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Routing vertex disjoint Steiner-trees in a cubic grid and connections to VLSI

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Abstract

Consider a planar grid of size $w \times n$. The vertices of the grid are called terminals and pairwise disjoint sets of terminals are called nets. We aim at routing all nets in a cubic grid (above the original grid holding the terminals) in a vertex-disjoint way. However, to ensure solvability, it is allowed to extend the length and the width of the original grid to $w' = sw$ and $n' = sn$ by introducing $s - 1$ pieces of empty rows and columns between every two consecutive rows and columns containing the terminals. Hence the routing is to be realized in a cubic grid of size $(s \cdot n) \times (s \cdot w) \times h$. The objective is to minimize the height h. It is easy to show that the required height can be as large as $h = \Omega(\max(n, w))$ in the worst case. In this paper we show that if $s \geq 2$ then a routing with height $h = 6 \max(n, w)$ can always be found in polynomial time. Furthermore, the constant factor '6' can be improved either by increasing the value of s or by limiting the number of terminals in a net. Possible trade-offs between s and h are discussed and the various constructions presented are compared by measuring the volumes of the routings obtained.

Keywords: VLSI layout; 3-D routing

1. Introduction

Traditionally, the detailed routing phase of the design of VLSI (very large scale integrated) circuits was considered as *a two-dimensional problem*, gradually extended to 2, 3, ... layers. Even within this problem single row routing and channel routing (where the terminals to be interconnected are on one side, or on two opposite sides, respectively, of a rectangle) are the better understood subproblems, where the inputs are essentially one-dimensional (one or two lists of terminals of length n, also called the *length* of the channel). Here, the main aim is to realize the routing, and since its 'horizontal' size is given, its 'vertical' size, or *width*, w should be minimized. An important quantity is the *density* of the problem: the maximum number d so that there exists a vertical straight line cutting d nets into two.

A much investigated restriction of single row routing and channel routing is to look at these problems in the *Manhattan model*. Here, it is assumed that each layer contains either horizontal (that is, parallel with the length) or vertical (that is, parallel with the width) wire segments and consecutive layers contain wire segments of different directions.

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(The Manhattan model is preferable for certain technologies, because in such a routing wires do not bend within a layer and no parallel wire segments arise on consecutive layers.) In case of the 2-layer Manhattan model

(1) single row routing is always possible in $O(n)$ time with $w = d$, see [12,18];
(2) channel routing is NP-hard [19,30];
(3) however, if it is allowed to extend the length of the channel by introducing additional columns, then the latter becomes solvable in $O(nd)$ time such that width of the obtained solution is at most constant times the optimum [3,6,14]. In fact, the number of additional columns that have to be added may be as large as $O(\sqrt{n})$, see [3, pp. 212–213].

In spite of (2), there are plenty of practically effective algorithms available, which can solve 'difficult' problems (of length 150 ... 200) with width around 20 [10,18]. In case of switchbox routing (where the terminals are on all the four sides of a rectangle—a real two-dimensional problem) instances with length 23 and width 15 ... 16 are already 'difficult' problems on two layers [5,8,16,23].

If more than two layers are permitted, both channel and switchbox routing become easier, see for example [7,4,29], respectively.

As technology permits more and more layers, a 'real' *three-dimensional* approach becomes reasonable. There are plenty of deep results in this area, see [1,2,9,11,13,20–22,26,28], for example. Most of them embed certain 'universal-purpose' graphs (like n-permuters, n-rearrangeable permutation networks, shuffle-exchange graphs) into the three-dimensional grid, ensuring that *pairs* of terminals can be connected, moreover, in some papers along *edge-disjoint* paths. Our results below are of much simpler structure but they allow *multiterminal nets* as well, and ensure *vertex disjoint* paths (or Steiner-trees) for the interconnections of the terminals within each net.

In this paper we consider the *single active layer routing problem* where the terminals occupy certain vertices of a $w \times n$ planar grid and the third dimension above this plane (with *height h*) is for interconnections only. The objective is to minimize the height h. One can easily see even in small instances like 4×1 or 2×2 that a routing is usually impossible (with an arbitrary height). Therefore, it is allowed to extend the length and the width of the original grid containing the terminals to $w' = s_w \cdot w$ and $n' = s_n \cdot n$, respectively (compare with (3) above). s_w and s_n, called the *spacing*, are given constants. Extending the dimensions of the grid is done by introducing empty rows and columns into the planar grid holding the terminals. Hence the final routing is realized in a cubic grid of size $(s_w \cdot w) \times (s_n \cdot n) \times h$.

