Square Grids with Long "Diagonals"

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March 29, 1999

Abstract

Bolker and Crapo gave a graph theoretical model of square grid frameworks with diagonal rods of certain squares. Baglivo and Graver solved the problem of tensegrity frameworks where diagonal cables may be used in the square grid to make it rigid. The problem of one-story buildings in both cases can be reduced to the planar problems. These results are generalized if some longer rods, respectively some longer cables are also permitted.

Keywords: grids, rigidity, frameworks, graphs, polyhedral methods.

1 Preliminaries

Throughout, every *framework* consists of rigid rods and rotatable joints, first in the *plane*, and later – in the case of the one-story buildings – in the *3-dimensional space*, and *rigidity* stands for infinitesimal rigidity. Elementary squares of a square grid will be called *rooms*, each room will be located by the numbers of its row and column. All rooms with the same row number or the same column number will be called a horizontal, respectively vertical *corridor*.

Let F be a framework composed of a $k \times l$ grid with certain additional rods forming diagonals of some rooms. Its *graph* $G(F)$ is a bipartite graph with vertex set $V(G(F)) = A \cup B$ where $|A| = k, |B| = l$; the vertices correspond to the corridors in a straightforward way and two vertices X_i, Y_j with $X_i \in A$ and $Y_j \in B$ are connected by an edge if and only if there is a diagonal in the room, formed by the intersection of row i and column j. For example, Figures 1a and 1b show a framework and its graph, respectively.

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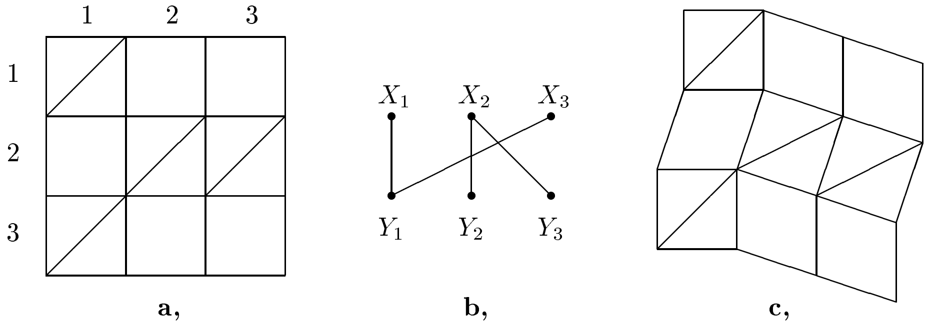


Figure 1

The main result of Bolker and Crapo [1] is the following:

Theorem 1.1 *Such a framework F is rigid if and only if its graph $G(F)$ is connected. Hence a collection C of certain diagonals is a minimum set making F rigid if and only if $|C| = k+l-1$ and the corresponding edges form a spanning tree of $G(F)$.*

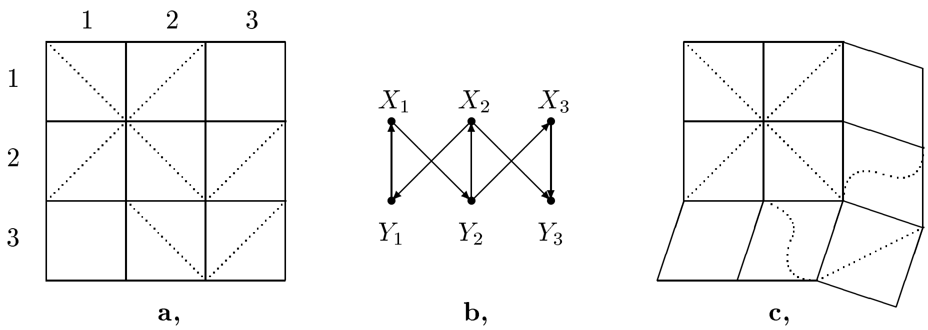
For example, the graph of Figure 1b is disconnected, hence the framework of Figure 1a has deformations, like that of Figure 1c.

