu_i(y) \end{aligned} \tag{5.4}$$

As earlier we should handle with care the paths ending at the same vertex (sharing a boundary crossing edge). The weights split on edges and hence the measure on vertexes add up as in (5.3)

$$d\mu_z(y) = \sum_{i:\, z\in B_y\cap l_i} d\mu_i(z),$$

while for $z \in B_y \cap l_i$

$$\begin{aligned} g^F(x,y) = g^F(y,x) &= \frac{R}{R_i} g^z(y,x) \\ &= \frac{R}{R_i} g_i^z(y,x) = \frac{R}{R_i} g_i^z(x,y). \end{aligned}$$

As a consequence of (5.4), (5.2) and the notation

$$e_l = \int_l g_l^z(x,y)\, d\mu_z(y),$$

we have that

$$E_x(F) = \sum_{z\in\partial F} \sum_{i:z\in l_i} \int_{l_i} g_{l_i}^F(x,y)\, d\mu_i(y) = \sum_l \frac{R}{R_l} e_l.$$

As a result we have the following observation.

**Lemma 5.2** *For the set $F$ the exit time $E_x(F)$ has the form*

$$E_x(F) = R \sum_l \frac{e_l}{R_l}.$$

We introduce the following notations: $C = \mathrm{Cap}(x,F) = 1/R(x,F)$

$$C_l = \frac{1}{R_l}, \quad \gamma_l = \frac{C_l}{C}, \quad \tilde{e} = \sum_l \gamma_l e_l, \quad \varphi_z = \frac{\delta_l e_l}{\tilde{e}},$$

where $\delta_z$ is such that $\mu_l R_l = (1+\delta_l)\, e_l$ holds.

**Theorem 5.3** *If $F$ is optimal for Problem 1, then it satisfies for all the B-K path $l$ and its endpoint $z \in l \cap \partial F$ that*

$$\frac{R_l(x,z)\, \mu(z)}{(1+\varphi_l)^{1/2}} = \text{const}.$$

The proof is deferred to the Appendix.

As in case of the capacity problem, the obtained solution is not necessarily optimal or unique, since it is only a necessary and not a sufficient condition for optimality in general (see Remark 4 2. and 3.).

## Appendix

*Proof of Proposition 4.2* Let us recall that we assume that $D$ is an optimal set and slightly change its boundary along the border crossing edges. We consider the Lagrange function with multiplier $\lambda \in \mathbb{R}$ :

$$\mathrm{Cap}_L^{\mathbb{P}}\left(D_\xi, F\right) + \lambda\mu\left(D_\xi\right).$$

Denote $\xi_l = \xi_l(x, y) : w_l = (\xi_l, x, y) \in \partial D$ forming the perturbation vector $\xi = [\xi_l]$. Let $z_l = \partial F \cap l$ be the endpoint of the path $l$ at the boundary of $F$.

$$\frac{\partial}{\partial \xi_l}\left[\sum_l \mathrm{Cap}_L^{\mathbb{P}}(w_l, z_l) + \lambda\mu\left(D_\xi\right)\right].$$

Setting the derivative zero and using $r_l = R^l(w_l, z_l)$ we have that

$$\begin{aligned}
0 &= \frac{\partial}{\partial \xi_l}\left[\sum_l \mathrm{Cap}_L^{\mathbb{P}}(w_l, z_l) + \lambda\mu\left(D_\xi\right)\right] = \\
&= \frac{\partial}{\partial \xi_l}\left[\frac{1}{r_l} + \lambda\mu_l\right] \\
&= -\frac{R^l(x_l, y_l)}{r_l^2} + \lambda\mu^l(x_l, y_l)
\end{aligned}$$

for all path $l \in L$ and

$$\mu^l(x_l, y_l)\, r_l = \text{const}$$

is a necessary condition for the optimality. □

*Proof of Theorem 5.3* We consider the variational problem

$$\max_{F' : \mu(F') \le M} E_x\left(F'\right).$$

Assume that $F$ is optimal with a path system $L$ and the probability $\mathbb{P}$ on it. As in the case of the capacity we perturb $F$ in a small neighborhood. The maximal solution should satisfy for a suitable $\lambda$ and for all path $l$ that

$$\frac{\partial}{\partial s_l}\left[E_x(F)+\lambda\mu(F)\right]=0$$

$$\frac{\partial}{\partial s_l}\left[R\sum_p\frac{e_p}{R_p}+\lambda\mu(F)\right]=\frac{\partial}{\partial s_l}\left[R\sum_p\left(\frac{e_p}{R_p}+\lambda\mu_p\right)\right]=0,$$