If $s_w \geqslant 2$ and $s_n \geqslant 2$ (that is, both dimensions of the grid can be at least doubled) then there is a trivial upper bound of $h = O(wn)$ for the height of the routing; see Lemma 2. The truth of the same statement is not at all clear if s_n is restricted to be 1 (that is, the length of the circuit board is fixed). However, in a previous paper [25] we proved that for any fixed value of w a routing of height $O(n)$ can be attained even if the new length n' satisfies $n \leqslant n' \leqslant n + 1$ and only w is extended to $w' = s_w \cdot w$, where s_w is a suitably chosen constant. In view of Lemma 1, this linear bound is best possible. Moreover, our algorithm realized this in $O(n)$ time, which is also essentially best possible. In that paper we considered w as fixed and obtained bounds for the height as a function of n only.

In this paper we consider the case where $s_w, s_n \geqslant 2$ is assumed, that is, both horizontal dimensions w and n can be at least doubled. We prove that every problem instance can be solved with height $h = O(\max(n, w))$, even in linear time (see Section 5). Furthermore, the constant factor hidden inside the O sign will not be too high. For example, in the most general case, a routing with height $h = 6\max(n, w)$ will be obtained (see Corollary 10). We will show, however, that this constant factor can be reduced either by increasing one of the values s_w and s_n or by limiting the number of terminals in a net. In Section 6 we compare the constructions presented by measuring the volumes of the obtained routings. The main tool of the constructions of this paper, the (edge-)coloring of a suitably defined conflict graph, is a well-known technique; see e.g. [15].

As mentioned above, our work is motivated by the research of VLSI design, where three-dimensional routing and, in particular, single active layer routing have been much investigated topics in the past two decades. However, as it is very often the case with algorithms having provably good performance in VLSI routing, our constructions are not intended for designing specific, ready-to-use routing patterns for given specifications. Contrarily, we rather aim at understanding the nature of the problem, exploring the capacity of the three-dimensional grid for solving such problems and thus helping heuristics to be designed and tested.

2. Definitions and preliminary results

The vertices of a given (planar) grid of size $w \times n$ (consisting of w rows and n columns) are called *terminals*. A *net* N is a set of terminals. A *single active layer routing problem* (or *SALRP* for short) is a set $\mathcal{N} = \{N_1, N_2, \ldots, N_t\}$ of pairwise disjoint nets. n and w are the *length* and the *width* of the routing problem, respectively.

By a *spacing of s_w in direction w* we are going to mean that we introduce $s_w - 1$ pieces of extra columns between every two consecutive columns (and also to the right-hand side of the rightmost column) of the original grid. This way the width of the grid is extended to $w' = s_w \cdot w$. A *spacing of s_n in direction n* is defined analogously.

A *solution with a given spacing s_w and s_n* of a routing problem $\mathcal{N} = \{N_1, N_2, \ldots, N_t\}$ is a set $\mathcal{T} = \{T_1, T_2, \ldots, T_t\}$ of pairwise vertex-disjoint Steiner-trees in the cubic grid of size $(w \cdot s_w) \times (n \cdot s_n) \times h$ (above the original planar grid containing the terminals) such that the terminal set of T_i is N_i for every $1 \leqslant i \leqslant t$. The Steiner-trees T_i are called *wires*, h is called the *height* of the routing and a cross-section of the cubic grid perpendicular to the height (and therefore of size $(w \cdot s_w) \times (n \cdot s_n)$) is called a *layer*.

In order to simplify the description of the routings, we are going to use the term *w-wire segment* to refer to a wire segment that is parallel with the width w of the grid. The meaning of the terms *n-wire segment* and *h-wire segment* is analogous.

Lemma 1. *For any given n there exists a routing problem that cannot be solved with height h smaller than $n/2s_w$.*

Proof. Let, for simplicity, the width and the length be even, let $w = 2a$ and $n = 2b$. Consider the following example (the idea is very similar to that in [4,17]). Suppose that each net consists of two terminals in central-symmetric position as shown in Fig. 1.