Remark. Since a $k \times l$ grid, as a framework, clearly has $j = (k+1)(l+1)$ joints and $r = k(l+1)+l(k+1)$ rods and since a minimum rigid planar framework with j joints (≥ 2) must have $R = 2j-3$ rods, the relation $|C| = R-r = k+l-1$ is obvious. However, given a framework composed of a $k \times l$ square grid plus a system of $k+l-1$ diagonals, the size of the rigidity matrix would be $O(kl)$, hence checking its rank with, say, Gaussian elimination would require $O(k^3l^3)$ operations. By contrast, checking the condition of Theorem 1.1 requires $O(k+l)$ operations only.

In the second case we can define a directed bipartite graph to the framework where $V(\vec{G}(F)) = A \cup B$ as above, and there is a directed edge from $X_i \in A$ to $Y_j \in B$ if and only if there is a rod or a Southwest-Northeast position cable in the room $(i;j)$, and similarly there is a directed edge from $Y_j \in B$ to $X_i \in A$ if and only if there is a rod or a Southeast-Northwest position cable in the room $(i;j)$.

Theorem 1.2 (Baglivo and Graver [2]) *A tensegrity framework is infinitesimally rigid if and only if its graph $\vec{G}(F)$ is strongly connected.*

For example in Figure 2 the directed graph of the system is not strongly connected, so we can see a deformation of the system in Figure 2c.



a, **b,** **c,**

Figure 2

Remark. Since each point must have indegree and outdegree at least 1, it requires at least $2\max(k,l)$ cables to make the grid rigid if we use only cables, and so many cables are sufficient.

The *one-story building* means that the joints of a $k \times l$ grid are connected to the ground by rods of uniform length. If each external vertical wall contains a diagonal rod then the problem of determining the rigidity of a one-story building reduces to the problem of determining whether the square grid is infinitesimally rigid if its corners are fixed to the plane.

Theorem 1.3 **([3])** *A framework of a one-story building which has rods in the external vertical walls is made rigid by certain diagonal rods if and only if $G(F)$ is connected or is an asymmetric 2-component graph where asymmetric means that*

$$\begin{vmatrix} |G_1 \cap A| & |G_1 \cap B| \\ |G_2 \cap A| & |G_2 \cap B| \end{vmatrix} \neq 0$$

where G_1 and G_2 are the vertex sets of the two connected components of G, A and B are the two vertex set of the bipartite graph.

So the minimum number of diagonal rods required is $k + l - 2$ (in addition to the 4 diagonals in the walls).

Using also diagonal cables yields the following

Theorem 1.4 **([4])** *Let $k, l \geq 2$ and $k + l \geq 5$. Then at least $k + l - 1$ diagonal cables are required to make the system infinitesimally rigid, and that number will always do.*

We have to mention that all theorems in this section are true if we consider a grid of rectangles.

Consider now a framework, composed again of a $k \times l$ square grid plus some longer rods or longer cables but some of these latter are not diagonals of rooms but connect joints of distance greater than $\sqrt{2}$ times the grid unit. To what extent can we generalize the previous theorems for such frameworks?

In Section 2 we give a full solution for the case of long rods in the planar square grid. The case of long cables is presented in Section 3. These results are then generalized for the case of one-story buildings in Section 4. Finally we show some algorithmic aspects of these problems.

2 Long rods in the framework

Let us consider first if there is only one long rod and there are several other "short" diagonal rods in a planar square grid. It is easy to see that a horizontal or a vertical long rod won't help us, because they don't prevent the infinitesimal motions. So we are not interested in these cases.

