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Commute times of random walks on trees

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ABSTRACT

In this paper we provide exact formula for the commute times of random walks on spherically symmetric random trees. Using this formula we sharpen some of the results presented in Al-Awadhi et al. to the form of equalities rather than inequalities.

1. Introduction

The commute time is a particular measure of random walks on weighted graphs. It has several nice properties which has been revealed independently partly or fully by many authors, see for example [5,2,3,14,21,1]. It is still in the focus of the research of computer scientists, probabilists and physicists as well. As examples, consider the tasks of graph embedding [9,16,18,22], graph sparsification [20], social network analysis, [13], proximity search [19], collaborative filtering [8], clustering [23], semisupervised learning [24], dimensionality reduction [10] image processing [17], graph labeling [11], and theoretical computer science [4,6]. For an extensive list of literature we refer the reader to [15]. Random walks on random graphs have been subject of permanent interest in the last three decades. Interestingly enough, very little is published on commute times of random walks on random graphs. The present paper studies commute times on very simple random objects, on spherically symmetric random trees, *SSRT*. Explicit results are presented in the annealed case, averaged commute times over the probability field of trees.

2. Commute times

Consider a random walk on a weighted graph $G = (\mathbf{V}, \mathbf{E})$ where a weight (conductivity) $c_{xy} = c_{yx}$ is assigned to edge $xy \in \mathbf{E}$. The commute time between two vertices r and s is the mean number of steps it takes the random walk to go from r to s and back to r and will be denoted by $\mathbb{E}(\tau) = \mathbb{E}(\tau_{r,s})$. We know, see [5], for a finite connected graph,

$$\mathbb{E}(\tau) = 2\rho_{rs}\mu_{rs}, \tag{2.1}$$

where $\rho = \rho_{rs}$ is the effective resistance between r and s and $\mu = \mu_{rs} = \frac{1}{2}\sum_{e\in\mathbf{E}} c_e$. If the assigned weights are all equal 1, then

$$\mathbb{E}(\tau) = 2\rho m,$$

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where $m = |\mathbf{E}|$ is the number of undirected edges of G. We confine our study to investigating the commute time of random walk on spherically symmetric random trees *SSRT* in which the degree of a vertex depends only on its distance from the root r.

The second probability space is given on the spherically symmetric trees of infinite heights and the corresponding probability and expectation will be denoted by $\mathbb{P}$ and $\mathbb{E}$.

This type of trees is completely determined by its degree sequence $\{d_n; n \geq 0\}$ where d_n is the degree of every node at level n. Let $\Im_n = \sigma(d_1, d_2, \dots, d_n)$. Then, for each realization $\mathbb{T}(\omega)$ of a random tree $\mathbb{T}$,

$$\mathbb{E}(\tau|\Im_n) = 2\rho\mu.$$

We are interested in the expected value with respect to $\mathbb{P}(\omega)$, probability distribution on the set of all possible trees $\mathbb{T}$. In such a case,

$$\mathbb{E}(\tau) = 2\mathbb{E}(\rho\mu).$$

It was shown in [1] that $\mathbb{E}(\tau) \leq 2\mathbb{E}(\rho)\,\mathbb{E}(\mu)$. It can easily be seen that this inequality can not be strengthened to equality. We first note that for positive nondegenerate random variable X, the function $f(X) = \frac{1}{X}$ is strictly convex and hence $\mathbb{E}(1/X) \gneqq 1/\mathbb{E}(X)$. Consider now a tree T of height 1 rooted at r which has random degree d_0. Then $m = d_0$ and $\rho_{rs} = 1/d_0$. Hence, $\mathbb{E}(\tau) = 2\mathbb{E}(\rho m) = 2$. On the other hand, $2\mathbb{E}(\rho)\,\mathbb{E}(m) \gneqq 2$.

