Maps of matroids with applications

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Abstract: Various (mostly well known) properties of strong and weak maps are surveyed and some probably new strong map constructions are presented. Then a civil engineering problem is described which served as the original motivation of this study. Since some of the tools were originally developed for applying matroids in electrical engineering, the analogue of some results is also given in network analysis.

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1 Maps of matroids

Let $\mathcal{A} = (S, \mathcal{I}_A)$ and $\mathcal{B} = (S, \mathcal{I}_B)$ be two matroids with the same underlying set S and with collections $\mathcal{I}_A, \mathcal{I}_B$ of their respective independent sets. (For concepts and notations in matroid theory the reader is referred to [10, 12, 17].) Their rank functions will be denoted by r_A and r_B, respectively. A subset $X \subseteq S$ is *closed* if $r(X \cup \{x\}) > r(X)$ for every $x \in S - X$.

Recall that these matroids are in the *strong map relation* $\mathcal{A} \Rightarrow \mathcal{B}$ if every closed set of $\mathcal{B}$ is closed in $\mathcal{A}$ as well and they are in the *weak map relation* $\mathcal{A} \to \mathcal{B}$ if every independent set of $\mathcal{B}$ is independent in $\mathcal{A}$ as well. Some basic properties of these maps are surveyed first, see also [2, 6, 7]

Lemma 1 *The strong map relation implies the weak one.*

Proof: Suppose that $X \subseteq S$ belongs to $\mathcal{I}_B$ but not to $\mathcal{I}_A$. Then there is a subset $C \subseteq X$ which is a circuit in $\mathcal{A}$ but independent in $\mathcal{B}$. Let $x \in C$ and consider the closure D of $C - \{x\}$ in $\mathcal{B}$. Clearly, $x \notin D$ since $C \in \mathcal{I}_B$ implies $r_B(C) > r_B(D)$. But the strong map relation implies that D is closed in $\mathcal{A}$ as well, leading to $x \in D$, a contradiction. $\square$

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Both maps imply that $r_B(S) \leq r_A(S)$. Equality implies $\mathcal{A} = \mathcal{B}$ in case of strong maps but there are nontrivial rank-preserving weak maps as well, see some examples later.

Probably the simplest way for constructing strong maps is the single element contraction. (Unlike in some texts, here we suppose that the contracted element remains in the matroid as a single element circuit or *loop*.)

Lemma 2 *Let $\mathcal{A} = (S, \mathcal{I}_A)$ and $x \in S$ and consider the contraction $\mathcal{B} = \mathcal{A}/x$. Then $\mathcal{A} \Rightarrow \mathcal{B}$.*

PROOF: If X is a closed set of $\mathcal{B}$, it contains x. Suppose that X is not closed in $\mathcal{A}$, hence there exists an element $y \in S - X$ and a circuit C in $\mathcal{A}$ so that $C - \{y\} \subseteq X$. If $x \in C$ then $C - \{x\}$ is a circuit in $\mathcal{B}$ hence X cannot be closed. If $x \notin C$ then $(C - \{y\}) \cup \{x\}$ is independent in $\mathcal{A}$, thus $C - \{y\}$ is independent in $\mathcal{B}$. Then C is a circuit in $\mathcal{B}$ hence again, X cannot be closed, a contradiction. □

Another important example of strong maps is the *truncation* $\hat{\mathcal{A}} = (S, \mathcal{I})$ of a matroid $\mathcal{A}$ with $r_A(S) = r$ where $\mathcal{I} = \{X | X \in \mathcal{I}_A$ and $|X| \leq r - 1\}$.

Lemma 3 $\mathcal{A} \Rightarrow \hat{\mathcal{A}}$.