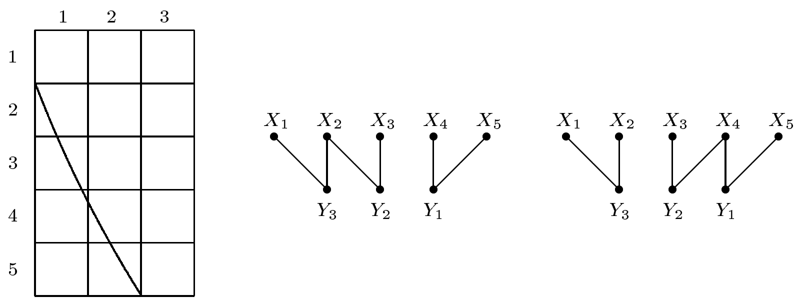
Now, let us see for example the framework in Figure 3a. The long rod intersects the horizontal corridors 2, 3, 4 and 5 and the vertical corridors 1 and 2. The corresponding subset of the vertex set of $G(F)$ $V_0 = \{X_2, X_3, X_4, X_5, Y_1, Y_2\}$ will be called the *range* of the long rod. It is obvious that both $A_0 = V_0 \cap A$ and $B_0 = V_0 \cap B$ are nonempty, i.e. that the long rod is neither horizontal nor vertical.

Let T be a 2-component graph on the vertex set of $G(F)$ with components T_1 and T_2. Let us denote their vertex sets $V(T_1)$ and $V(T_2)$ by V_1 and by V_2 respectively. Observe that if a subset, say B_0, intersects both V_1 and V_2 then

- either $|A_0 \cap V_1| \cdot |B_0| = |B_0 \cap V_1| \cdot |A_0|$ and $|A_0 \cap V_2| \cdot |B_0| = |B_0 \cap V_2| \cdot |A_0|$ hold simultaneously
- or none of them is true.

Accordingly, we shall call the 2-component graph T *symmetric* or *asymmetric*, respectively, *with respect to the range* $V_0 = A_0 \cup B_0$.

For example, $k = 5$ and $l = 3$ for the framework of Figure 3a. Every 2-component forest of $G(F)$ has $k + l - 2 = 6$ edges. That of Figure 3b is symmetric (since $2 \cdot 2 = 1 \cdot 4$ in both cases) and that of Figure 3c is asymmetric (since $1 \cdot 2 \neq 0 \cdot 4$ and $3 \cdot 2 \neq 2 \cdot 4$) with respect to the range of the given rod.

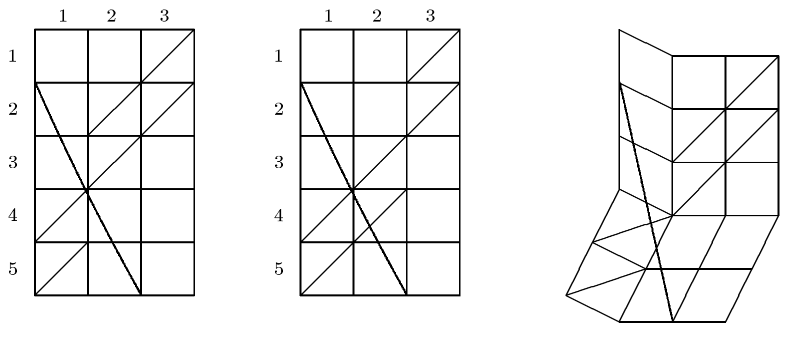


a, **b,** **c,**

Figure 3

Theorem 2.1 *Let a framework F consist of a $k \times l$ square grid, the diagonals of some rooms, plus a single long rod which is neither horizontal nor vertical. Then F is rigid if and only if $G(F)$ is either connected or is a 2-component graph which is asymmetric with respect to the range of the rod. (Accordingly, a collection C of certain diagonals is a minimum set making F (together with the long rod) rigid if and only if $|C| = k + l - 2$ and the corresponding edges form such an asymmetric 2-component forest of $G(F)$.)*

For example, the graphs of Figures 3b and 3c belong to the systems of Figures 4a and 4b, respectively. The second one is rigid, the first one is not; see a deformation in Figure 4c.



a, **b,** **c,**

Figure 4

Proof. **Step 1.** Suppose for a moment that we have a $k \times l$ square grid only, without any other rods. Then its deformations can be composed from elementary deformations of single corridors. For example, the deformation shown in Figure 4c was obtained from the translations of column 1 and rows 4 and 5. The translations of the rows and columns will be denoted by x_i's and y_j's, respectively.