Now we seek for asymptotic equality for $\mathbb{E}(\tau)$. Let S_i be the sphere of radius i and centered at r; that is the set of vertices at distance i from r. Let $\rho_i = \rho(S_{i-1}, S_i)$, $\mu_i = \mu(S_{i-1}, S_i)$, and $\mathbb{E}(\tau)$ is the commute time between the root and the sphere of radius n shorted in one vertex. Then

$$\begin{aligned}\mathbb{E}(\tau) &= 2\mathbb{E}(\rho\mu) = 2\mathbb{E}\left(\left(\sum_{i=1}^{n}\mu_i\right)\left(\sum_{j=1}^{n}\rho_j\right)\right)\\ &= 2\mathbb{E}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i\rho_j\right) = 2\mathbb{E}\left(\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_i\rho_j\right) + 2n, \qquad (2.2)\end{aligned}$$

where the last step uses the fact that $\rho_i = 1/\mu_i$. Below d_j^+ will denote the outdegree of state j; that is $d_j^+ = d_j - 1$. Now we concentrate on the double sum.

$$I_n + J_n := \mathbb{E}\left(\sum_{i=2}^{n}\sum_{j=1}^{i-1}\mu_i\rho_j\right) + \mathbb{E}\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mu_i\rho_j\right).$$

Now,

$$\begin{aligned}I_n &= \mathbb{E}\left(\sum_{i=2}^{n}\sum_{j=1}^{i-1}\mu_i\rho_j\right) = \mathbb{E}\left(\sum_{j=1}^{n-1}\rho_j\left(\sum_{i=j+1}^{n}\mu_i\right)\right)\\ &= \mathbb{E}\left(\sum_{j=1}^{n-1}\rho_j\mu\left(S_j, S_n\right)\right) = \sum_{j=1}^{n-1}\mathbb{E}\left(\rho_j\mu\left(S_j, S_n\right)\right)\\ &= \sum_{j=1}^{n-1}\sum_{k=1}^{\infty}\mathbb{E}\left(\rho_j\mu\left(S_j, S_n\right)|\mu_j = k\right)P\left(\mu_j = k\right)\\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{n-1}\mathbb{E}\left(\frac{1}{k}(\mu_{j+1} + \mu_{j+2} + \cdots + \mu_n)|\mu_j = k\right)P\left(\mu_j = k\right)\\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{n-1}\mathbb{E}\left(\frac{1}{k}k(d_j^+ + d_j^+d_{j+1}^+ + \cdots + d_j^+d_{j+1}^+\cdots d_{n-1}^+)\right)P\left(\mu_j = k\right)\\ &= \sum_{j=1}^{n-1}\sum_{x=j}^{n-1}\mathbb{E}\left(\Pi_{i=j}^{x}d_i^+\right)\\ &= \sum_{x=1}^{n-1}\sum_{j=1}^{x}\mathbb{E}\left(\Pi_{i=j}^{x}d_i^+\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}
J_n &= \mathbb{E}\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mu_i\rho_j\right)\\
&= \sum_{k=1}^{\infty}\mathbb{E}\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mu_i\rho_j|\mu_i=k\right)P\left(\mu_i=k\right)\\
&= \sum_{k=1}^{\infty}\mathbb{E}\left(\sum_{i=1}^{n-1}k(\rho_{i+1}+\rho_{i+2}+\cdots+\rho_n)|\mu_i=k\right)P\left(\mu_i=k\right)\\
&= \mathbb{E}\left(\sum_{i=1}^{n-1}k\frac{1}{k}\left(\frac{1}{d_i^+}+\frac{1}{d_i^+d_{i+1}^+}+\cdots+\frac{1}{d_i^+d_{i+1}^+\cdots d_{n-1}^+}\right)\right)\\
&= \sum_{i=1}^{n-1}\mathbb{E}\left(\frac{1}{d_i^+}+\frac{1}{d_i^+d_{i+1}^+}+\cdots+\frac{1}{d_i^+d_{i+1}^+\cdots d_{n-1}^+}\right)\\
&= \sum_{i=1}^{n-1}\sum_{x=i}^{n-1}\mathbb{E}\left(\Pi_{j=i}^{x}\frac{1}{d_j^+}\right)\\
&= \sum_{x=1}^{n-1}\sum_{i=1}^{x}\mathbb{E}\left(\Pi_{j=i}^{x}\frac{1}{d_j^+}\right),
\end{aligned}$$