PROOF: Let X be a closed set of $\hat{\mathcal{A}}$. If X is not closed in $\mathcal{A}$, it has a subset $C \subseteq X$ so that $C \cup \{x\}$ is a circuit in $\mathcal{A}$ for some $x \in S - X$. Hence $C \in \mathcal{I}_A$. If it is a base of $\mathcal{A}$ then its closure in $\hat{\mathcal{A}}$ were the whole set S contradicting to $x \notin X$. Otherwise C is independent in $\hat{\mathcal{A}}$ and $C \cup \{x\}$ is a circuit in $\hat{\mathcal{A}}$ as well, but then X could not be closed in $\hat{\mathcal{A}}$, a contradiction. □

If x is an arbitrary non-loop element of $\mathcal{A}$ and $\mathcal{B}$ denotes its contraction $\mathcal{A}/x$ (in the sense above) then $\hat{\mathcal{A}} \to \mathcal{B}$ holds since all the $\leq r-1$ element independent subsets of $\mathcal{A}$ are independent in $\hat{\mathcal{A}}$ but only some of them are independent in $\mathcal{B}$ (namely those which remain independent in $\mathcal{A}$ adding the element x). In fact, the truncation is the free-est among the nontrivial strong maps.

Any matroid $\mathcal{L}$ whose truncation is $\mathcal{A}$ is called a *lift* of $\mathcal{A}$. The *free lift* $\mathcal{A}^{FL}$ of $\mathcal{A}$ is defined so that $\{X | X \subseteq S, |X| = r + 1$ and X contains a base of $\mathcal{A}\}$ is the set of its bases. Clearly, $\mathcal{A}^{FL} \to \mathcal{L}$ for every lift $\mathcal{L}$ of $\mathcal{A}$. One can also see that $(\mathcal{A}^{FL})^*$ is just the truncation of $\mathcal{A}^*$.

A large family of other lifts of a matroid $\mathcal{A} = (S, \mathcal{I}_A)$ can be constructed as follows [16].

Theorem 4 *Fix a subset R of S (the "red" elements). If a subset $X \subseteq S$ with $|X| = r + 1$ contains a base of $\mathcal{A}$ then it contains a unique circuit of $\mathcal{A}$. Accept such a subset as a base of $\mathcal{A}^R$ only if this circuit has at least one red element. Then $\mathcal{A}^R$ is a matroid and $\mathcal{A}^R \Rightarrow \mathcal{A}$.*

PROOF: Define a rank one matroid $\mathcal{P}_R$ on S so that its bases are the red elements. One can easily see that $\mathcal{A}^R = \mathcal{A} \vee \mathcal{P}_R$. The second statement $\mathcal{A}^R \Rightarrow \mathcal{A}$ will follow from the statement of Lemma 8 below. □

2 Some further constructions for representable matroids

In what follows, suppose that $\mathcal{A} = (S, \mathcal{I}_A)$ is representable over the field $\mathbb{R}$ of the reals. Without loss of generality we may suppose that it is represented by a matrix $\mathbf{A}$ of size $r \times n$ (where

$r = r(S), n = |S|$) and then all the rows of $\mathbf{A}$ are linearly independent. If $\mathbf{T}$ is a nonsingular $r \times r$ matrix then the product $\mathbf{TA}$ represents the same matroid $\mathcal{A}$.

Let $\mathbf{T}$ be an arbitrary (possibly singular) $r \times r$ matrix and let $\mathcal{B}$ be the matroid represented by the matrix $\mathbf{TA} = \mathbf{B}$. One can immediately see that $\mathcal{A} \Rightarrow \mathcal{B}$. Let namely X be a closed subset of $\mathcal{B}$ and suppose that it is not closed in $\mathcal{A}$. Then there exists an element $x \in S - X$ so that the corresponding column vector $\mathbf{x}$ of $\mathbf{A}$ arises as the linear combination of some X-columns of $\mathbf{A}$. However, then $\mathbf{Tx}$ also arises as the linear combination of the corresponding columns of $\mathbf{TA}$, a contradiction.