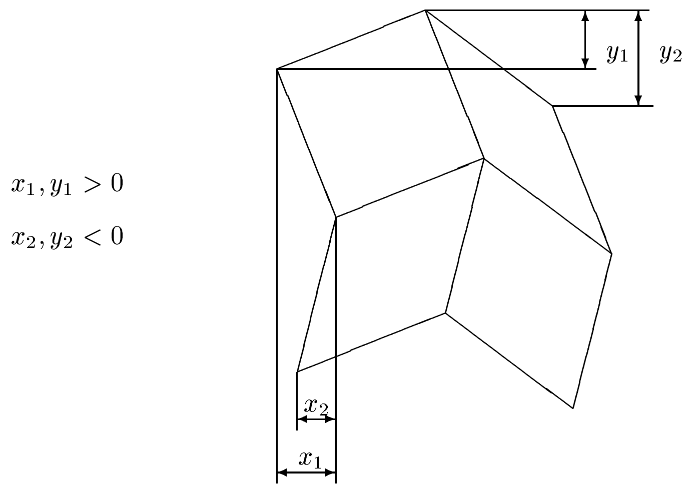


Figure 5

If a room (like (5;1) in Figures 4a and 4c) has a diagonal rod, it forces the deformations x_5 and y_1 to be of the same magnitude, which will be written, for short, as $x_5 = y_1$.

This is basically the proof of Theorem 1.1 – edges of $G(F)$ indicate that certain corridors can move only together and the connectedness of $G(F)$ means that $x_1 = x_2 = \cdots = x_k = y_1 = y_2 = \cdots = y_l$. Hence if this common value is nonzero then the motion of F is a rotation, i.e. a congruence. See also [5].

As a byproduct, we obtained the following:

Lemma 2.2 *If $G(F)$ in Theorem 2.1 is a forest with two components T_1 and T_2 then the only non-congruent motion of F has one magnitude, m_1, for each corridor belonging to $V(T_1)$ and another one, m_2, for each corridor belonging to $V(T_2)$.*

This statement is also illustrated by Figures 4a and 4c.

Step 2. If our long rod has range V_0 and we use the notation $A_0 = A \cap V_0$ and $B_0 = B \cap V_0$ then the physical effect of the rod can be described with the equation

$$|B_0| \cdot \sum_{i: X_i \in A_0} x_i = |A_0| \cdot \sum_{j: Y_j \in B_0} y_j \qquad (1)$$

To see this let us substitute a wide row for the rows of the range and similarly a wide column for the columns of the range for a moment. Now we have a rectangle with its diagonal, so it is rigid. Deforming the grid the vertices of this rectangle and the corresponding joints of the grid move together which results the equation above, because the effect of the diagonal rod in a rectangle is that $x/(\text{width of the row}) = y/(\text{width of the column})$.

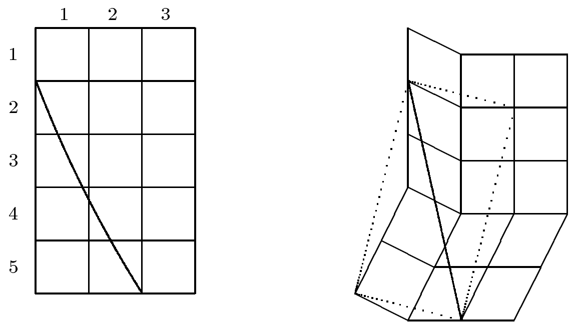


Figure 6

For example, the long rod in Figure 3a means $4(y_1+y_2) = 2(x_2+x_3+x_4+x_5)$. Observe that the equation $x_5 = y_1$ for a room diagonal, as mentioned in Step 1, is a special case of Eq. (1).