where in the third step the condition $\mu_i = k$ is used to calculate the resistance of k parallel branches starting in level i. Finally,

$$\mathbb{E}(\tau) = 2n + 2\sum_{x=1}^{n-1}\sum_{j=1}^{x}\left(\mathbb{E}\left(\Pi_{i=j}^{x}d_i^+\right)+\mathbb{E}\left(\Pi_{i=j}^{x}\frac{1}{d_i^+}\right)\right). \tag{2.3}$$

3. Spherically symmetric trees

In this section we use τ_{rs} to denote the commute time between the root r of a *SSRT* Γ and the level n shorted in one node s, while τ_{rx_n} denotes the commute time between r and a leaf x_n of level n. We assign a unit resistance to every edge of Γ. We also use d_n^+ to refer the outdegree of each node of level n and Z_n to refer to the number of vertices in the level n. Then $Z_n = \Pi_{k=0}^{n-1}d_k$. We will assume that d_n^+'s are independent random variables. We need the following lemma from [12].

Lemma 1. *Consider two nonnegative sequences a_n and b_n such that $\sum_n b_n$ is divergent. If $\lim_n \frac{a_n}{b_n} = L$, then $\lim_n \frac{\sum_{k=1}^n a_k}{\sum_{k=1}^n b_k} = L$.*

Notation 1. *We use $a_n = \Theta\left(b_n\right)$ for $\lim_n \frac{a_n}{b_n} \epsilon \left(0, \infty\right)$.*

The following theorem strengthen Theorem 1 of [1] that gives only an upper bound for the commute time.

Theorem 1. *Consider a spherically symmetric random tree Γ such that*

$$d_n^+ = \begin{cases} 1 & \text{with probab. } 1-q_n \\ 2 & \text{with probab. } q_n \end{cases}$$

and $\sum q_n < \infty$. Then

$$\tau_{rs} = \Theta\left(n^2\right) \quad \mathbb{P}\text{-a.s.}$$

and

$$\tau_{rx_n} = \Theta\left(n^2\right) \quad \mathbb{P}\text{-a.s..}$$

Proof. Applying the Borel Cantelli Lemma to the infinite outdegree sequence $\{d_n^+\}$ shows that $p\left(d_n = 1 \text{ eventually}\right) = 1$. As such, there is N such that $p\left(d_n = 1, n \geq N\right) = 1$. Then

$$\begin{aligned}
\mu_{rs} &= \sum_{k=1}^{n}\mathbb{Z}_k - 1\\
&= \sum_{k=1}^{N}\mathbb{Z}_k + \sum_{k=N+1}^{n}\mathbb{Z}_k - 1 = \Theta\left(n\right) \quad \text{a.s.}
\end{aligned}$$

where $\mathbb{Z}_k = |S_k|$. Similarly,

$$\rho_{rs} = \Theta(n) \quad \text{a.s.}$$

Therefore,

$$\mu_{rs}\rho_{rs} = \Theta\left(n^2\right) \quad \text{a.s.}$$

and the result follows from Eq. (2.1). $\square$

The following lemma is presented in [7, p. 63].

Lemma 2. *For any $c > 0$,*

$$\Pi_{j=1}^{n}\left(1+\frac{c}{j}\right) \sim n^c.$$

The following lemma is presented in [12, p. 66].