Using this observation one can obtain much shorter proofs for some statements of the previous section if the matroid $\mathcal{A}$ is representable. For example, if x is a non-loop element of S, we may suppose that the corresponding column vector $\mathbf{x}$ of $\mathcal{A}$ has only zeroes in the first $r-1$ entries (since either $\mathbf{x} = \mathbf{0}$ if x was a loop or $\mathbf{x}$ can be transformed to be the last unity vector $\mathbf{e}_r$). Then using the $r \times r$ diagonal matrix $\mathbf{T}_1 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}$ one can immediately see that $\mathbf{T}_1\mathbf{A}$ represents the contraction matroid $\mathcal{A}/x$ (cf. Lemma 2).

Similarly, one can obtain the representation of the truncation $\hat{\mathcal{A}}$ of the matroid $\mathcal{A}$ as follows. Let $\mathbf{A}$ represent the original matroid $\mathcal{A}$ so that each entry of $\mathbf{A}$ is an integer. Let $\mathbf{T}_2$ be an $r \times r$ matrix whose entries are algebraically independent over the field $\mathbb{Q}$ of the rationals except the relation $\det \mathbf{T}_2 = 0$. (This is possible, for example select $r^2 - 1$ algebraically independent reals for all the entries except $t_{r,r}$ and then express $t_{r,r}$ as their function from the equation $\det \mathbf{T}_2 = 0$.) In this case $\mathbf{T}_2\mathbf{A}$ represents the truncation $\hat{\mathcal{A}}$ of the matroid (cf. Lemma 3).

Finally, if $\mathcal{A}^R = \mathcal{A} \vee \mathcal{P}_R$ then, by [3] it can be represented over $\mathbb{R}$ by such an $(r+1) \times n$ matrix $\mathbf{M}$ whose first r rows (as an $r \times n$ matrix) represent $\mathcal{A}$ and whose last row is the characteristic vector of the set R. Then using the $(r+1) \times (r+1)$ diagonal matrix $\mathbf{T}_3 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}$ one obtains that $\mathcal{A}^R \Rightarrow \mathcal{A}$ since $\mathbf{T}_3\mathbf{M}$ represents $\mathcal{A}$ (cf. Theorem 4).

A "new" strong map construction can be obtained as follows. Let the $r \times n$ matrix $\mathbf{A}$ represent $\mathcal{A}$ over $\mathbb{R}$ and extend $\mathbf{A}$ with an arbitrary additional nonzero column vector $\mathbf{w}$. Using an appropriate nonsingular $r \times r$ matrix $\mathbf{T}_4$ one can obtain $\mathbf{T}_4(\mathbf{A}|\mathbf{w}) = (\mathbf{A}'|\mathbf{e}_r)$ where $\mathbf{e}_r$ is the rth unit vector. Then $\mathbf{A}'$ still represents $\mathcal{A}$ and if we multiply it with the above $r \times r$ diagonal matrix $\mathbf{T}_1 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}$ then $\mathbf{T}_1\mathbf{A}'$ represents a matroid $\mathcal{A}_w$ with $\mathcal{A} \Rightarrow \mathcal{A}_w$.

This construction can easily be visualized if $\mathcal{A}$ is graphic. Let $G = (V, E)$ be a connected graph with $|V| = r+1$ vertices and $|E| = n$ edges and $\mathcal{A}$ be its circuit matroid. Assign arbitrary orientation to the edges of G and let $\mathbf{A}$ be the vertex-edge incidence matrix of G. The sum of the row vectors is zero and $r(\mathbf{A}) = \mathbf{r}$. Define a weight function $w : V \to \mathbb{R}$ on the vertex set satisfying $\sum_{v\in V} w(v) = 0$ and $w \not\equiv 0$. Extend the matrix $\mathbf{A}$ with the vector $\mathbf{w}$ composed of the vertex weights. Then the above construction reduces to the following result [15]:

Theorem 5 *The asymmetric 2-component forests form the bases of a matroid on the edge set E where a forest is called* **asymmetric** *if the sums of the vertex weights $\sum w(v)$ taken separately to the two components are nonzero.*

Using this concept we can give one-line proofs for Lemmata 2–3 if $\mathcal{A}$ is graphic. Let $x = \{p, q\}$ be the edge to be contracted. Then $\mathcal{A}/x$ is obtained by the weight function $\{w(p) = 1, w(q) = -1, w(r) = 0$ for all the other vertices$\}$. Similarly, if p is an arbitrary vertex then the truncation $\hat{\mathcal{A}}$ is obtained by the following weight function: Let $\{w(q)|q \neq p\}$ be algebraically independent reals and $w(p)$ be their negative sum.