The number of nets is an. Since each net is cut into two by the central vertical line e, any routing with width $w' = s_w \cdot w$ and height h must satisfy $w'h \geqslant an$. Therefore, $h \geqslant (w/2w')n$, hence $h \geqslant n/2s_w$. $\square$

Lemma 2. *If $s_w \geqslant 2$ and $s_n \geqslant 2$ then every routing problem can be solved with height $h \leqslant wn/2$.*

Proof. We assign a separate layer to each net. For every terminal we introduce an h-wire segment to connect the terminal with the layer of its net. The interconnection of the terminals of each net can now be performed trivially on its layer using the extra rows and columns guaranteed by the spacing in both directions.

Since 1-terminal nets can be disregarded, the number of nets is at most $\frac{1}{2}nw$ thus $h \leqslant wn/2$ follows immediately. $\square$

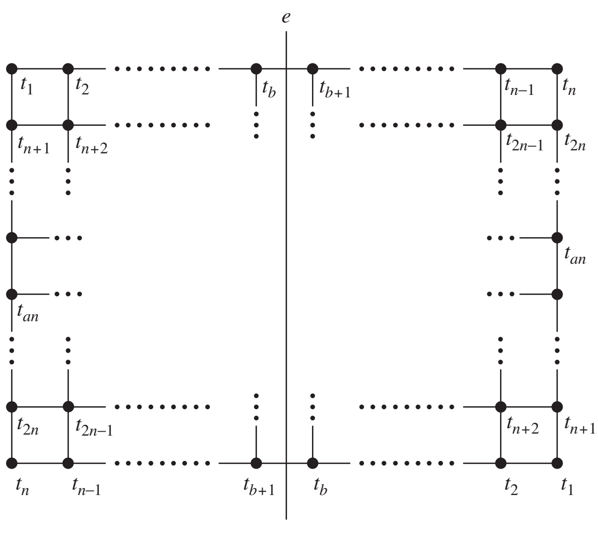

Fig. 1.

In 2000 Aggarwal et al. [2] proved the following theorem using advanced probabilistic methods. (More precisely, they presented an algorithm that depends on randomly chosen parameters and they proved that for some choice of these parameters the algorithm attains the given bounds).

Theorem 3 (*Aggarwal et al. [2]*). *If each net consists of two terminals only then the nets of an* $n \times n$ *SALRP can be partitioned into* $\mathrm{O}(n\log^2 n)$ *classes such that each class of nets can be routed on a copy of the grid* (*of size* $n \times n$). *Furthermore, the weaker bound of* $\mathrm{O}(n^{7/5})$ *can be reached for the same problem if each path of the routing is restricted to have at most 6 bends*.

An easy corollary of this theorem is that if $s_w = s_n = 2$ and each net consists of two terminals only then every SALRP can be solved with height $h = \mathrm{O}(n\log^2 n)$; see Proposition 5 (with $h_0 = 1$) for a proof of this corollary from the above theorem.

The main aim of the next section is to further improve over the above result by showing that $h = \mathrm{O}(n)$ essentially suffices.

3. 2-Terminal Nets

In this section we restrict ourselves to the special case in which every net contains two terminals only.

Theorem 4. *If each net consists of two terminals only then the nets of a SALRP can be partitioned into* $\lfloor \frac{3}{2}n \rfloor$ *classes such that each class of nets can be routed as a separate SALRP with* $s_n = \lceil w/2n \rceil$ *and height* $h = 2$.

Proof. Assume that a single active layer routing problem with w rows and n columns is given. Let the set of rows be $R = \{r_1, r_2, \ldots, r_w\}$. We define a graph G with vertex set R where the edges correspond to the nets: if the two terminals of a net N are located in the rows r_i and r_j then we add the edge $e_N = r_i r_j$ to the edge set E of G. (Note that parallel edges and loops are possible in G).

The definition of G implies that the degree of every vertex is at most n. Therefore, by Shannon's classic theorem [27], the edges can be coloured with $\lfloor \frac{3}{2}n \rfloor$ colours (such that no two adjacent edges share the same colour). Since the edges of G correspond to the nets, this colouring induces a partition on the set of nets. We claim that this partition fulfils the requirements of the theorem, that is, each partition class can be routed with $s_n = \lceil w/2n \rceil$ and height $h = 2$.