Step 3. If $G(F)$ is connected then F is rigid even if the long rod is removed. If $G(F)$ is a 2-component forest with components T_1 and T_2 then the only possible motion would be the one, described in Lemma 2.2. Recall that we denoted the vertex sets of T_1 and T_2 by V_1 and V_2 respectively. Substituting the quantities m_1, m_2 into Eq. (1) we obtain

$$|B_0|\cdot[|A_0\cap V_1|\cdot m_1+|A_0\cap V_2|\cdot m_2]=|A_0|\cdot[|B_0\cap V_1|\cdot m_1+|B_0\cap V_2|\cdot m_2]$$

or, equivalently,

$$[|B_0|\cdot|A_0\cap V_1|-|A_0|\cdot|B_0\cap V_1|]\cdot m_1=[|A_0|\cdot|B_0\cap V_2|-|B_0|\cdot|A_0\cap V_2|]\cdot m_2 \tag{2}$$

If $G(F)$ is symmetric with respect to V_0 then this equation reduces to $0 = 0$, that is, Eq. (1) is the consequence of the equations of the room diagonals. Thus the long rod does not prevent the only non-congruent motion, described in Lemma 2.2. Hence F is nonrigid.

Step 4. Finally, let $G(F)$ be an asymmetric 2-component forest with respect to V_0. Then neither the coefficient of m_1, nor that of m_2 in Eq. (2) is zero. Observe that these two nonzero coefficients are equal. This can easily be seen by re-writing $|A_0| \cdot |B_0|$ either as $|B_0| \cdot [|A_0 \cap V_1| + |A_0 \cap V_2|]$ or as $|A_0| \cdot [|B_0 \cap V_1| + |B_0 \cap V_2|]$. Then dividing both sides of Eq. (2) by this common value we obtain $m_1 = m_2$ which means that F is rigid (see the last paragraph preceeding Lemma 2.2). This completes the proof of Theorem 2.1.

Now let us consider the problem generally, so that we can use rods of any length in the square grid which are neither horizontal nor vertical. Each rod determines an equation of variables of the corridors, so we have a system of equations with $k+l$ variables and (in case of s long rods) with s equations. But its determinant is a 0,1-determinant and has a very simple structure.

To describe this structure we define the functions $\Delta_t : A \cup B \mapsto \{0,1\}$ for the rods R_t $(t = 1, \ldots, s)$. The value of $\Delta_t(X_i)$ or $\Delta_t(Y_j)$ is 1 if and only if X_i (or Y_j, respectively) is in the range of R_t (i.e. $X_i \in A_t$ or $Y_j \in B_t$ where V_t is the range of the rod R_t and $A_t = A \cap V_t$, $B_t = B \cap V_t$).

The system of equations is the following:

$$\begin{pmatrix} \Delta_1(X_1)|B_1| & \cdots & \Delta_1(X_k)|B_1| & -\Delta_1(Y_1)|A_1| & \cdots & -\Delta_1(Y_l)|A_1| \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Delta_s(X_1)|B_s| & \cdots & \Delta_s(X_k)|B_s| & -\Delta_s(Y_1)|A_s| & \cdots & -\Delta_s(Y_l)|A_s| \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ y_l \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Theorem 2.3 *The long rods $R_1, \ldots, R_s$ make the square grid infinitesimally rigid if and only if the rank of the matrix of coefficients, $r(M(R_1, \ldots, R_s)) = k + l - 1$, where*

$$M(R_1,\ldots,R_s) = \begin{pmatrix} \Delta_1(X_1)|B_1| & \cdots & \Delta_1(X_k)|B_1| & -\Delta_1(Y_1)|A_1| & \cdots & -\Delta_1(Y_l)|A_1| \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Delta_s(X_1)|B_s| & \cdots & \Delta_s(X_k)|B_s| & -\Delta_s(Y_1)|A_s| & \cdots & -\Delta_s(Y_l)|A_s| \end{pmatrix}$$

Proof. The framework is rigid if and only if the solutions of the system of equations form the 1-dimensional subspace of $\mathbf{R}^{k+l}$ induced by $(1, 1, \ldots, 1)$, because the elements of this subspace correspond exactly to the rotations of the framework. So the rank of the matrix must be $k + l - 1$.