Although we shall not need it for the forthcoming applications, we present one more construction. Like in case of Lemma 4, let $\mathcal{A} = (S, \mathcal{I}_A)$ be a rank r matroid and let $R \subset S$ be the subset of its "red" elements. Suppose that $\mathcal{A}$ is representable over $\mathbb{R}$ and let $\mathbf{A}$ be an $r \times n$ matrix with integer entries only, which represents $\mathcal{A}$. Extend $\mathbf{A}$ with an additional row obtained as the sum of the other rows. Replace the entries of the "red" columns of the extended matrix with algebraically independent reals. The resulting matrix represent another matroid $\mathcal{A}^{[R]}$.

Lemma 6 $\mathcal{A}^{[R]} \to \mathcal{A}$ *and if* $|R| = 1$ *then* $\mathcal{A}^{[R]} \cong \mathcal{A}^R$.

The statements are obvious. However, unlike in case of $\mathcal{A}^R$ the strong map relation $\mathcal{A}^{[R]} \Rightarrow \mathcal{A}$ does not always hold. For example, let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ and let R be formed by the third and sixth columns. Then the subset of the first three columns is closed in $\mathcal{A}$ but will not be closed in $\mathcal{A}^{[R]}$.

3 Further strong map constructions

Lemma 7 *If $\mathcal{A} \Rightarrow \mathcal{B}$ then for their duals $\mathcal{B}^* \Rightarrow \mathcal{A}^*$ holds.*

Proof: Let X be closed in $\mathcal{A}^*$. Then for every element $x \in S - X$ the rank of $X \cup \{x\}$ is greater than that of X in $\mathcal{A}^*$. Using the formula $r^*(X) = |X| + r(S - X) - r(S)$ we obtain $r_A(S-X) = r_A(S-X-\{x\})$. Suppose indirectly that there were an $x \in S - X$ so that the rank of $X \cup \{x\}$ is equal to that of X in $\mathcal{B}^*$. In the same way this implies $r_B(S-X) > r_B(S-X-\{x\})$.

Consider the closure of $S - X - \{x\}$ in $\mathcal{B}$. This is closed in $\mathcal{B}$ (hence in $\mathcal{A}$ as well) and does not contain x, leading to $r_A(S - X) > r_A(S - X - \{x\})$, a contradiction. $\square$

Lemma 8 *If $\mathcal{A}, \mathcal{B}$ are arbitrary matroids then for their union (or sum) $\mathcal{A} \vee \mathcal{B} \Rightarrow \mathcal{A}$ holds.*

PROOF: Let X be a closed set in $\mathcal{A}$ and let $r_{A\vee B}(X)=k$. Indirectly suppose that there exists an element $x \in S-X$ so that $r_{A\vee B}(X\cup\{x\})=k$. Let T be a k-element subset of X which is independent in $\mathcal{A}\vee\mathcal{B}$. T arises as $T_A\cup T_B$ where $T_A\in\mathcal{I}_A$ and $T_B\in\mathcal{I}_B$ and without loss of generality we may suppose that $|T_A|=r_A(X)$. Now $|T\cup\{x\}|>k=r_{A\vee B}(X\cup\{x\})$ implies that $T\cup\{x\}\notin\mathcal{I}_{A\vee B}$, hence $T_A\cup\{x\}\notin\mathcal{I}_A$. Thus X cannot be closed in $\mathcal{A}$, a contradiction. □

The operation $\mathcal{A}\wedge\mathcal{B}=(\mathcal{A}^*\vee\mathcal{B}^*)^*$, sometimes also called the *product* of two matroids, has independently been introduced by several authors [5, 9, 11]. A straightforward consequence of the above two lemmata is that $\mathcal{A}\Rightarrow\mathcal{A}\wedge\mathcal{B}$ holds.