Assume that the nets $\mathcal{N}' = \{N_{i_1}, N_{i_2}, \ldots, N_{i_k}\}$ form one of the partition classes. Call a net *trivial* if its two terminals are in the same row. It follows from the construction of G that every row contains either at most one terminal belonging to a nontrivial net in $\mathcal{N}'$, or two terminals that form a trivial net of $\mathcal{N}'$. We route each trivial net by a single n-wire segment on the bottom layer connecting the two terminals. We assign a column of the top layer to every nontrivial net arbitrarily (each column to at most one net). Since the number of nontrivial nets in $\mathcal{N}'$ is at most $w/2$ and the number of columns is $ns_n \geqslant nw/2n = w/2$, such an assignment exists. The routing of a nontrivial net N_{i_j} consists of two n-wire segments on the bottom layer that connect the two terminals with the column assigned to N_{i_j}, a w-wire segment on the top layer running in the column assigned to N_{i_j} that connects the two rows of the two terminals, and at the endpoints of this w-segment two additional 1-unit h-wire segments. It is easy to verify that the above described routing is correct. $\square$

Figs. 2 and 3 illustrate the above proof. Fig. 2 shows an example for the part of such a routing (with $s_n = 1$). We have two layers of size $w \times (n \cdot s_n)$, the terminals are located in, say, the bottom layer and the top layer is for the wires only. In Fig. 2 wires of the bottom and top layer are represented by continuous and dashed lines, respectively; solid dots denote the terminals; empty dots denote the location of the vias (that is, 1-unit h-wire segments between the two layers); terminal pairs marked with the same number form the nets; $r_1, r_2, \ldots$ and $c_1, c_2, \ldots$ denote the rows and columns, respectively. Fig. 3 shows the corresponding part of the graph G associated with the routing problem of Fig. 2; for every edge, the number of the corresponding net is shown in brackets.

Proposition 5. *Assume that the nets of a SALRP can be partitioned into* h_1 *classes such that each class of nets can be routed as a separate SALRP with a given* $s_w = s'$, $s_n = s''$ *and height* h_0. *Then the original SALRP can be solved with* $s_w = s' + 1$, $s_n = s'' + 1$ *and height* $h = h_1 \cdot h_0$.

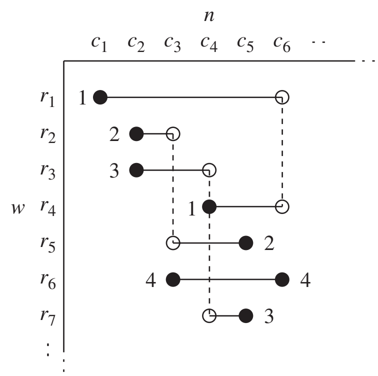

Fig. 2.

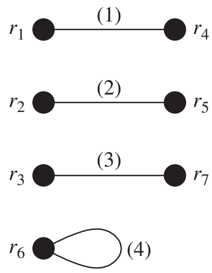

Fig. 3.

Proof. By a *primary row* we mean one of the 'original' w rows that contain the terminals. The remaining $(s_w - 1)w = s'w$ rows will be called *secondary rows*. We are also going to use the terms *primary column* and *secondary column* with an analogous meaning.

We introduce a 'long' h-wire segment starting from every terminal. We route each class of nets on h_0 consecutive layers (thus resulting in a total height of $h_1 \cdot h_0$). For a given class we temporarily introduce a virtual terminal for the terminals belonging to the nets of this class: a virtual terminal is always moved one unit to the right and one unit back from its original position and is placed on the lowest of the h_0 layers belonging to this class. We apply the routing guaranteed by the assumption of the theorem to connect the virtual terminals using the secondary rows and columns only. Finally, we connect every virtual terminal with the h-wire segment coming from its original version by a 1-unit w-wire segment and a 1-unit n-wire segment. (It is easy to verify that these two 1-unit segments cannot cross the routing of any other net. However, one of them can be unnecessary depending on the routing of the virtual terminals). □

Fig. 4 illustrates the above proof. The nets of the SALRP with $w = n = 3$ on the left-hand side are first partitioned into $h_1 = 2$ classes and each class of nets is routed with height $h_0 = 1$ (and $s_w = s_n = 1$). Then the two routings are combined using the above construction to give a solution of the original problem with $s_w = s_n = 2$ and height $h = 2$. Primary and secondary rows are denoted by dashed and dotted lines, respectively; bigger dots denote the terminals, while smaller dots denote the virtual terminals; for each net N, virtual terminals corresponding to the terminals of N are denoted by N'.