Lemma 3. *For $0 < \alpha < 1$,*

$$\sum_{k=1}^{n} \frac{1}{k^\alpha} \sim \frac{n^{1-\alpha}}{1-\alpha}.$$

Theorem 2. *Consider a SSRT Γ such that*

$$d_n^+ = \begin{cases} 1 & \text{with probab. } 1-q_n \\ 2 & \text{with probab. } q_n \end{cases}$$

where $q_n = \min(1, c/n)$, $c > 0$. Then

$$\mathbb{E}(\tau_{rs}) = \Theta\left(n^2 \log n\right) \quad \text{if } c = 1,$$
$$\mathbb{E}(\tau_{rs}) = \Theta\left(n^2\right) \quad \text{if } c < 1$$
$$\mathbb{E}(\tau_{rs}) = \Theta\left(n^{c+1}\right) \quad \text{if } c > 1.$$

Moreover, for any $c > 0$,

$$E\left(\tau_{rx_n}\right) = \Theta\left(n^{c+2}\right).$$

Proof. We first note that as long as the Θ-asymptotic behavior of the commute time is our concern and since I_n is greater than J_n, it is enough to calculate I_n. It follows for $c = 1$ that

$$\mathbb{E}\left(\Pi_{i=j}^{x} d_i^+\right) = \Pi_{i=j}^{x}\left(1+\frac{1}{i}\right) = \frac{x+1}{j}.$$

Then,

$$\begin{aligned} I_n &= \sum_{x=1}^{n-1}\sum_{j=1}^{x}\frac{x+1}{j} \\ &= \sum_{x=1}^{n-1}(x+1)\,\alpha_x \log x; \quad \alpha_j \longrightarrow 1 \\ &\sim \sum_{x=1}^{n-1}(x+1)\log x \\ &= \Theta\left(n^2 \log n\right), \end{aligned}$$

where the last equality follows from the proof of Theorem 11 of [1]. Let us recall that in J_n the product of expected values is $\frac{x+\frac{1}{2}}{j}$ while in $I_n \frac{x+1}{j}$. It follows that

$$J_n = I_n - \sum_{x=1}^{n-1}\sum_{j=1}^{x}\frac{1}{2j}$$

which means that

$$I_n - J_n \sim \sum_{x=1}^{n-1} \frac{1}{2} \log x.$$

But $\sum_{x=1}^{n-1} \frac{1}{2} \log x = o\left(n^2 \log n\right)$ and since $\lim_n \frac{I_n}{n^2 \log n} \in (0, \infty)$ we have that $\lim_n \frac{I_n + J_n}{n^2 \log n} = \lim_n \frac{2I_n}{n^2 \log n} \in (0, \infty)$.

We consider now the case $c < 1$. It follows from Lemma 2 that

$$\Pi_{i=j}^{x} \left(1 + \frac{c}{i}\right) = \frac{x^c \alpha_x}{j^c \alpha_j}; \quad \alpha_x \longrightarrow \alpha \in (0, \infty),$$

we see

$$\begin{aligned} I_n &= \sum_{x=1}^{n-1} \sum_{j=1}^{x} \frac{x^c \alpha_x}{j^c \alpha_j} \\ &= \sum_{x=1}^{n-1} x^c \alpha_x \sum_{j=1}^{x} \frac{1}{j^c \alpha_j}. \end{aligned} \tag{3.1}$$

It follows from Lemmas 1 and 3 that

$$\begin{aligned} I_n &\sim \sum_{x=1}^{n-1} \alpha_x x^c \lambda_x \frac{x^{-c+1}}{-c+1}; \quad \lambda_x \to \lambda \in (0, \infty) \\ &= \Theta\left(n^2\right). \end{aligned}$$