Theorem 9 *Let $\mathcal{A}=(S,\mathcal{I}_A)$ be a rank r matroid. Let $S=B\cup R$ be a decomposition of the underlying set into disjoint "black" and "red" subsets and suppose that $r(B)=r$. Then if the set $\{X|X\subseteq B, X\in\mathcal{I}_A, |X|=r-1$ and there exists an $x\in R$ so that $X\cup\{x\}\in\mathcal{I}_A\}$ is nonempty then it is a collection of bases of a matroid $\mathcal{A}_R$ on B and $\mathcal{A}\setminus R\Rightarrow\mathcal{A}_R$ where $\setminus$ denotes the deletion of a subset from a matroid.*

PROOF: A set $X\subseteq B$ is in $\mathcal{X}$ if and only if $X\cup\{x\}$ is a base in $\mathcal{A}$, that is, $S-X-\{x\}$ is a base in $\mathcal{A}^*$. Using the matroid $\mathcal{P}_R$ (see the proof of Lemma 4) this is equivalent to $S-X$ being a base in $\mathcal{A}^*\vee\mathcal{P}_R$. Hence $\mathcal{A}_R$ is simply $(\mathcal{A}\wedge{\mathcal{P}_R}^*)\setminus R$. □

To avoid any possible confusion with $\mathcal{A}^R$ introduced in Lemma 4 observe that the rank of S increases in $\mathcal{A}^R$ and decreases in $\mathcal{A}_R$.

Theorem 10 *Let $\mathcal{A}=(S,\mathcal{I}_A)$ be a rank r matroid. Let $S=B\cup R\cup G$ be a decomposition of the underlying set into disjoint "black", "red" and "green" subsets and suppose that $r(B)=r$. Then if the set $\{X|X\subseteq B, X\in\mathcal{I}_A, |X|=r-2$ and there exists an $x\in R$ and a $y\in G$ so that $X\cup\{x,y\}\in\mathcal{I}_A\}$ is nonempty then it is a collection of bases of a matroid $\mathcal{A}_{R,G}$ on B and $\mathcal{A}\setminus(R\cup G)\Rightarrow\mathcal{A}_{R,G}$.*

The *proof* follows the same line, leading to $\mathcal{A}_{R,G}=[(\mathcal{A}\wedge{\mathcal{P}_R}^*)\wedge{\mathcal{P}_G}^*]\setminus(R\cup G)$. □

4 An application for bar and joint frameworks

Consider a $k\times l$ square grid as a planar framework composed of rigid bars and flexible joints. If diagonal rods are added to some squares then the resulting framework may become rigid (Figure 1a) or may remain nonrigid (Figures 1b and 1c).

Deformations like on Figure 1c may be obtained by successive deformations of whole rows or columns of squares and the presence of a diagonal in a square only means that the magnitude of the deformations were the same for the row and for the column of that square. Hence if the rows and the columns correspond to the vertices of a bipartite graph and the diagonals correspond to edges of this graph in a straightforward way (see the bottom part of Figure 1) then the framework is rigid if and only if the corresponding bipartite graph is connected [1]. In particular, $k+l-1$ is the minimum number of diagonal braces required for the rigidity of the framework (corresponding to the trees of the complete bipartite graph $K_{k,l}$).

Consider now the more complicated framework of Figure 2 which was obtained from a $k\times l$ square grid by removing an $s\times u$ portion $(s,u\geq 2)^*$. Then a minimal system of diagonals which

*See also [4] for rigidity properties of square grids containing "holes".