where $s_l$ is the length of $l$ and we use $\mu_l$ for the volume of the path $l$. Let $E=\sum\frac{e_l}{R_l}$, the density of $\mu$ is $\mu(z_l)=\frac{d\mu}{ds}|_{s_l}$, where $s$ is the arc length parametrization of $l_z : w(s_l)=z_l$. Furthermore, $\mu_l=\mu(z_l)$ and the density of resistance is $\rho(z_l)=1/\mu(z_l)$, then the derivative is as follows

$$\frac{\partial}{\partial s_l}\left[R\sum_p\left(\frac{e_p}{R_p}+\lambda\mu_p\right)\right]=\left(\frac{\partial}{\partial s_l}R\right)E+R\frac{\partial}{\partial s_l}E+\lambda\mu(z_l).$$

One can find that

$$\left(\frac{\partial}{\partial s_l}R\right)=\frac{\partial}{\partial s_l}\frac{1}{\sum_p\frac{1}{R_p}}=R^2\frac{\rho(z_l)}{R_l^2}$$

and

$$\begin{aligned}\frac{\partial}{\partial s_l}e_l&=\frac{\partial}{\partial s_l}\int r(w_s,z_l)\,\mu(w(s))\,ds\\&=\frac{\partial}{\partial s_l}\int_0^{s_l}\int_s^{s_l}\rho(w(t))\,dt\mu(w(s))\,ds\\&=\rho(z_l)\,\mu_l,\end{aligned}$$

$$\frac{\partial}{\partial s_l}E=\frac{\partial}{\partial s_l}\frac{e_l}{R_l}=\rho(z_l)\,\frac{\mu_lR_l-e_l}{R_l^2}.$$

It is trivial that $e_l\le\mu_lR_l$, so the defined $\delta_l$ is nonnegative. Furthermore,

$$\frac{\partial}{\partial s_l}E=\frac{\rho(z_l)}{R_l^2}\delta_le_l.$$

$$\frac{\partial}{\partial s_l}E_x(F)=R^2\frac{\rho(z_l)}{R_l^2}E+R\frac{\rho(z_l)}{R_l^2}\delta_le_l+\lambda\mu(z_l)=0$$

$$R\frac{\rho(z_l)^2}{R_l^2}E+\frac{\rho(z_l)^2}{R_l^2}\delta_le_l=\text{const}.$$

□

## References


1. R. Bañuelos, Isoperimetric inequalities in probability, geometry, and PDE's, http://www.math.purdue.edu/~banuelos/btlecture.pdf
2. M. Barlow, J. Pitman, M. Yor, *On Walsh's Brownian Motions*. Lecture Notes in Mathematics, vol. 1372 (Springer, Berlin, 1989), pp. 275–293
3. M.T. Barlow, R.F. Bass, T. Kumagai, Stability of parabolic Harnack inequalities on metric measure spaces. J. Math. Soc. Jpn. **58**(2), 485–519 (2006)
4. K.A. Berman, M.H. Konsowa, Random paths and cuts, electrical networks, and reversible Markov chains. SIAM J. Discret. Math. **3**(3), 311–319 (1990)
5. T. Carroll, Old and new on the Bass note, the Torsion function and the hyperbolic metric. Irish Math. Soc. Bull. **47**, 41–65 (2001)
6. P.G. Doyle, J.L. Snell, *Random Walks and Electric Networks*. Carus Mathematical, Monographs, 22 (Mathematical Association of America, Washington, 1984)
7. A. Grigory'an, A. Telcs, Two-sided estimates of heat kernels on metric measure spaces. Ann. Probab. **40**(3), 1212–1284 (1990)
8. F.T. den Hollander, S. Jensen, Berman-Konsowa Principles for Reversible Markov Jump Processes (preprint). http://arxiv.org/abs/1309.1305
9. G. Pólya, G. Szegő, *Isoperimetric Inequalities in Mathematical Physics (No. 27)* (Princeton University Press, Princeton, 1951)
10. A. Telcs, Isoperimetric inequalities for Random Walks. Potential Anal. **19**(3), 237–249 (2003)
11. A. Telcs, *The Art of Random Walks*. Lecture Notes in Mathematics, vol. 1885 (Springer, Berlin/New York, 2006)