While for $c > 1$,

$$\sum_{j=1}^{x} \frac{1}{j^c \alpha_j} = \Theta(1)$$

and then, from Eq. (3.1) and Lemma 1,

$$\begin{aligned} I_n &= \sum_{x=1}^{n-1} x^c \alpha_x \sum_{j=1}^{x} \frac{1}{j^c \alpha_j} \\ &\sim \sum_{x=1}^{n-1} x^c \alpha_x \sim \sum_{x=1}^{n-1} x^c. \end{aligned}$$

Hence,

$$I_n = \Theta\left(n^{c+1}\right).$$

The result for $\mathbb{E}(\tau_{rx_n})$ follows from the fact that $\rho_{rx_n} = n$ and applying Lemma 2 gives

$$\begin{aligned} \sum_{k=1}^{n} \mathbb{E}(Z_k) &= \sum_{k=1}^{n} \Pi_{i=1}^{k} \left(1 + \frac{c}{i}\right) \\ &\sim \sum_{k=1}^{n} k^c = \Theta\left(n^{c+1}\right). \quad \square \end{aligned}$$

The following lemma is analogous to Theorem 3, p. 64 of [12].

Lemma 4. *Consider a positive decreasing function f and define a sequence a_k, $k = 1, 2, \ldots$ such that $f(t) = a_t$. Let $\mathbb{I}_n = \int_1^n f(t)dt$ and $\mathbb{S}_n = \sum_{k=1}^{n} a_k$. If $\lim_n \mathbb{I}_n = \infty$ then*

$$\lim_n \frac{\mathbb{S}_n}{\mathbb{I}_n} = 1.$$

Proof. Since f is decreasing, then for $j = 2, 3, \ldots$

$$\int_j^{j+1} f(x)dx \le a_j \le \int_{j-1}^{j} f(x)dx.$$

By summing over j, we obtain

$$\int_2^{n+1} f(x)dx \le \mathbb{S}_n - a_1 \le \int_1^n f(x)dx.$$

That is,

$$\mathbb{I}_{n+1} - \mathbb{I}_2 \le \mathbb{S}_n - a_1 \le \mathbb{I}_n. \tag{3.2}$$

It follows also that

$$\int_n^{n+1} f(x)dx \le \int_1^2 f(x)dx \le C.$$

As such,

$$\lim_n \frac{\int_n^{n+1} f(x)dx}{\mathbb{I}_n} = 0,$$

which implies that

$$\lim_n \frac{\mathbb{I}_{n+1}}{\mathbb{I}_n} = \lim_n \left(1 + \frac{\int_n^{n+1} f(x)dx}{\mathbb{I}_n}\right) = 1,$$

and the result follows from Eq. (3.2). □

Theorem 3. *Consider a SSRT Γ such that for $0 < \alpha < 1$*

$$d_n^+ = \begin{cases} 1 & \textit{with probab. } 1 - \dfrac{1}{n^\alpha} \\ 2 & \textit{with probab. } \dfrac{1}{n^\alpha}. \end{cases}$$

Then for any $\epsilon < \frac{1}{1-\alpha}$, there exists N such that for $n \ge N$, the following inequalities hold

$$\text{(i) } \Theta\left(n^{\alpha+1}\exp\left(\left(\frac{1}{1-\alpha} - \epsilon\right)n^{1-\alpha}\right)\right) \le \mathbb{E}\left(\tau_{rx_n}\right) \le \Theta\left(n^{\alpha+1}\exp\left(\left(\frac{1}{1-\alpha} + \epsilon\right)n^{1-\alpha}\right)\right),$$

$$\text{(ii) } \Theta\left(n^{\alpha}\exp\left(\left(\frac{1}{1-\alpha} - \epsilon\right)n^{1-\alpha}\right)\right) \le \mathbb{E}\left(\tau_{rs}\right) \le \Theta\left(n^{\alpha}\exp\left(\left(\frac{1}{1-\alpha} + \epsilon\right)n^{1-\alpha}\right)\right).$$

Remark 1. The case $\alpha > 1$ is covered by Theorem 1 and the case $\alpha = 1$ is covered by Theorem 2.