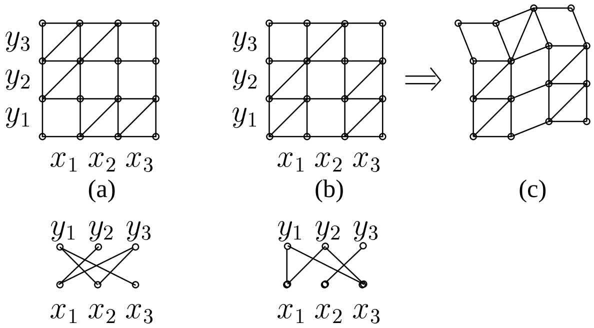


Figure 1

can make this system rigid will consist of $k + l + s + u - 3$ elements. In order to characterize these minimal systems, first we replace the previous bipartite graph model $K_{k,l}$ with a more complicated one $G_{k,l,s,u}$ as follows. (Observe that the edges of $G_{k,l,s,u}$ will remain the possible candidates of diagonals.)

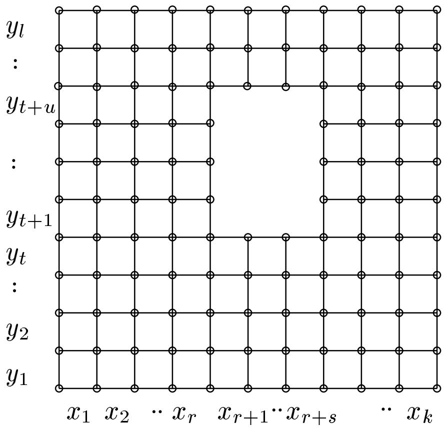


Figure 2

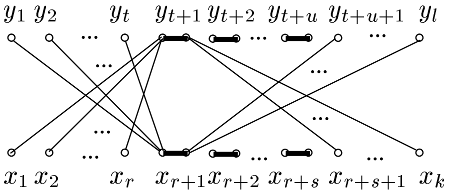


Figure 3

Let us replace the vertices $y_{t+1}, y_{t+2}, \ldots, y_{t+u}$ each by a pair of vertices (referring to the left hand side rows and the right hand side rows of level $t+1, t+2, \ldots, t+u$, respectively) and join these pairs of vertices by "red" edges. Similarly, replace the vertices $x_{r+1}, x_{r+2}, \ldots, x_{r+s}$ by pairs of vertices joined by "green" edges. (See Figure 3 where both the red and the green edges are denoted by heavy lines.) The original pairs of vertices $\{x_i, y_j\}$ remain adjacent if either $i \leq r$ or $i > r + s$ and either $j \leq t$ or $j > t + u$. Furthermore join each vertex x_i with $i \leq r$ (or $i > r + s$) to the left hand side (right hand side, respectively) copy of the red edges and join each vertex y_j with $j \leq t$ (or $j > t+u$) with the left hand side (right hand side, respectively) copies of the green edges.

Theorem 11 *Remove an $s \times u$ portion ($s, u \geq 2$) from a $k \times l$ square grid (Figure 2). Then a set of $k+l+s+u-3$ diagonals makes the above framework rigid if and only if the corresponding edge set is a base in $\mathcal{A}_{R,G}$, see Theorem 10 where $\mathcal{A}$ is the circuit matroid of the graph $G_{k,l,s,u}$ and R, G are the subsets of the red and green edges, respectively. In other words, the diagonals must form a 3-component forest F and there must exist a red edge x and a green edge y so that $F \cup \{x, y\}$ is a tree in $G_{k,l,s,u}$.*

Example 12 *Let $k = 5, l = 4, s = 3, t = 2$ and consider the three different bracings with 11 diagonals each in Figure 4, each with the corresponding 11-edge subgraphs of $G_{5,4,3,2}$. The first subgraph has 4 components – one of them contains a circuit (x_1, y_1, x_5, y_4)–, the second one is a 3-component forest but both red edges form circuits when added to the component with 9 vertices. Hence the first two frameworks are nonrigid, see deformations in Figure 5, while the last one is rigid.*