Corollary 6. *If each net consists of two terminals only then a SALRP can be solved with* $s_n = \lceil w/2n \rceil + 1$, $s_w = 2$ *and height* $h = 3n$.

Proof. Apply Theorem 4 and Proposition 5. □

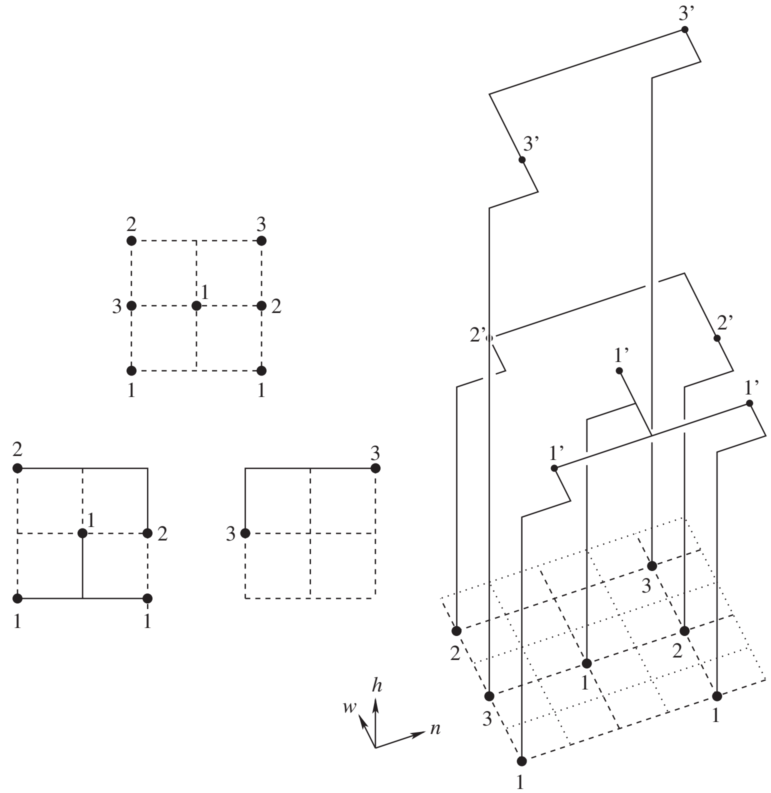

Fig. 4.

Corollary 7. *If each net consists of two terminals only then a SALRP can be solved with* $s_w = s_n = 2$ *and height* $h = 3\max(n, w)$.

Proof. Since the role of w and n is symmetric, $n \geqslant w$ can be assumed without loss of generality. Now the statement follows from Corollary 6. $\square$

If w is much larger than n then the statement of Corollary 6 offers two options: either the height is relatively small ($h = 3n$) and the spacing is large ($s_n = \lceil w/2n \rceil + 1$), or the height is large ($h = 3w$) and then the spacing is only 2. The following theorem claims that a trade-off between these two extremities can be found.

Theorem 8. *If each net consists of two terminals only then a SALRP can be solved with* $s_n \geqslant \lceil w/4n \rceil + 1$, $s_w = 2$ *and height* $h = \left\lfloor \frac{9}{2} n \right\rfloor$.

Proof. Consider the partition of the nets defined in the proof of Theorem 4. We claim that the same partition can also be used to prove the statement of this theorem. Namely we show that each class can be routed as a separate 'nearly SALRP' with $s_n \geqslant \lceil w/4n \rceil$ and height $h = 3$. By a 'nearly SALRP' we mean that the terminals are located on the middle of the three layers, and not on the bottom one. However, this modification will not affect significantly the proof of Proposition 5 (the only change is that virtual terminals have to be placed on the middle layer rather than on the lowest one) and thus will prove the theorem.