Since the vectors of the above subspace are always solutions of the equations, if the framework is not rigid, the subspace of the solutions is at least 2-dimensional. So, the rank of matrix is at most $k + l - 2$.

Corollary. We need at least $k+l-1$ rods to make the framework rigid, and this amount is sufficient. For example in Figure 7 there is a structure without "short" diagonal rods.

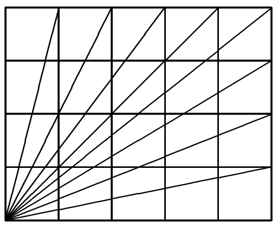


Figure 7

Let us consider now the one-story buildings. In this case we have to put two more equations into the system of equations [3], namely:

$$\begin{aligned} x_1 + x_2 + \ldots + x_k &= 0 \\ y_1 + y_2 + \ldots + y_l &= 0 \end{aligned} \tag{3}$$

Since the four corners are fixed, the framework is rigid if and only if the solution of the new system is unique: the zero vector.

Theorem 2.4 *The one-story building based on a $k \times l$ square grid is made rigid by (neither horizontal nor vertical) long rods $R_1, R_2, \ldots, R_s$ if and only if the rank of the corresponding matrix $\overline{M}(R_1, \ldots, R_s)$ is $k + l$, where*

$$\overline{M}(R_1,\ldots,R_s) = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ \Delta_1(X_1)|B_1| & \cdots & \Delta_1(X_k)|B_1| & -\Delta_1(Y_1)|A_1| & \cdots & -\Delta_1(Y_l)|A_1| \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Delta_s(X_1)|B_s| & \cdots & \Delta_s(X_k)|B_s| & -\Delta_s(Y_1)|A_s| & \cdots & -\Delta_s(Y_l)|A_s| \end{pmatrix}$$

Corollary. One needs $k + l - 2$ long rods to make the one-story building rigid (of course in addition to the 4 rods in the external vertical walls).

3 Long cables in the framework

In this section we discuss the problem of long cables: we would like to make the given frameworks rigid inserting long cables between some pairs of joints. But on the analogy of the problem of long rods we have to exclude the horizontal or vertical cables.

The *range* of a cable is defined in the same way as in the previous section, but now we have to know its direction too, because the effect of the cable is different in the two cases. So, if the direction of the cable L_t is southwest-northeast, let the number D_t be 1, and otherwise let D_t be -1. The meaning of A_t and B_t will be the range of the cable, just as in the former cases.

For example in Figure 8a $A_1 = \{X_2, X_3, X_4\}$; $B_1 = \{Y_1, Y_2\}$; $D_1 = 1$, while in Figure 8b $A_2 = \{X_2, X_3\}$; $B_2 = \{Y_2, Y_3, Y_4\}$; $D_2 = -1$.

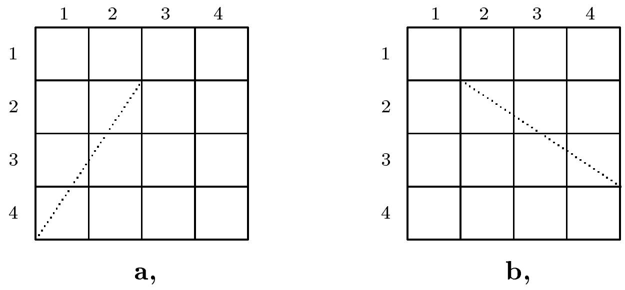


Figure 8

Following the same way as in case of the long rods we can observe that the physical effect of a long cable L_t can be described with this inequality:

$$D_t \cdot |B_t| \cdot \sum_{i:X_i \in A_t} x_i \geq D_t \cdot |A_t| \cdot \sum_{j:Y_j \in B_t} y_j \tag{4}$$

First, let us consider a simple problem: we have a $k \times l$ square grid and $k + l - 1$ short diagonal rods which form a spanning tree in $G(F)$, and we want to substitute one of the short diagonals for cable(s).