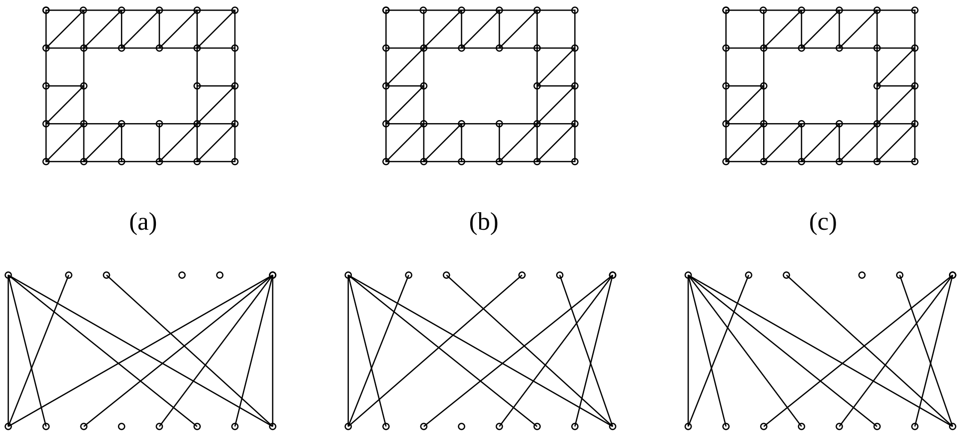


Figure 4

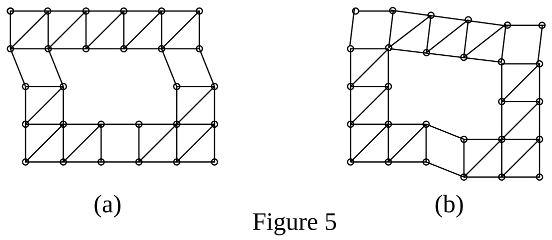


Figure 5

If the four corners of a $k \times l$ square grid are pinned down to the plane, we only need $k+l-2$ diagonals to prevent any (infinitesimal) motions†. Such a collection of $k+l-2$ braces

†If we consider a one-story building – the simplest 3-dimensional cubic grid – and all the four external vertical walls contain at least one diagonal brace each then the roof can be modelled with such a square grid [1]. This remark motivates the study of such configurations.

is appropriate if and only if the corresponding subgraph of $K_{k,l}$ is an asymmetric 2-component forest (see also Theorem 5). In general, a 2-component subgraph of a bipartite graph is called *asymmetric* if the ratios of the sizes of the bipartition classes in the subgraphs are different.

Theorem 13 *Remove an $s \times u$ portion ($s, u \geq 2$) from a $k \times l$ square grid (Figure 3) and fix its four external corners to the plane. Then a set of $k + l + s + u - 4$ diagonals makes this system rigid if and only if*

(1) the corresponding edge set is a 4-component forest F in the graph $G_{k,l,s,u}$,

(2) there exist a red edge x and a green edge y so that $F \cup \{x, y\}$ is a 2-component forest in $G_{k,l,s,u}$,

(3) and contracting all the red and green edges the result is an asymmetric 2-component subgraph in $K_{k,l}$.

Example 14 *Let $k = 6, l = 4, s = 3, t = 2$ and consider the two bracings with 11 diagonals each in Figure 6, with the corresponding 11-edge subgraphs of $G_{6,4,3,2}$. Both systems satisfy the first two conditions but the second system violates (3), leading to the deformation shown in Figure 7.*