Proof. Let $\mathbb{S}_n = \log \mathbb{E}(\mathbb{Z}_n) = \sum_{k=0}^{n-1} \log\left(1 + \frac{1}{k^\alpha}\right)$. We first show that

$$\lim_n \frac{\mathbb{S}_n}{n^{1-\alpha}} = \frac{1}{1-\alpha}. \tag{3.3}$$

Since

$$\mathbb{I}_n = \int_1^n \log\left(1 + \frac{1}{x^\alpha}\right)dx = n\log\left(1 + \frac{1}{n^\alpha}\right) - \log 2 + \alpha\int_1^n \frac{1}{1+x^\alpha}dx, \tag{3.4}$$

and

$$\lim_n \frac{n\log\left(1 + \frac{1}{n^\alpha}\right)}{n^{1-\alpha}} = 1,$$

and also

$$\lim_n \frac{\int_1^n \frac{1}{1+x^\alpha}dx}{n^{1-\alpha}} = \lim_n \frac{\frac{1}{1+n^\alpha}}{(1-\alpha)n^{-\alpha}} = \frac{1}{1-\alpha},$$

then, from (3.4),

$$\lim_n \frac{\mathbb{I}_n}{n^{1-\alpha}} = 1 + \frac{\alpha}{1-\alpha} = \frac{1}{1-\alpha}.$$

It follows from Lemma 4 that

$$\lim_n \frac{\mathbb{S}_n}{n^{1-\alpha}} = \lim_n \frac{\mathbb{S}_n}{\mathbb{I}_n} \cdot \frac{\mathbb{I}_n}{n^{1-\alpha}} = \lim_n \frac{\mathbb{I}_n}{n^{1-\alpha}} = \frac{1}{1-\alpha}.$$

As such,

$$\mathbb{E}(\mathbb{Z}_n) = \exp\left(\gamma_n n^{1-\alpha}\right); \quad \gamma_n \to \frac{1}{1-\alpha}. \tag{3.5}$$

That is, for arbitrary small $\epsilon > 0$, there is a sufficiently large N such that for $n \geq N$,

$$\exp\left(\left(\frac{1}{1-\alpha} - \epsilon\right) n^{1-\alpha}\right) \leq \mathbb{E}(\mathbb{Z}_n) \leq \exp\left(\left(\frac{1}{1-\alpha} + \epsilon\right) n^{1-\alpha}\right), \tag{3.6}$$

and

$$\sum_{k=N}^{n} \exp\left(\left(\frac{1}{1-\alpha} - \epsilon\right) k^{1-\alpha}\right) \leq \sum_{k=N}^{n} \mathbb{E}(\mathbb{Z}_k) \leq \sum_{k=N}^{n} \exp\left(\left(\frac{1}{1-\alpha} + \epsilon\right) k^{1-\alpha}\right).$$

Since,

$$\lim_n \frac{\int_N^n \exp\left(\left(\frac{1}{1-\alpha} + \epsilon\right) x^{1-\alpha}\right) dx}{n^{\alpha} \exp\left(\left(\frac{1}{1-\alpha} + \epsilon\right) n^{1-\alpha}\right)} = \frac{1}{1 + \epsilon\,(1-\alpha)},$$

and

$$\frac{\int_N^n \exp\left(\left(\frac{1}{1-\alpha} - \epsilon\right) x^{1-\alpha}\right) dx}{n^{\alpha} \exp\left(\left(\frac{1}{1-\alpha} - \epsilon\right) n^{1-\alpha}\right)} = \frac{1}{1 - \epsilon\,(1-\alpha)},$$

then

$$\Theta\left(n^{\alpha} \exp\left(\left(\frac{1}{1-\alpha} - \epsilon\right) n^{1-\alpha}\right)\right) \leq \sum_{k=N}^{n} \mathbb{E}(\mathbb{Z}_k) \leq \Theta\left(n^{\alpha} \exp\left(\left(\frac{1}{1-\alpha} + \epsilon\right) n^{1-\alpha}\right)\right)$$

and the result for $\mathbb{E}\left(\tau_{rx_n}\right)$ follows since $\rho_{rx_n} = n$.