The routing of the classes (each class on 3 layers) is again going to be similar to the one described in the proof of Theorem 4 (and in Fig. 2). The only difference is that now we assign columns of the top and bottom layers to nontrivial nets arbitrarily. Since the number of nontrivial nets in each class is again at most $w/2$, and the number of columns on the top and bottom layers altogether is $2n\lceil w/4n \rceil \geqslant w/2$, such an assignment exists. The routing of nontrivial nets is then modified in an obvious way: the two n-wire segments are placed on the middle layer, the single w-wire segment is placed either on the top or on the bottom layer (depending on where the column of the net is) and the 1-unit h-wire segments connect the w-segment with the two n-segments. □

To conclude this section we mention that in case of the constructions presented by Corollaries 6, 7 and Theorem 8 the path connecting the two terminals of a net contains at most 8 bends: connecting a terminal to the corresponding virtual terminal requires 2 bends for each terminal (see Fig. 4) and the realization of the vias (empty dots) of Fig. 2 requires another two bends for each via. It is worth mentioning that applying Proposition 5 to the routing provided by the second part of Theorem 3 yields a routing with height $h = \mathrm{O}(n^{7/5})$ such that each path also contains at most eight bends.

4. Multiterminal nets

In this section we consider the general case in which the nets may have an arbitrary number of terminals. We mention that allowing multiterminal nets is a crucial point in VLSI design, but in practical problems the distribution of the number of terminals in a net is usually not arbitrary (for example, the number of large nets is relatively small). However, it will turn out that allowing arbitrary multiterminal nets will not affect the heights of the obtained routings fundamentally (that is, the height stays within at most twice the height of the 2-terminal case). Therefore, we disregard this aspect of the problem henceforth with reference to what we said in the last paragraph of Section 1. We restrict ourselves to mentioning that if each net contains two terminals only except for a fixed (small) number e of exceptional nets then the bounds of Section 3 remain to be asymptotically true. For example, in case of Corollary 7 a routing with height $h = 3\max(n, w) + e$ can easily be obtained by preserving a separate layer for each net having more than 2 terminals, routing these nets using the extra rows and columns provided by the spacing (as in the proof of Lemma 2) and applying the construction of Corollary 7 for 2-terminal nets.

Getting on to the general problem of multiterminal nets, we first modify the proof of Corollary 6 to get a similar result.

Theorem 9. *Any SALRP can be solved with* $s_n \geqslant \lceil w/2n \rceil + 1$, $s_w = 2$ *and height* $h = 6n$.

Proof. Assume that a single active layer routing problem with w rows and n columns is given. We define the notion of *subnet* for further use. For each net N, we fix an arbitrary order of its terminals: $N = \{t_1, t_2, \ldots, t_u\}$. Now the terminal pairs $\{t_i, t_{i+1}\}$ $(i = 1, 2, \ldots, u - 1)$ are going to be called *subnets* of N. (Thus each net has one less subnet than the number of its terminals).

In what follows, we are going to route each subnet of every net such that the routings of two subnets can intersect only if they belong to the same net. Thus, we obtain a routing of the whole problem. We proceed similarly as in the proof of Theorem 4.

Let again the set of rows be $R = \{r_1, r_2, \ldots, r_w\}$. We define a graph $G(R, E)$ on R with the edges corresponding to the subnets: if the two terminals of a subnet S are located in the rows r_i and r_j then we add the edge $e_S = r_i r_j$ to the edge set E of G. (Again, parallel edges and loops are possible in G).

Now the degree of every vertex of G is at most $2n$ (since each terminal in a row belongs to at most two subnets). By Shannon's theorem [27], the edges can be coloured with $3n$ colours. The edges of G correspond to the subnets, so this colouring induces a partition on the set of subnets.

Again, we introduce a 'long' h-wire segment starting from every terminal. We assign two consecutive layers to each partition class and route the subnets belonging to this class on these two layers. (Thus, the total height of the routing will be $6n$.) Given a partition class, we place virtual terminals for every terminal belonging to any of the subnets of this class on the lowest of the two layers assigned to the class; the position of the virtual terminals is the same as in Proposition 5: it is one unit to the right and one unit behind its original position. Now we route the virtual terminals using the secondary rows and columns of the two layers. The routing is identical to the one described in Theorem 4 (and Fig. 2). To complete the routing, we connect every virtual terminal with the h-wire segment coming from its original version in the same way as in Proposition 5: through a 1-unit w-wire segment and a 1-unit n-wire segment. □

Corollary 10. *Any SALRP can be solved with* $s_w = s_n = 2$ *and height* $h = 6\max(n, w)$.