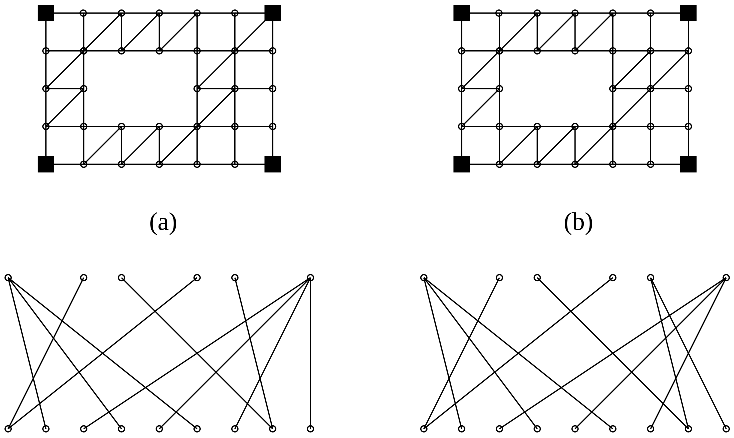


Figure 6

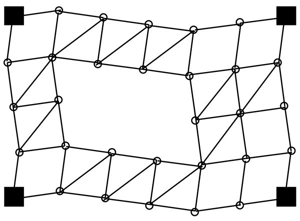
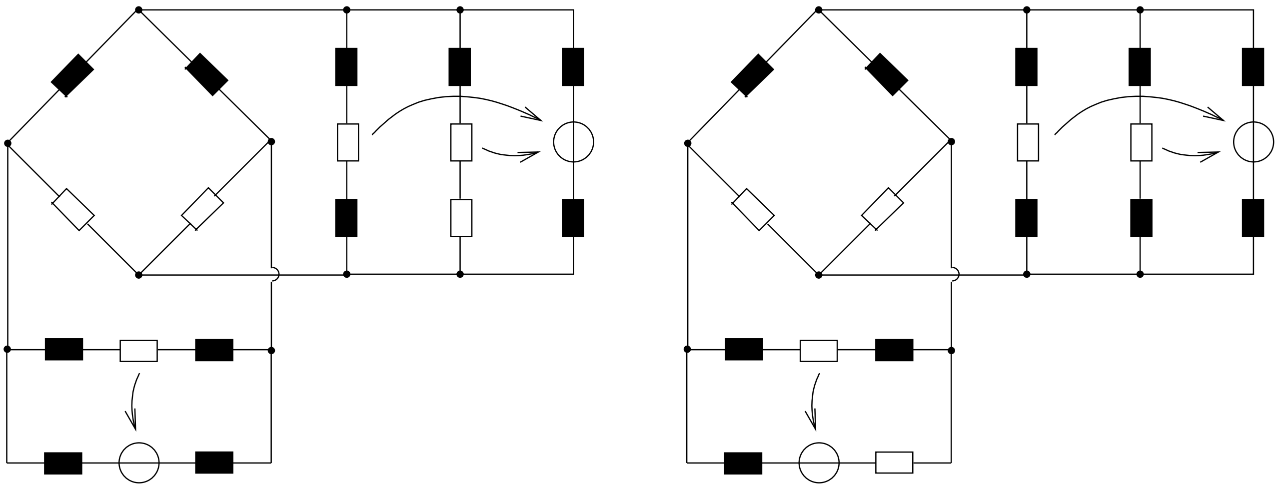


Figure 7

5 An analogous phenomenon for electric networks

The matroid sum and product proved to be the appropriate operations for describing the rigidity of these frameworks. These matroidal tools have been widely used to describe the independence of the voltages and currents in electric networks containing controlled sources [5, 8, 9, 11], see also some attempts to study these two areas in a unified way [13, 14]. In particular, the effects of controlled voltage (and current) sources could be modelled by the product (or sum, respectively) of the circuit matroid of the graph of the network and the rank-one matroid $\mathcal{P}_R$ where R is the set of elements involved in the control. For curiosity, consider again the second and third frameworks of Figure 4. Given the "almost bipartite" structure of $G_{5,4,3,2}$ one can find an electric network with four resistors corresponding to the corners, two parallel branches each with two series resistors plus a controlled voltage source, corresponding to the red edges and three similar parallel branches, corresponding to the green edges. Then the bracing of the two frameworks is equivalent to prescribing the voltages of the shaded resistors of Figure 9, which uniquely determines every voltage and current in case of system (c) but leads to a contradiction in case of system (b).



(b) (c)

Figure 8

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