For $\mathbb{E}\left(\tau_{rs}\right)$, we follow the same argument of computing I_n as in the proof of Theorem 2. From Eq. (3.5), we see that

$$\begin{aligned}\mathbb{E}(\Pi_{i=j}^{x} d_i^{+}) &= \Pi_{i=j}^{x}\left(1 + \frac{1}{i^{\alpha}}\right)\\ &= \exp\left(\gamma_x x^{1-\alpha} - \gamma_j j^{1-\alpha}\right); \quad \gamma_x \to \frac{1}{1-\alpha}.\end{aligned}$$

As such,

$$\begin{aligned}\sum_{j=1}^{x} \mathbb{E}(\Pi_{i=j}^{x} d_i^{+}) &= \sum_{j=1}^{x} \exp\left(\gamma_x (x^{1-\alpha}) - \gamma_j (j^{1-\alpha})\right)\\ &= \left(\exp \gamma_x \left(x^{1-\alpha}\right)\right) \sum_{j=1}^{x} \exp\left(-\gamma_j (j^{1-\alpha})\right).\end{aligned}$$

Using the fact that the two series $\sum a_n$ and $\sum 2^{\nu} a_{2^{\nu}}$ have the same convergence behavior, we can see that

$$\sum_{j=1}^{x} \exp\left(-\gamma_j (j^{1-\alpha})\right) = \Theta(1)$$

and hence,

$$\sum_{j=1}^{x} \mathbb{E}(\Pi_{i=j}^{x} d_i^{+}) \sim \exp(\gamma_x (x^{1-\alpha})).$$

It follows then that for arbitrary small $\epsilon > 0$, and sufficiently large N,

$$\exp\left(\left(\frac{1}{1-\alpha} - \epsilon\right) x^{1-\alpha}\right) \leq \sum_{j=1}^{x} \mathbb{E}(\Pi_{i=j}^{x} d_i^{+}) \leq \exp\left(\left(\frac{1}{1-\alpha} + \epsilon\right) x^{1-\alpha}\right); \quad x \geq N$$

$$\sum_{x=N}^{n} \exp\left(\left(\frac{1}{1-\alpha} - \epsilon\right) x^{1-\alpha}\right) \leq \sum_{x=N}^{n-1} \sum_{j=1}^{x} \mathbb{E}(\Pi_{i=j}^{x} d_i^{+}) \leq \sum_{x=N}^{n} \exp\left(\left(\frac{1}{1-\alpha} + \epsilon\right) x^{1-\alpha}\right).$$

The same argument of part (i) shows that

$$\Theta\left(n^{\alpha}\exp\left(\left(\frac{1}{1-\alpha}-\epsilon\right)n^{1-\alpha}\right)\right)\le I_n\le\Theta\left(n^{\alpha}\exp\left(\left(\frac{1}{1-\alpha}+\epsilon\right)n^{1-\alpha}\right)\right),$$

and this proves part (ii). □

References

[1] F. Al-Awadhi, M. Konsowa, Z. Najeh, Commute times and the effective resistances of random trees, Probability in the Engineering and Informational Sciences 23 (4) (2009) 649–660.

[2] D. Aldous, Random walk covering of some special trees, Journal of Mathematical Analysis and Applications 157 (1991) 271–283.

[3] D. Aldous, J. Fill, Reversible Markov chains and random walks on graphs, in: Monograph. Available at: http://www.stat.berkeley.edu/~aldous/RWG/book.html (in preparation).

[4] R. ALeliunas, R. Karp, R. Lipton, L. Lovasz, C. Rackoff, Random walks, universal traversal sequences, and complexity of maze problem, in: 20th Annual Symposium on Foundations of Computer Science, 1979, pp. 218–223.