If we omit one of the short diagonals the remaining graph will be a 2-component forest, so we have to use at least one long cable. Obviously, only one is not sufficient, because we can choose the common value of the two components which satisfy the inequality corresponding to the cable. However, two cables will be already sufficient.

Theorem 3.1 *Two long cables, L_1 and L_2 make our framework rigid if and only if the 2-component forest formed by the $k+l-2$ short diagonals is asymmetric with respect to the ranges of both cables, and the quantities $P = D_1 \cdot (|B_1||A_1 \cap G_1| - |A_1||B_1 \cap G_1|)$ and $Q = D_2 \cdot (|B_2||A_2 \cap G_1| - |A_2||B_2 \cap G_1|)$ satisfy the equality: $P \cdot Q < 0$. (G_1 denotes the vertex set of the first component of the forest.)*

Proof. Both of the two components of the forest is connected, so the deformation of their corridors must be the same, m_1 and m_2, respectively. The cables determine the following system of inequalities:

$$\begin{pmatrix} P & -P \\ Q & -Q \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

because it was already seen that

$$|B_i||A_i \cap G_1| - |A_i||B_i \cap G_1| = -1 \cdot (|B_i||A_i \cap G_2| - |A_i||B_i \cap G_2|).$$

We need that the only solutions of this system will be the pairs $m_1 = m_2$, which correspond to the rotations. But it will be realized if and only if $P \neq 0$, $Q \neq 0$ (i.e. the forest is asymmetric both cases) and the two inequalities are opposite of each other (i.e. $P \cdot Q \leq 0$).

Now we can consider the real problem of this section: if we can use any kind of cables wich are neither horizontal nor vertical to make the framework rigid. At first we study the case of the $k \times l$ square grid when we use s cables in it. The effect of the cables can be described with a system of s inequalities corresponding to the cables:

$$\begin{pmatrix} D_1\Delta_1(X_1)|B_1| & \cdots & -D_1\Delta_1(Y_l)|A_1| \\ \vdots & \ddots & \vdots \\ D_s\Delta_s(X_1)|B_s| & \cdots & -D_s\Delta_s(Y_l)|A_s| \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ y_l \end{pmatrix} \geq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the function Δ_i is the same as in Section 2.

All the inequalities determine a halfspace in the $(k+l)$-dimensional space, and all the bordering hyperplanes include the straight line of $(c, c, \ldots, c)$ vectors. So the intersection of these halfspaces form an unbounded polyhedron or consists of only the common straight line. It's obvious that our framework is rigid if and only if this intersection equals the straight line.

We needed $k+l-1$ long rods to make the framework rigid, so we need at least as many cables, too. However, $k+l-1$ cables are not enough. In order to see this let us project the polyhedron to the hyperplane with normal vector $(1, 1, \ldots, 1)$. So our problem is reduced to the following: when will our polyhedron consist of only the origin? We know that a bounded polyhedron has at least (dimension $+1$) of bounding hyperplanes, and in this case it means $(k+l-1)+1 = k+l$ hyperplanes. Thus, we need at least $k+l$ long cables for the rigidity, and this amount will be sufficient. For example let us consider the cable-system in Figure 9.

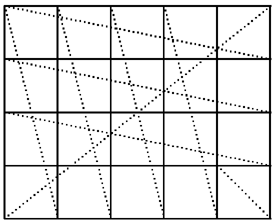


Figure 9

As we can see, there is a cable in each row of the grid (except the last one) crossing the whole corridor. It means that

$$x_i \leq \frac{\sum_{j=1}^{l} y_j}{l} \tag{5}$$

for every $1 \leq i \leq k-1$. The long cable in the diagonal of the whole grid causes that

$$\frac{\sum_{i=1}^{k} x_i}{k} \geq \frac{\sum_{j=1}^{l} y_j}{l}$$

which, together with Eq.(5) gives

$$x_k \geq \frac{\sum_{j=1}^{l} y_j}{l} \quad \text{and so} \quad x_k \geq \frac{\sum_{i=1}^{k} x_i}{k} \tag{6}$$

Since the cables have the same situation in the vertical corridors we have two more equalities like

$$y_j \geq \frac{\sum_{i=1}^{k} x_i}{k} \quad \text{for} \quad 1 \leq j \leq l-1 \tag{7}$$

$$y_l \leq \frac{\sum_{i=1}^{k} x_i}{k} \tag{8}$$

The effect of the remaining short diagonal cable is that $x_k \leq y_l$. This relation and the inequalities (6) and (8) imply that $x_k = y_l$. Hence the inequalities (5) and (7) are realized as equations, so $x_i = y_j$ for every pair (i, j).