Proof. Assume $n \geqslant w$ without loss of generality and apply Theorem 9. □

Similarly as it was the case with Theorem 8, the statement and the proof of Theorem 9 can again be modified to achieve a smaller spacing in return for an increase in the height.

Theorem 11. *Any SALRP can be solved with* $s_n \geqslant \lceil w/4n \rceil + 1$, $s_w = 2$ *and height* $h = 9n$.

5. Algorithmic aspects

In order to actually generate the routings provided by the above proofs, the only real task is to edge-colour the graph assigned to the routing problems. We omit all the other details of the algorithms, we restrict ourselves to mentioning that these can be performed in linear time in the area $A = w \cdot n$ of the (planar) grid, that is, in the size of the input.

The folklore proof of Shannon's theorem using Kempe-chains (see e.g. [24]) involves an algorithm to edge-colour a multigraph with v vertices, m edges and maximum degree Δ in $O(m \cdot (\Delta + v))$ time (using at most $\lfloor \frac{3}{2}\Delta \rfloor$ colours). This implies that, if t denotes the number of nets, each of the above constructions can be performed in $O(t \cdot (w + n))$ time. Since the size of the input is $A = w \cdot n$, the presented algorithms work in $O(A^{3/2})$ time (in the $w = \Theta(n)$ case).

However, it is an alternative to use the trivial greedy method for edge-colouring. In this case the number of colours increases to at most $2\Delta - 1$ and therefore the height of the routing also increases (in case of Theorem 9, for example, from $h = 6n$ to $h = 8n$), but remains to be linear in $\max(n, w)$. As a consequence of the increase in the height, the routing algorithm also becomes linear in the size A of the input.

6. Summary

All the results of this paper show that if $s_w, s_n \geqslant 2$ then the three-dimensional single active layer routing problem can be solved with linear height in $\max(n, w)$. However, the exact height of the routings provided by the various constructions depend on different conditions: whether we restrict ourselves to 2-terminal nets or we allow multiterminal nets, the width w and length n of the routing problem and the spacing s_w and s_n.

An obvious measure to compare various solutions of the same routing problem is the volume of the routing, that is, the product $(w \cdot s_w) \cdot (n \cdot s_n) \cdot h$. Assume $w \geqslant n$ and denote the ratio w/n by m. The following table shows the volume of the routings provided by the constructions of this paper (in each case, the solution with the smaller volume is shown in bold letters).

	2-terminal nets		Multiterminal nets	
	Corollary 6	Theorem 8	Theorem 9	Theorem 11
$m \leqslant 2$	$\mathbf{12wn^2}$	$18wn^2$	$\mathbf{24wn^2}$	$36wn^2$
$2 < m \leqslant 4$	$\mathbf{18wn^2}$	$\mathbf{18wn^2}$	$\mathbf{36wn^2}$	$\mathbf{36wn^2}$
$4 < m \leqslant 6$	$\mathbf{24wn^2}$	$27wn^2$	$\mathbf{48wn^2}$	$54wn^2$
$6 < m \leqslant 8$	$30wn^2$	$\mathbf{27wn^2}$	$60wn^2$	$\mathbf{54wn^2}$
$8 < m \leqslant 10$	$\mathbf{36wn^2}$	$\mathbf{36wn^2}$	$\mathbf{72wn^2}$	$\mathbf{72wn^2}$
$10 < m \leqslant 12$	$42wn^2$	$\mathbf{36wn^2}$	$84wn^2$	$\mathbf{72wn^2}$
$12 < m \leqslant 14$	$48wn^2$	$\mathbf{45wn^2}$	$96wn^2$	$\mathbf{90wn^2}$
$14 < m \leqslant 16$	$54wn^2$	$\mathbf{45wn^2}$	$108wn^2$	$\mathbf{90wn^2}$

We mention, however, that the routings provided by Corollaries 6, 7 and 10 and Theorem 9 belong to the multilayer Manhattan model, that is, each layer contains either w-wire segments or n-wire segments only and consecutive layers contain wire segments of different directions. On the other hand, this does not hold for the routings of Theorems 8 and 11, because these involve consecutive layers containing w-segments (although it is still true that no layer contains both w-wire segments and n-wire segments).

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