[5] A. Chandra, P. Raghavan, W. Ruzzo, R. Smolensky, P. Tiwari, The electrical resistance of a graph captures its commute and cover times, in: Proceedings of the 21st Annual Symposium on Theory of Computing, ACM Association for Computing Machinery, New York, 1989, pp. 574–586.

[6] C. Cooper, A. Frieze, The cover time of random geometric graphs, SODA (2009) 48–57.

[7] W. Feller, An Introduction to Probability Theory and Its Applications, Vol. I, second ed., John Wiley & Sons, New York, 1968.

[8] A. Fouss, J. Pirotte, M. Renders, M. Saerens, A novel way of computing dissimilarities between nodes of a graph with application to collaborative filtering and subspace projection of the graph nodes, Technical Report IAG WP 06/08, Universite cathlolique de Louvain, 2006.

[9] S. Guattery, Graph embeddings, symmetric real matrices, and generalized inverses, Technical Report, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, 1998.

[10] J. Ham, D.D. Lee, S. Mika, B. Scholkopf, A kernel view of the dimensionality reduction of manifolds, in: International Conference on Machine Learning, ACM, 2004, pp. 369–376.

[11] M. Herbster, M. Pontil, Prediction on a graph with a perceptron, in: Bernhard Scholkopf, John Platt, Thomas Hoffman (Eds.), New Information Processing Systems, 2006, pp. 577–584.

[12] K. Knopp, Infinite Sequences and Series, Dover Publications, INC., New York, 1956.

[13] Liben-Nowell, J. Kleinberg, The link prediction problem for social networks, in: Proceedings of the 2003 ACM CIKM International Conference of Information and Knowledge Management, CIKM-03, 2003, pp. 556–559.

[14] L. Lovasz, Random walks on graphs: a survey, combinatorics, Bolyai Society Mathematical Studies (1993) 353–397.

[15] U. Luxburg, A. Radl, M. Hein, Hitting and commute times in large graphs are often misleading, arXiv:1003.1266v2 [CS.ds], 2011.

[16] H. Qiu, E. Hancock, Graph embedding using commute time, Structural, Syntactic, and Statistical Pattern Recognition (2006) 441–449.

[17] H. Qiu, E.R. Hancock, Image segmentation using commute times, in: Proceedings of the 16th British Machine Vision Conference, 2005, pp. 929–938.

[18] M. Saerens, F. Fouss, L. Yen, P. Dupont, The principal components analysis of a graph and its relationship to spectral clustering, in: Proceedings of the 15th European Conference on Machine Learning, Springer, Berlin, 2004, pp. 371–383.

[19] P. Sarkar, A. Moore, A. Prakash, Fast incremental proximity search in large graphs, in: Proceedings of the 25th International Conference of Machine Learning, 2008, pp. 896–903.

[20] D. Spielman, N. Srivastava, Graph sparsification by effective resistances, in: Proceedings of the 40th Annual Symposium on Theory of Computing, 2008, pp. 563–568.

[21] P. Tetali, Random walks and effective resistances of networks, Journal of Theoretical Probability 4 (1) (1991) 101–109.

[22] D.M. Wittmann, D. Schmidl, F. Blochl, F.J. Theis, Reconstruction of graphs based on random walks, Theoretical Computer Science 410 (38–40) (2009) 3826–3838.

[23] L. Yen, D. Vanvyve, F. Wouters, F. Fouss, M. Verleysen, M. Saerens, Proceedings of the 13th Annual Symposium on Artificial Neural Networks, 2005, pp. 317–324.

[24] D. Zhou, B. Scholkopf, Learning from labeled and unlabeled data using random walks, Pattern Recognition, in: Proceedings of the 26th DAGM Symposium, Berlin, Germany, 2004, pp. 237–244.