Now we know that $k+l$ long cables are sufficient to make a $k \times l$ square grid rigid. We can consider a new problem: we have $k+l$ long cables, but we know only their ranges, their directions must be chosen by us. The question is: have we a good choice of directions ensuring rigidity?

Theorem 3.2 *The given $k+l$ long cables have right directions for rigidity if and only if any $k+l-1$ rows of the corresponding matrix without the D_t multipliers are independent. If there exists an appropriate direction it is unique apart from reversing the directions of all the cables.*

Proof. Let us suppose first that we found a rigid system of cables. For each cable the intersection of the corresponding halfspace in the $k+l$-dimensional space and the polyhedron bounded by the remaining $k+l-1$ hyperplanes consists of the common straight line only. So, the straight line must be a real edge of the polyhedron, which involves that the corresponding rows of the matrix are independent.

Now let us suppose that the conditions of the theorem referring to the matrix are satisfied. To see the unicity we will show that fixing the direction of an arbitrary cable determines the directions of all the other cables. Consider any cable together with the fixed one. The faces of the polyhedron determined by the remaining $k+l-2$ cables are at least 2-dimensional, and there is exactly one 2-dimensional face which contains the common straight line. The two hyperplanes corresponding to the chosen cables partition the $k+l$-dimensional space into four parts from which two opposite parts are intersected by the 2-dimensional face since, by the conditions, none of the two hyperplanes contains this face.

The direction is appropriate if and only if the space quarter determined by the two cables is empty, so the direction of the fixed cable determine the other direction.

On the other hand, if we have $k+l$ cables with the given conditions we can fix the direction of one of them and we can determine the directions of the other cables as above. If it is not a good system of cables there must be two cables so that the intersection of the corresponding halfspaces contains the 2-dimensional face of the polyhedron of the remaining cables. But the 2-dimensional face is in the hyperplane corresponding to the fixed cable, and when we determined one of the directions of these two cables the actual face was contained by the hyperplane of the other cable. So, one of the two directions contradicts to the definition above.

We saw in the previous section that we need just as many long rods as of short diagonals to make the framework rigid. But it is not true for the cables. We know that at least $2\max(k,l)$ short diagonal cables are necessary to the rigidity, but we saw above that "only" $k+l$ long cables are sufficient. And it is a great decrease, in the case of $l=1$ this number is about the half of the number of short diagonal cables.

The problem of one-story building with long cables is similar to that with long rods, so $k+l-1$ cables are not sufficient but one more can be. A simple example can be obtained from Figure 9. Let us consider the system of the planar case. Then we can omit the longest cable (which crosses the whole grid) if the four corners were fixed (see Figure 10).

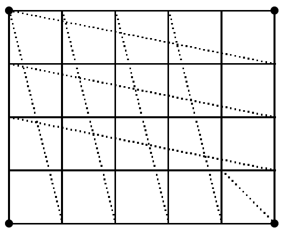


Figure 10

4 Algorithmic aspects

We mentioned in Section 1 that checking the rank of the rigidity matrix requires $O(k^3l^3)$ operations. It is true for the long rods or cables, too, because the size of the matrix is the same, $O(kl)$. But the size of the matrices in our results is $O(k+l)$, so to check their rank requires $O(\max(k^3,l^3))$ operations only.

Acknowledgements: Research partly supported by grants No. OTKA 17181, 17580 and 24037 of the Hungarian National Science Foundation and by grant No. FKFP 409/1997 of the Hungarian Ministry of Education.

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